Foundations
A Model Theoretic Perspective

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Outline

A Data point for PBPL

Practice based philosophy of logic

Are model theorists logicians?

They do not analyze methods of reasoning.

A Data point for PBPL

Practice based philosophy of logic

Are model theorists logicians?

They do not analyze methods of reasoning.

A model theorist is a self conscious mathematician

Rather they utilize various formal languages and semantics to prove mathematical theorems.

Sociology

The discussion below will focus on two currents of 'main-stream' model theory.
It does not encompass:

1. models of arithmetic
2. finite model theory /model theory in computer science
3. generalized quantifiers
4. universal algebra

Most of the emphasis is on first order– for brevity. There are important extensions to infinitary logic and even 'syntax deprived' model theory.

Mid-Atlantic Model Theory 2008

1. Model theory and non-archimedean geometry
2. The valuation inequality for complex analytic structure
3. Cherlin's Conjecture and Generix's Adventures in Groupland
4. ω-stable semi-Abelian varieties
5. O-minimal triangulation respecting a standard part map
6. Some modest attempts at defining the notions of groups and fields of dimension one, and establishing their algebraic properties
7. Dependent theories: limit model existence and recounting the number of types
8. The non-elementary model theory of analytic Zariski structures
9. Difference fields, model theory and applications
10. Model Theory of the Adeles

A tool for PBPM

Practice Based Philosophy of Mathematics

foundationS of mathematics

If X is any field of study, "foundations of X" refers to a more-or-less systematic analysis:

1. of the most basic or fundamental concepts of field X.
2. of the basic methodologies and proof techniques of the subject.
3. comparison with other fields

Below, I expound the use of logic as a tool for such an analysis.

Thesis I:

Studying the model of different(complete first order) theories provides a framework for the understanding of the foundations of specific areas of mathematics.

Three types of model theoretic analysis:

1. Properties of first order logic (1930-1965)
2. Properties of complete theories (1950-present)
3. Properties of classes of theories (1970-present)

Review of 'The Birth of Model Theory' by C. Badesa
http://www2.math.uic.edu/~ jbaldwin/pub/birthbbltrev.pdf
google Baldwin Badesa

Properties of first order logic (1930-1965):

1. Completeness and Compactness
2. Lowenheim-Skolem
3. syntactic characterization of preservation theorems
4. Interpolation

Complete Theory

A theory T is complete if for every sentence ϕ,

$$T \vdash \phi$$

or

$$T \vdash \neg\phi$$

Note that for any structure M,

$$\mathrm{Th}(M) = \{\phi : M \models \phi\}$$

is a complete theory.

Properties of complete theories (1950's):

1. Complete Theories (A. Robinson)
2. Elementary Extension (Tarski-Vaught)
3. Models Generated by Indiscernibles (Ehrenfeucht-Mostoski)
4. quantifier elimination and model completeness (Robinson and Tarski)

Algebraic examples: complete theories

Algebraic Geometry

Algebraic geometry is the study of definable subsets of algebraically closed fields
Not quite: definable by positive formulas

Chevalley-Tarski Theorem

Chevalley: The projection of a constructible set is constructible.
Tarski: Acf is admits elimination of quantifiers.

More precisely, this describes 'Weil' style algebraic geometry.

Algebraic consequences for complete theories

1. Artin-Schreier theorem (A. Robinson)
2. Decidability and qe of the real field (Tarski)
3. Decidability and qe of the complex field (Tarski)
4. Decidability and model completeness of valued fields (Ax-Kochen-Ershov)
5. quantifier elimination for p-adic fields (Macintyre)
6. o-minimality of the real exponential field (Wilkie)

On Mathematical methodology

Thesis II:

Studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics.

Properties of classes of theories (1970-present)

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

The stability hierarchy: examples

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields, complex compact manifolds

strictly superstable

$(\mathcal{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$,

strictly stable

$(\mathcal{Z}, +)^\omega$, separably closed fields,

unstable

Arithmetic, Real closed fields, complex exponentiation, random graph

Concept Analysis: Dimension

Two Senses of Dimension

Size The reals have uncountable dimension as $\mathbb{Q}$-vector space.

Rank A surface is a two-dimensional set.

Size is a measure of a model.

rank is a measure of a definable set.

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
A4. $cl(cl(X)) = cl(X)$
If points are closed the structure is called a geometry.

Generalization

STRONGLY MINIMAL

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

The complex field is strongly minimal.

Strongly minimal set are the building blocks of structures whose first order theories are categorical in uncountable power.

Quasiminimality

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. Closure of countable sets is countable

Theorem

A quasiminimal class is $\aleph_1$-categorical.

Conjecturally Complex exponential field.

Quasiminimal sets are the building blocks of structures whose $L_{\omega_1,\omega}$-theories are categorical in uncountable power. (VWGCH)

(Families) of Combinatorial Geometries

Two important generalizations occur in stability theory. The key is to define a family of 'almost combinatorial geometries': $a \in \mathrm{cl}_A(B)$
For each A, $\mathrm{cl}_A(*)$ satisfies the first three conditions of a combinatorial geometry (and coherence conditions among the cl_A's).

Recall Definition

Approximate dimension

Dimension of Uncountable Structures

One can assign dimension (size) to uncountable models by replacing

$cl(cl(X)) = cl(X)$

with

For every B, there is a finite B_0 such that a is independent from B over B_0.

(This condition holds in superstable theories.)

Precise dimension

Consider relations on and among regular types.

Regaining a combinatorial geometry

$p \in S(A)$ is regular if on realizations of p, if $a \in \mathrm{cl}_A(BC)$ and each $c \in C$ satisfies $c \in \mathrm{cl}_A(B)$ then $a \in \mathrm{cl}_A(B)$.

Now one can give an exact dimension to models (by means of the relations between various regular types) of superstable theories satisfying certain additional conditions.

Consequences: Main Gap

Using these ideas of dimension Shelah proved:

Main Gap

For every first order theory T, either

1. Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or
2. The theory T has the maximal number of models in all uncountable cardinalities.

Connecting the two notions of dimension

Suppose T has a good notion of independence. Then define $R(\phi(\mathbf{x})) = n$ iff the maximal dimension of a solution $\mathbf{a}$ if ϕ is n.

Limitations on Sense II (Rank)

If a model admits a pairing function, it has no rank (dimension in sense I)
e.g. Arithmetic

Theorem

If T admits a pairing function then T is not superstable.

Various notions of rank

1. Zariski dimension in algebraic geometry is a special case.
2. This is generalized to the Hrushovski-Zilber notion of Zariski Geometries.
3. In the class of *o*-minimal theories there is a Rank on definable sets. This allows a general framework answering Grothendieck's call for 'tame topologies'.

‘Tame Theories’: Rank

There is a Rank on definable sets.
There are regularity properties connecting the different dimensions.
Roughly, a definable set ϕ has rank n if there is a definable bijection between M^n and $\phi(M)$.

o-minimality

An ordered structure is *o*-minimal if every definable set is a Boolean Combination of intervals.
o-minimal structures are a natural solution to Grothendieck’s request to isolate ‘tame topologies.

Conclusion

Thesis II:

Studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics.

We have illustrated this thesis by connecting the notion of dimension in the study of

1. algebraic geometry
2. complex exponentiation
3. general model theory
4. tame topology