How to Cover without Lifting Relations

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Introduction

It has by now been recognised that coalgebras of various different endofunctors can be used to give semantics to a wide range of modal logics [5] and the last decade has seen an intensive study of modal languages based on predicate liftings [10] and logical languages based on the so-called *cover modality*, first introduced in [8]. Given an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ on the category of sets and functions, the role of *frames* is played by T-coalgebras (C, γ) where C is a set (the carrier) and $\gamma : C \to TC$ is a (transition) function. Different choices for T then define different types of frame classes. In this framework, Kripke frames arise as T-coalgebras for the covariant powerset functor $T = \mathcal{P}$, (monotone) neighbourhood frames arise as T-coalgebras for $T = \mathcal{E}$ (resp. $T = \mathcal{M}$) where $\mathcal{E}X = \mathcal{P} \circ \mathcal{P}$ is the composition of the (contravariant) powerset functor with itself, and $\mathcal{M}(X) = \{N \in \mathcal{E}X \mid N \text{ upclosed}\}$, and probabilistic frames arise as coalgebras for $T = \mathcal{D}$ where $\mathcal{D}X$ is the set of finitely supported probability distributions over X.

The idea behind the cover modality is to provide a generic modal syntax that can be developed for all classes of frames that arise as T-coalgebras, and the endofunctor T that defines the frame classes serves double duty as a syntax constructor. In case of probabilistic frame, for example, the modal formulas are (finitely supported) probability distributions over (already constructed) formulas. In general, given $T : \mathsf{Set} \to \mathsf{Set}$, the induced language $\mathcal{L}(T)$ contains the formula $\nabla\Phi \in \mathcal{L}$ whenever $\Phi \in T\mathcal{L}$, the semantics of which is given by *relation lifting*. If (C, γ) is a T-coalgebra and $c \in C$, we have that $(C, \gamma), c \models \nabla\Phi$ iff $\gamma(c)\overline{T}(\models)\Phi$ where $\overline{T}$ lifts the action of T to *relations*, and can therefore be applied to the satisfaction relation $\models \subseteq C \times \mathcal{L}(T)$ in a meaningful way.

Languages arising in this way are by now well-developed: Venema presents automata for fixpoint logics based on the cover modality [12], complete axiomatisations may be found in [3, 7] and Moss' original paper [8] establishes the Hennessy-Milner property.

While the cover modality provides a generic treatment of many different classes of frames, all of the above results rely on the assumption that the functor that defines the frame classes *preserves weak pullbacks* which precludes the instantiation of the generic theory to important examples such as (monotone) neighbourhood frames [6] or conditional frames [4]. The conceptual reason for requiring that the underlying endofunctor preserve weak pullbacks lies at the very heart of the definition of the semantics of the cover modality: the endofunctor needs to be lifted to *relations* and weak pullback preservation is required to ensure functoriality of this lifting.

The question of possible ways to extend the treatment of the cover modality to classes of (coalgebraic) frames without requiring the preservation of weak pullbacks has so far remained open. Taking formulas to be elements of the final sequence of the underlying endofunctor removes the requirement for weak pullback preservation [9], but does not provide a syntactial notion of modal operator. Other advances have been made for monotone neighbourhood frames [11] (where weak pullback preservation fails) by means of changing the definition of relation lifting, however it seems difficult to obtain a generalisation to arbitrary classes of frames.

Here, we take a different approach. Given that the extension of an arbitrary endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ to relations is problematic, we argue that the semantics of modal formulas should *not* be defined by extending T to *relations*, *but* simply by applying T to *functions*, where logical properties can be derived in terms of the functor laws. Rather than defining the semantics of the cover modality by applying (the lifting of) T to the *satisfaction relation* we propose an alternative semantics that arises by applying T to the *theory map*. In particular, this entails that logics conceived in this way are no longer monotonic which precludes e.g. fixpoint extensions in the style of [12]. Our main findings indicate that this is nonetheless a viable approach that leads to a rich theory: we are able to establish a Hennessy-Milner property, a logical distributive law and a complete tableau calculus without any assumptions on the endofunctor that defines the underlying frame class.

Syntax, Semantics and Properties

Suppose that $T : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor. A *T-frame* is a T-coalgebra (C, γ) where $C \in \mathsf{Set}$ and $\gamma : C \to TC$ is a function. A *T-model* is a triple (C, γ, π) where (C, γ) is a T-frame and $\pi : C \to \mathcal{P}(\mathcal{V})$ is a valuation (of propositional variables $\mathcal{V}$). We may assume without loss of generality that T preserves set-theoretic inclusions and intersections [1, 2] (including empty intersections).

Definition 1. The *language induced by T* is given by the grammar

$$\mathcal{L}_{\mathcal{V}}(T) \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \blacktriangledown\Phi$$

where $p \in \mathcal{V}$ and $\Phi \in T\mathcal{P}(\Sigma)$ for some finite set $\Sigma \subseteq \mathcal{L}_{\mathcal{V}}(T)$ and a set $\mathcal{V}$ of propositional variables. If $\mathbb{C} = (C, \gamma, \pi)$ is a T-model and $c \in C$, then $\mathbb{C}, c \models \blacktriangledown\Phi \iff T(t) \circ \gamma(c) = \Phi$ where $t = t^{\mathbb{C}}_{\Phi} : C \to \mathcal{P}(\mathsf{supp}(\Phi))$ is the *local theory map* defined by $t(c) = \{\phi \in \mathsf{supp}(\Phi) \mid \mathbb{C}, c \models \phi\}$ and $\mathsf{supp}(\Phi) = \bigcap\{\Psi \subseteq \mathcal{L}_{\mathcal{V}}(T) \mid \Phi \in T\Psi\}$ is the support of Φ.

Applied to Kripke frames and monotone neighbourhood frames, we obtain equi-expressive languages witnessed by the following translations:

Example 2. If $T = \mathcal{P}$, then T-models are Kripke models. If $\mathcal{L}(\Box)$ is the standard modal language, we have the equivalence on the left below

$$\Box\phi = \blacktriangledown\varnothing \vee \blacktriangledown\{\{\phi\}\} \qquad \Box\phi = \blacktriangledown\uparrow\{\alpha_0\} \vee \blacktriangledown\uparrow\{\alpha_0, \alpha_1\}$$

which states that the theory-set of all successors either has to be empty or just contain ϕ itself. Similarly over monotone neighbourhood frames (coalgebras for $T = \mathcal{M}$) and again with the standard, but now different reading of $\Box$ we have the equivalence on the right above, where $\alpha_0 = \{\{\phi\}\}$, $\alpha_1 = \{\varnothing\}$ and $\uparrow$ denotes upward closure.

The resulting language is invariant under (coalgebraic) behavioural equivalence and characterises the latter in case of finitely branching systems.

Proposition 3. *If $\mathbb{C} = (C, \gamma, \pi)$ and $\mathbb{D} = (D, \delta, \sigma)$ are T-models then any two behaviourally equivalent points $c \in C$ and $d \in D$ have the same logical theory. If T is finitary then the converse holds as well.*

Similarly to the situation with modal languages in terms of Moss' cover modality [13], one can establish that – if T maps finite sets to finite sets – conjunction can be eliminated and negation can be pushed to atomic propositions. For example $\neg\blacktriangledown\Phi$ can be replaced by the disjunction of $\blacktriangledown\Psi$ where Ψ ranges over the complement of Φ under T. This allows us to view formulae as finite non-deterministic automata, where disjunctions represent non-determinism and $\blacktriangledown$-formulas constrain the behaviour of a system, ultimately leading to a characterisation of $\mathcal{L}_{\mathcal{V}}(T)$ in terms of non-deterministic finite automata.

Proposition 4. *Suppose that T maps finite sets to finite sets. Then every formula of $\mathcal{L}_{\mathcal{V}}(T)$ is equivalent to a formula without conjunctions where negations only occur in front of atomic propositions.*

For completeness, the situation is particularly pleasant, and is most conveniently described in terms of a tableau calculus that only consists of invertible rules. The key here to first remove negations and conjunctions to obtain a disjunctive formula, which is satisfiable if and only if each of the local theories it depends on is satisfiable.

Proposition 5. *If T preserves finite sets, then satisfiability of a formula in $\mathcal{L}_{\mathcal{V}}(T)$ can be characterised by a tableau calculus consisting of invertible rules only.*

We currently investigate to what extent $\mathcal{L}_{\mathcal{V}}(T)$ also admits bisimulation quantifiers, following [11], and hope to make the relationship to finite automata precise.

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