On the Expressivity of Dynamic Topological Logics

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Kutaisi 2011: Ninth International Tbilisi Symposium on Language, Logic and Computation

September 26-30,2011,
Kutaisi, Georgia

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Motivating Idea

Our aim is to investigate expressive power of dynamic topological languages depending on the ability for expressing the orbit behaviors.

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In this direction,

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- Then, in search for the right language to capture orbit behaviors we investigate various ways to enrich the language and we obtain first language which has the intended expressive power,

- Finally, we propose a different interpretation of key modal connectives over dynamical systems and by enriching this new basic dynamic topological language we obtain second language which also has the intended expressive power but does not include very powerful operators.

Our results indicate that the new interpretation is more suitable for reasoning about topological dynamical systems.

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Our results indicate that the new interpretation is more suitable for reasoning about topological dynamical systems.

Definition

(Brown, J., R., 1976) and (Katok, A. and Hasselblatt, B., 1998) A *topological dynamical system (tds)* is an ordered pair (X, f) where $X = (X, \tau)$ is a topological space and f is a continuous function on X.

Definition

The *orbit* of a point $x \in X$ in a tds (X, f) is a set

$$O_x = \{x, f(x), f^2(x), ..., f^n(x), ...\}$$

consisting of x and all iterates of f on x.

For example, when the function that we use to iterate is $f(x) = x^2$, the orbit of the starting point $x_0 = 2$ is $O_2 = \{2, 4, 16, ..., 2^n, ...\}$.

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Example

With $f(x) = x - x^2$ and usual topology τ on $\mathbb{R}$,

i) $O_0 = \{0, 0, 0, ...\}$ is finite set, so the orbit of 0 is finite,

ii) $O_{\frac{1}{3}} = \{\frac{1}{3}, \frac{2}{9}, \frac{14}{81}, ...\}$ is infinite set, orbit of $\frac{1}{3}$ is infinite

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Let (X, f) be a tds. We classify the orbits of the point $x \in X$ under the function f as follows:

- **Constant orbit:** O_x is called *constant* and x itself is called *fixed point* iff $f(x) = x$.
- ***n*-periodic orbit:** O_x is called *n-periodic* iff it is not constant, $f^n(x) = x$ and for any $1 < m < n$, $f^m(x) \neq x$.
- **Periodic orbit:** O_x is called *periodic* iff it is n-periodic for some $n > 1$ or it is constant.

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For any property P of the orbits,

- ***m*-step P orbit:** For any $m > 0$, O_x is called *m-step P* iff $O_{f^m(x)}$ has the property P and for any $n < m$, $O_{f^n(x)}$ has property not$-P$.
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- **Eventually *P* orbit:** O_x is called *eventually P* iff it is m-step P for some $m > 0$.

- **Convergent orbit:** O_x is called *convergent* iff it is infinite (not periodic, not eventually periodic) and it has at least one limit point.
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Now under the consideration of the classification above we can present the following examples:

Example

With $f(x) = x^2$ and usual topology τ on $\mathbb{R}$,

i) $O_1 = \{1, 1, 1, ...\}$ is constant orbit, 1 is a fixed point,

ii) $O_{-1} = \{-1, 1, 1, ...\}$ is 1-step constant orbit and also eventually constant orbit,

iii) $O_{\frac{1}{3}} = \{\frac{1}{3}, \frac{1}{9}, \frac{1}{81}, ...\}$ is convergent orbit,

iv) $O_5 = \{5, 25, 625, ...\}$ is divergent orbit

on tds $(\mathbb{R}, f)$.

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Definition

(Kremer, P. and Mints, G., 1997) Basic dynamic language $\mathtt{DL}_{\langle\to\rangle}$ consists of propositional letters $PROP = \{p, q, r, ...\}$, the boolean connectives $\wedge, \neg$, the constant truth $\top$, a modal box operator $[c]$ and the temporal operators $\langle\to\rangle$(next time), $\langle\to\rangle^+$(eventually). More precisely, $\mathtt{DL}_{\langle\to\rangle}$ formulas are given by the following definition:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \psi \mid [c]\phi \mid \langle\to\rangle\phi \mid \langle\to\rangle^+\phi$$

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For a given property P of orbits, we will say that P is expressible on the pointed structure $(X, f), x$ in a given language L and a given interpretation $\models$, if there exists a formula $\phi \in L$ such that

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Constant and periodic orbits are expressible in basic dynamic language $\mathtt{DL}_{\langle\rightarrow\rangle}$ by the following formulas, respectively:

- $p \rightarrow \langle\rightarrow\rangle p$
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By defining a suitable notion of bisimulation and a consequent notion of validity preserving morphisms between tds, we show that all the other considered orbit behaviors are not expressible in $\mathtt{DL}_{\langle\rightarrow\rangle}$ even in the presence of an additional global modality E.

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- Adding nominals,
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- *Nominals* are propositional variables that denote singleton sets and usually they are shown by the letters '$i, j, k, \ldots$' . Truth definition for nominals is the same as for propositional letters:

$$M, x \models i \;\Leftrightarrow\; v(i) = \{x\}$$

Modal languages containing nominals are called *hybrid languages*. Hybrid formulas with no occurrences of propositional variables are called *pure*.

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Dynamic hybrid language $\mathtt{DHL}_{\langle\to\rangle}$ which is the extension of $\mathtt{DL}_{\langle\to\rangle}$ with nominals and satisfaction operator is defined as follows:

Definition

Given a countable set of ordinary propositional letters $PROP$ and a countable set of nominals $NOM = \{i, j, k, ...\}$ disjoint from $PROP$, we define the formulas of $\mathtt{DHL}_{\langle\to\rangle}$ to be

$$\phi ::= \top \mid p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid [c]\phi \mid @_i\phi \mid \langle\to\rangle\phi \mid \langle\to\rangle^+\phi$$

where $p \in PROP$ and $i \in NOM$. A substitution instance of ϕ is defined as usual in propositional logic with the additional requirement that nominals can be replaced only with nominals.

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Let M be a tdm. The semantic of @ is as follows:

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n-periodic orbits are expressible in $\mathtt{DHL}_{\langle\to\rangle}$ by the following formula:

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For a given property P of orbits, we will say that P is expressible on the level of models by pure hybrid formula $\phi(i)$ having a nominal i, if for any tdm $M = (X, f, v)$, the following holds

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If some property P of the orbits is expressible on the level of models by pure hybrid formula $\phi(i)$, then $m-$step P is expressible in $\mathtt{DHL}_{\langle\rightarrow\rangle}$ by the formula

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The state variables can be bound to the current point of evaluation by using $\downarrow$ binder.

Given a countable set of ordinary propositional letters *PROP*, a countable set of nominals *NOM* disjoint from *PROP* and a countably infinite set of state variables $SVAR = \{u, v, w, ...\}$ disjoint from *PROP* and *NOM*. The formulas of $\mathtt{DHL}_{\langle\rightarrow\rangle,\downarrow}$ are given by the following recursive definition:

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$$\begin{array}{lcl} M, g, x \models u & \Leftrightarrow & g(u) = x \\ M, g, x \models @_u \varphi & \Leftrightarrow & M, g, g(u) \models \varphi \\ M, g, x \models \downarrow u.\phi & \Leftrightarrow & M, g^{[u \mapsto x]}, x \models \phi; \end{array}$$

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By defining a suitable notion of bisimulation, we show that convergent orbits are not expressible in $\mathtt{DHL}_{\langle\to\rangle,\downarrow}$.

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Definition

The global modality E allows us to express that a formula holds somewhere in the model: $E\phi$ is true at point x iff there is a point y (not necessarily related to x) satisfying ϕ.

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Convergent and divergent orbits are expressible in $\mathtt{DHL}_{\langle\rightarrow\rangle,\downarrow,E}$ by the following formulas, respectively:

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- In this section, we propose to change the interpretation of the 'temporal' modalities. We consider the operator $\langle\leftarrow\rangle$ which is interpreted as f rather than f^{-1} on a tds (X, f). The language with $\langle\leftarrow\rangle$ and $\langle\leftarrow\rangle^+$ is denoted by $\mathtt{DL}_{\langle\leftarrow\rangle}$ and we study the expressive power of this language.

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Constant and periodic orbits are expressible in the basic dynamic language $\mathtt{DL}_{\langle\leftarrow\rangle}$ by the following formulas, respectively:

- $p \to \langle\leftarrow\rangle p$
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By defining a suitable notion of bisimulation and a consequent notion of validity preserving morphisms between tds, we show that all the other considered orbit behaviors are not expressible in $\mathtt{DL}_{\langle\leftarrow\rangle}$ even in the presence of an additional global modality E.

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where $p \in PROP$ and $i \in NOM$.

If some property P of the orbits is expressible on the level of models by pure hybrid formula $\phi(i)$, then $m-$step P and eventually P are expressible in $\mathtt{DHL}_{\langle\leftarrow\rangle,E}$ by the following formulas, respectively:

- $i \to (\neg\phi(i) \wedge E((\langle\leftarrow\rangle^m i \wedge j) \to \phi(j)) \wedge A(\bigwedge_{s=1}^{m-1}((\langle\leftarrow\rangle^s i \wedge k) \to \neg\phi(k)))$
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Definition

Finally, to increase expressive power of the language $\mathtt{DHL}_{\langle\leftarrow\rangle,E}$, we change the interpretation of topological box operator interior to limit. Corresponding operator is shown by $[d]$.

Given a countable set of ordinary propositional letters *PROP* and a countable set of nominals *NOM* disjoint from *PROP*. We define the formulas of $\mathtt{DHL}_{\langle d\rangle,\langle\leftarrow\rangle,E}$ to be

$$\phi ::= \top \mid p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid [d]\phi \mid @_i\phi \mid \langle\leftarrow\rangle\phi \mid \langle\leftarrow\rangle^+\phi \mid E\phi$$

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Let M be a tdm. Truth definition for $[d]$ is as follows:

$M, x \models [d]\phi$ iff $\exists U_x \in \tau$ s.t. $\forall y \in U_x$, $y \neq x$ implies $M, y \models \phi$.

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