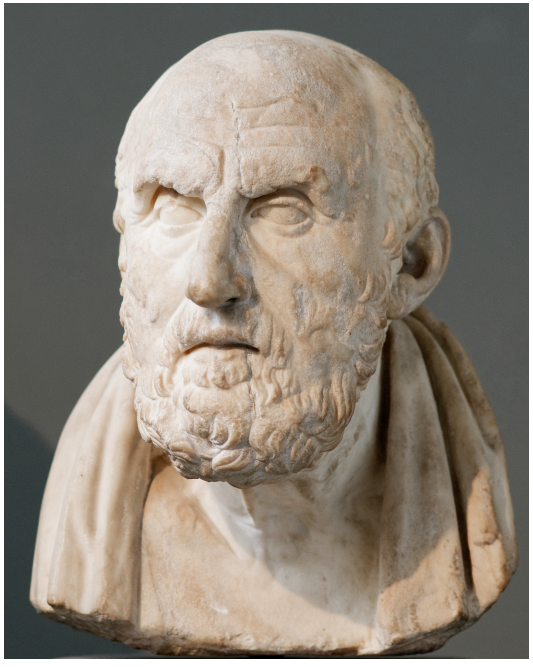
A review of classical propositional logic



Chrysippus, c. 279 BC – c. 206 BC.

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To construct compound formulæ we use the logical connectives:

- $\vee$, for disjunction ("inclusive or", Latin *vel*, unlike Latin *aut*);
- $\wedge$, for conjunction ("and", Latin *et*);
- $\rightarrow$, for implication ("if. . . then. . . ", conditional assertions);
- $\neg$, for negation ("not", negative assertions).

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Let us write $\textsc{Form}$ for the set of all formulæ constructed over the countable language $X_1, \ldots, X_n, \ldots$, and $\textsc{Form}_n$ for the set of formulæ constructed over the first n propositional variables $X_1, \ldots, X_n$, at the most.

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- $w(\neg\alpha) = 1$ if and only if $w(\alpha) = 0$.

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Following a well-established tradition going back at least to G. W. Leibniz, it is useful to think of an evaluation $w \colon \textsc{Form} \to \{0, 1\}$ as a possible world in which each proposition α either holds — i.e., $w(\alpha) = 1$ — or fails to hold – i.e., $w(\alpha) = 0$.

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Because of truth-functionality, then, a classical logician's world is something very ethereal indeed: it is uniquely determined by the collection of true atomic propositions that can be uttered.

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Next we select a set of deduction rules that tell us that if we already established that formulæ $\alpha_1, \ldots, \alpha_n$ are provable, and these have a certain shape, then a specific formula β is also a provable formula.

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(A0–A5) read as shown next.

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Deduction rule for classical logic.

(R1)	$\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$	*Modus ponens.*

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- $\alpha \vee \beta \;\equiv\; (\alpha \to \beta) \to \beta$
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Exercise. Show that $\vdash \neg\neg\alpha \to \alpha$. Then pause to appreciate why proof-theorists develop Gentzen-style calculi to replace Hilbert-style calculi such as the one above.

Easy to show:

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Theorem (Validity or Soundness Theorem for CL)

For any $\alpha \in \textsc{Form}$,

$$\vdash \alpha \quad \textit{implies} \quad \models \alpha\,.$$

That is,

$$\textsc{Thm} \subseteq \textsc{Taut}\,.$$

(Similarly, $\textsc{Thm}_n \subseteq \textsc{Taut}_n$).

Harder to show:

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Theorem (Completeness Theorem for CL)

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Theorem (Soundness and Completeness Theorem for CL)

For any $\alpha \in \textsc{Form}$,

$$\vDash \alpha \quad \textit{if, and only if,} \quad \vdash \alpha\,.$$

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Immediate consequence:

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Corollary (Decidability of CL)

There is a an algorithm (for definiteness, a Turing machine) that, on input $\alpha \in \textsc{Form}$, *outputs* $\textsc{Yes}$ *if* $\alpha \in \textsc{Thm}$, *and* $\textsc{No}$ *otherwise.*

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In light of completeness, one such algorithm is: check all possible $\{0,1\}$-valued assignments to the n variables occurring in α. Since there is 2^n of them, computationally this is a pretty silly algorithm.

An yet, it is hard to do structurally better:

Theorem (S. Cook, 1971; L. Levin, 1973)

The problem of deciding classical logic is co-$\textsc{np}$-complete.

Łukasiewicz infinite-valued propositional logic

Jan Łukasiewicz, 1878–1956.

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The formulæ of Łukasiewicz logic are the same as the ones in CL: $\textsc{Form}$ and $\textsc{Form}_n$ are as before.

Axiom system for Łukasiewicz logic.

(A0)	$\bot \to \alpha$	*Ex falso quodlibet.*
(A1)	$\alpha \to (\beta \to \alpha)$	*A fortiori.*
(A2)	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	Implication is transitive.
(A3)	$(\alpha \vee \beta) \to (\beta \vee \alpha)$	Disjunction is commutative.
(A4)	$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$	Contraposition.
~~(A5)~~	~~$\alpha \vee \neg\alpha$~~	~~*Tertium non datur / Consequentia Mirabilis.*~~

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~~(A5)~~ ~~$\alpha \vee \neg\alpha$~~ ~~*Tertium non datur / Consequentia Mirabilis.*~~

Deduction rule for Łukasiewicz logic.

(R1) $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$ *Modus ponens.*

Łukasiewicz logic can be succinctly described as classical logic without the Aristotelian law of Tertium non datur, but with the Ex falso quodlibet law.

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> ~~*Łukasiewicz logic*~~ *Intuitionistic logic can be succinctly described as classical logic without the Aristotelian law of Tertium non datur, but with the Ex falso quodlibet law.*[1]

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[1] Almost *verbatim* from J. Moschovakis, *Intuitionistic Logic*, The Stanford Encyclopedia of Philosophy, 2010, Edward N. Zalta (ed.).

> ~~*Łukasiewicz logic*~~ *Intuitionistic logic can be succinctly described as classical logic without the Aristotelian law of Tertium non datur, but with the Ex falso quodlibet law.*[1]

Such "succint descriptions" can be polysemous to a surprising extent indeed.

Moral: The import of removing an axiom from an axiom system depends on the axiom system itself. In particular, Hilbert-style systems are of little use to analyse the structural properties of logics in terms of a specific axiomatisation.

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Notation	**Definition**	**Name**	**Idempotent**
$\bot$	–	*Falsum*	–
$\top$	$\neg\bot$	*Verum*	–
$\neg\alpha$	–	Negation	–
$\alpha \to \beta$	–	Implication	–
$\alpha \vee \beta$	$(\alpha \to \beta) \to \beta$	(Lattice) Disjunction	Yes
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	(Lattice) Conjunction	Yes
$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$	Biconditional	–
$\alpha \oplus \beta$	$\neg\alpha \to \beta$	Strong disjunction	No
$\alpha \odot \beta$	$\neg(\alpha \to \neg\beta)$	Strong conjunction	No
$\alpha \ominus \beta$	$\neg(\alpha \to \beta)$	Co-implication	–

Table: Connectives in Łukasiewicz logic.

We now turn to semantics. Łukasiewicz logic has a many-valued semantics: specifically, *pace* Frege, we take $[0, 1] \subseteq \mathbb{R}$ as a set of "truth values".

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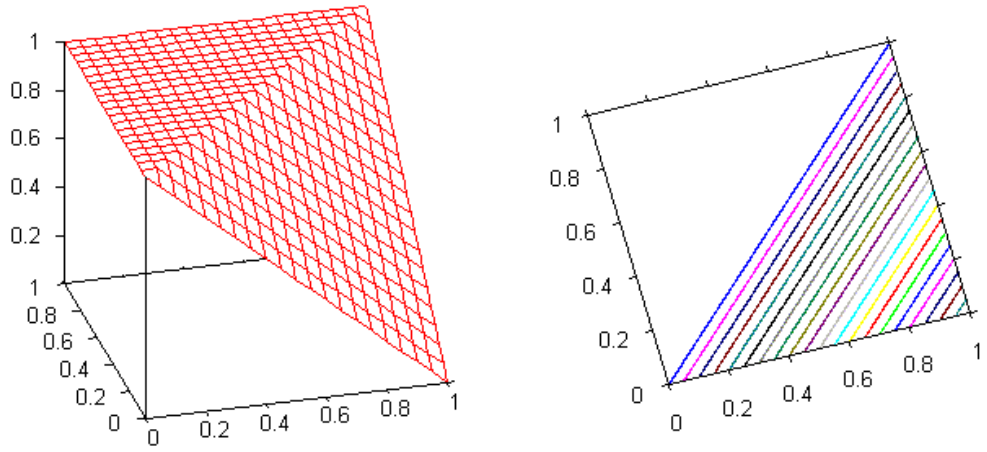
- $w(\bot) = 0$.
- $w(\neg\alpha) = 1 - w(\alpha)$.
- $w(\alpha \to \beta) = \begin{cases} 1 & \text{if } w(\alpha) \leqslant w(\beta) \\ 1 - (w(\alpha) - w(\beta)) & \text{otherwise.} \end{cases}$



Truth-function of Łukasiewicz implication.

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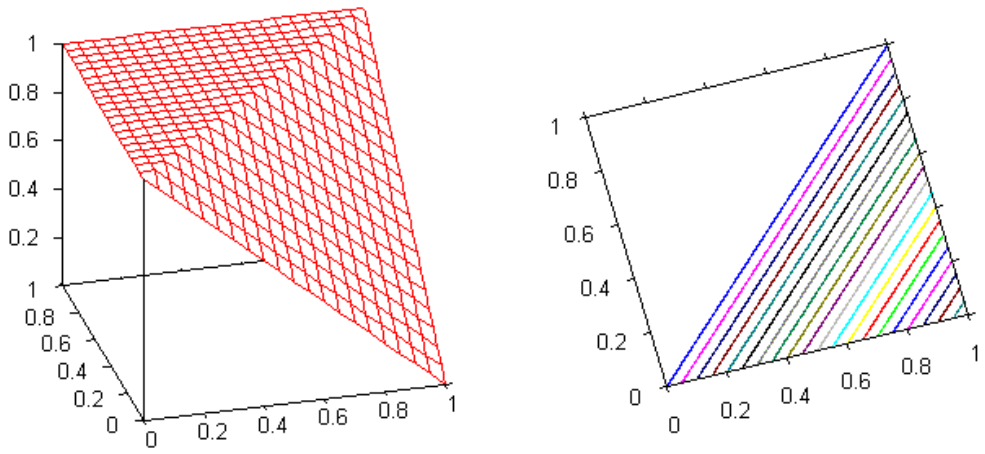
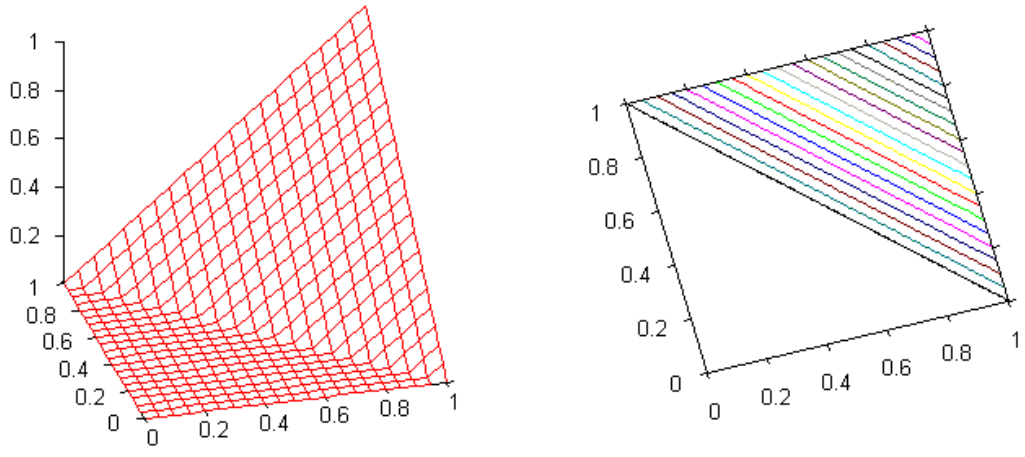


Truth-function of Łukasiewicz implication.

$$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$$

Notation	Formal semantics
$\bot$	$w(\bot) = 0$
$\top$	$w(\top) = 1$
$\neg\alpha$	$w(\neg\alpha) = 1 - w(\alpha)$
$\alpha \to \beta$	$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$
$\alpha \vee \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$w(\alpha \leftrightarrow \beta) = 1 - \lvert w(\alpha) - w(\beta)\rvert$
$\alpha \oplus \beta$	$w(\alpha \oplus \beta) = \min\{1, w(\alpha) + w(\beta)\}$
$\alpha \odot \beta$	$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$
$\alpha \ominus \beta$	$w(\alpha \ominus \beta) = \max\{0, w(\alpha) - w(\beta)\}$

Table: Formal semantics of connectives in Łukasiewicz logic.



Truth-function of Łukasiewicz "strong conjunction" $\odot$.
(*Note: Non-idempotent operation.*)

$$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$$

As in classical logic, a tautology is a formula α that receives value 1 in every possible world, i.e. for every assignment $w\colon \textsc{Form} \to [0,1]$ it so happens that $w(\alpha) = 1$. So 1 is the only designated value.

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Notation.

- $\vdash_{\text{Ł}} \alpha$ — α is provable.
- $\models_{\text{Ł}} \alpha$ — α is a tautology.
- $\mathrm{THM}^{\text{Ł}}$ — Set of provable formulæ.
- $\mathrm{TAUT}^{\text{Ł}}$ — Set of tautologies.
- Similarly for $\mathrm{THM}^{\text{Ł}}_n$ and $\mathrm{TAUT}^{\text{Ł}}_n$.

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Validity or Soundness Theorem for Ł

For any $\alpha \in \textsc{Form}$,

$$\vdash_{Ł} \alpha \quad \text{implies} \quad \models_{Ł} \alpha\,.$$

That is,

$$\textsc{Thm}^{Ł} \subseteq \textsc{Taut}^{Ł}\,.$$

(Similarly, $\textsc{Thm}^{Ł}_{n} \subseteq \textsc{Taut}^{Ł}_{n}$).

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(*Proof.* Take $w(p) = \frac{1}{2}$.)

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The (syntactic) inclusion $\mathrm{THM}^{Ł} \subseteq \mathrm{THM}$ also is proper. The full relationship between tautologies and theorems follows from the next result, a substantial piece of mathematics.

Completeness Theorem for Ł (A. Rose and J. B. Rosser, 1958)

For any $\alpha \in \textsc{Form}$,

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In summary:

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Soundness and Completeness Theorem for Ł

For any $\alpha \in \textsc{Form}$,

$$\models_{Ł} \alpha \quad \text{if, and only if,} \quad \vdash_{Ł} \alpha\,.$$

That is,

$$\textsc{Taut}^{Ł} = \textsc{Thm}^{Ł}\,.$$

(Similarly, $\textsc{Taut}^{Ł}_{n} = \textsc{Thm}^{Ł}_{n}$).

A consequence, though far from immediate:

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Corollary (D. Mundici, 1987)

There is a an algorithm (for definiteness, a Turing machine) that, on input $\alpha \in \textsc{Form}$, *outputs* $\textsc{Yes}$ *if* $\alpha \in \textsc{Thm}^{Ł}$, *and* $\textsc{No}$ *otherwise.*

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But even with completeness, a brute-force check of all $[0,1]$-valued assignments to the n variables in α is impossible. Something cleverer is needed (we will perhaps hint at a proof later).[2]

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But even with completeness, a brute-force check of all $[0,1]$-valued assignments to the n variables in α is impossible. Something cleverer is needed (we will perhaps hint at a proof later).[2]

And yet, structurally, the situation is no worse than in the classical case:

Theorem (D. Mundici, 1987)

The problem of deciding Łukasiewicz logic is co-$\textsc{np}$-complete.

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Thank you for your attention.