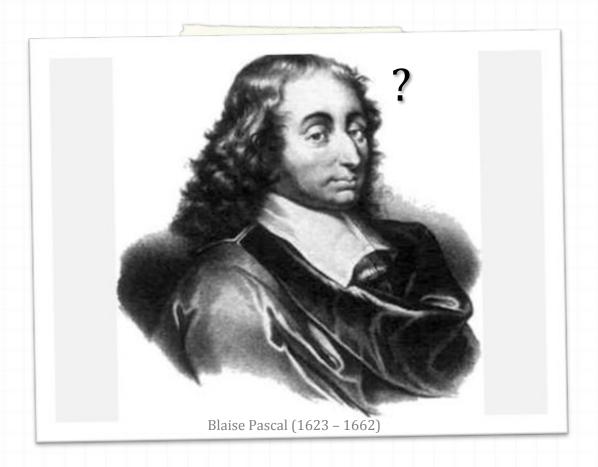


Łukasiewicz logic: an introduction

Vincenzo Marra Università degli Studi di Milano Kutaisi2011 26–30 .IX.2011



Prologue

Betting on Vague Propositions?

E= "Italy scores in the match against France"



Blaise Pascal (1623 – 1662)

E = "Italy scores in the match against France"

Blaise's Stake

Proposition	Stake
Е	1€
not E	0€



Blaise Pascal (1623 - 1662)

E = "Italy scores in the match against France"

Proposition	Stake
E	1€
not E	0€
	man and

Blaise's Stake

Ada's	Book
Aua S	DOOK

Proposition	Price of Bet
E	0.25
not E	0.75



Blaise Pascal (1623 – 1662)



Ada Lovelace (1815 - 1852)

E = "Italy scores in the match against France"

Proposition	Stake
E	1€
not E	0€
	To and the

Blaise's Stake

Proposition Pri

Proposition	Price of Bet
E	0.25
not E	0.75

Ada's Book



Blaise Pascal (1623 - 1662)

25 cents now.



Ada Lovelace (1815 - 1852)

E = "Italy scores in the match against France"

Blaise's Stake

Proposition	Stake
Е	1€
not E	0€

Ada's Book

Proposition	Price of Bet
Е	0.25
not E	0.75



Blaise Pascal (1623 - 1662)

25 cents now.



1€ if Italy scores, 0€ otherwise



Ada Lovelace (1815 - 1852)

E = "Italy scores in the match against France"

Probability Theory deals with Events described by Formulae in Classical Logic (=Boolean algebras of Events).

E = "Italy scores late in the match against France"



Blaise Pascal (1623 - 1662)

E = "Italy scores **late** in the match against France"

Blaise's Stake

Proposition	Stake
E	1€
not E	
	Manager 1



Blaise Pascal (1623 - 1662)

E = "Italy scores late in the match against France"

Proposition	Stake
Е	1€
not E	

Ada's Book Blaise's Stake

Proposition	Price of Bet
Е	?
not E	?



Blaise Pascal (1623 - 1662)





Ada Lovelace (1815 - 1852)

E = "Italy scores **late** in the match against France"

Blaise's Stake

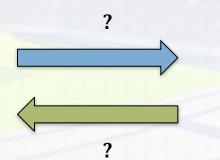
Proposition	Stake
Е	1€
not E	?
	Distance N

Ada's Book

Proposition	Price of Bet
E	?
not E	?



Blaise Pascal (1623 - 1662)







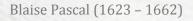
Ada Lovelace (1815 - 1852)

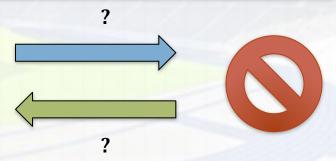
E = "Italy scores **late** in the match against France"

Is There a Probability Theory of
Events described by
such Vague Propositions?

(=non-Boolean algebras of Events?)







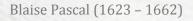


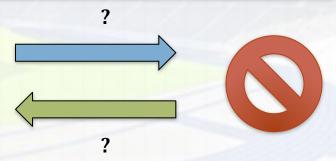
Ada Lovelace (1815 – 1852)

E = "Italy scores **late** in the match against France"

Is There a Logic of Vague
Propositions?
Is it just Classical Logic?
If not, which Non-Classical Logic is it?









Ada Lovelace (1815 – 1852)



Who Cares, Anyway?

Some Motivating Remarks

Main Motivation

To develop a

Theory of Probabilities of Vague Events

bridging the gap from Foundations to Applications

Why is it Important?

Because

Vague (or Non-Classical) Events are everywhere

and
Classical Probability Theory does <u>not</u> cope well with them

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 1: Insist that vagueness be ruled out.

Classical Probability Theory does not cope well with Vague Events

Solution 1: Insist that vagueness be ruled out.

O You can precisify a vague event: "Italy will score against France in the last 15' of the match."

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 1: Insist that vagueness be ruled out.

- O You can precisify a vague event: "Italy will score against France in the last 15' of the match."
- O Then Blaise loses the whole stake if Italy scores at 74'59", and wins the whole stake if Italy scores at 75'01".

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 1: Insist that vagueness be ruled out.

- O You can precisify a vague event: "Italy will score against France in the last 15' of the match."
- O Then Blaise loses the whole stake if Italy scores at 74'59", and wins the whole stake if Italy scores at 75'01".
- O This violates <u>continuity</u>, our fundamental intuition about the proposition "to score late": there is no single instant of time that counts as "the first late one". (Cf. Eubulides' Sorites Paradox.)

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 2:

Classical Probability Theory does not cope well with Vague Events

Solution 2:

Use random variables/measure theory directly.

O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.

Classical Probability Theory does not cope well with Vague Events

Solution 2:

- O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.
- O Then Blaise gets back $\mu([0,t])$ × the stake if Italy scores at time t.

Classical Probability Theory does not cope well with Vague Events

Solution 2:

- O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.
- O Then Blaise gets back $\mu([0,t])$ × the stake if Italy scores at time t.
- O Problem 1: What are the <u>admissible/appropriate</u> measures μ?

Classical Probability Theory does not cope well with Vague Events

Solution 2:

- O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.
- O Then Blaise gets back $\mu([0,t])$ × the stake if Italy scores at time t.
- O Problem 1: What are the <u>admissible/appropriate</u> measures μ? Ad Hoc Models.

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 2:

- O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.
- O Then Blaise gets back $\mu([0,t])$ × the stake if Italy scores at time t.
- O Problem 1: What are the admissible/appropriate measures μ? Ad Hoc Models.
- O Problem 2: Even if μ is given by some Oracle, what is the <u>corresponding</u> measure for, say, "to score early"?

Classical Probability Theory does <u>not</u> cope well with Vague Events

Solution 2:

Use random variables/measure theory directly.

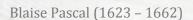
- O If [0,1] is the (normalized) duration of the match, choose a Borel probability measure μ on it that models "to score late" appropriately.
- O Then Blaise gets back $\mu([0,t])$ × the stake if Italy scores at time t.
- O Problem 1: What are the admissible/appropriate measures μ? Ad Hoc Models.
- O Problem 2: Even if μ is given by some Oracle, what is the <u>corresponding</u> measure for, say, "to score early"?

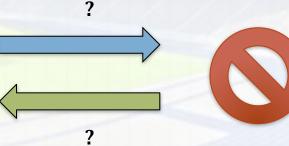
Algebra of r.v. ≠ Logic of Vague Events.

E = "Italy scores late in the match against France"

Is There a Logic of Vague **Propositions?** Is it just Classical Logic? If not, which Non-Classical Logic is it?









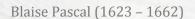


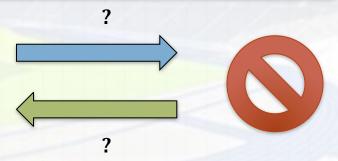
Ada Lovelace (1815 - 1852)

E = "Italy scores **late** in the match against France"

We will look at Łukasiewicz infinitevalued propositional logic. It is the best candidate I know for a logic of vague propositions.

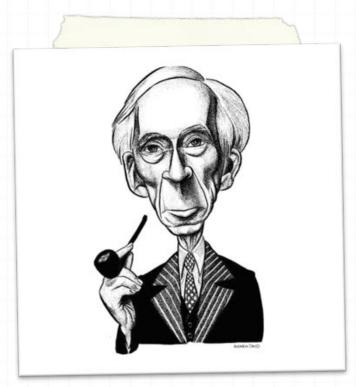








Ada Lovelace (1815 – 1852)



Bertrand Russell, 1872–1970

Theories of Vagueness

A Cursory Sketch

- **O** From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.

0

0

0

0

0

 C

- **O** From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- *1 grain of wheat does not make a heap.*

- **O** From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- **0** 1 grain of wheat does not make a heap.
- O If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.

0

0

0

- O From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- **0** 1 grain of wheat does not make a heap.
- O If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.
- O If 2 grains of wheat do not make a heap, then 3 grains do not.

0

0

- O From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- **0** 1 grain of wheat does not make a heap.
- O If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.
- O If 2 grains of wheat do not make a heap, then 3 grains do not.

0 ...

0

- O From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- **0** 1 grain of wheat does not make a heap.
- O If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.
- O If 2 grains of wheat do not make a heap, then 3 grains do not.
- 0 ...
- O If $(10^{100}-1)$ grains of wheat do not make a heap, then 10^{100} do not.

0

- O From ancient Greek: "The Paradox of the Heap".
- O Attributed to Eubulides of Miletus, 4th century BC.
- **0** 1 grain of wheat does not make a heap.
- O If 1 grain of wheat does not make a heap, then 2 grains of wheat do not.
- O If 2 grains of wheat do not make a heap, then 3 grains do not.
- 0 ...
- O If $(10^{100}-1)$ grains of wheat do not make a heap, then 10^{100} do not.
- 0 Hence: 10^{100} grains of wheat do not make a heap.

- O Chrysippus' response (inferred from Cicero's writings):
- O Q. Does 1 grain of wheat make a heap?
- 0 C. No.
- 0
- 0
- 0
- 0
- 0
- 0

- O Chrysippus' response (inferred from Cicero's writings):
- O Q. Does 1 grain of wheat make a heap?
- 0 C. No.
- O Q. Do 2 grains of wheat make a heap?

0

0

0

0

0

- O Chrysippus' response (inferred from Cicero's writings):
- O Q. Does 1 grain of wheat make a heap?
- 0 C. No.
- O Q. Do 2 grains of wheat make a heap?
- 0 C. No.
- 0 ...
- 0
- 0
- 0

```
O Chrysippus' response (inferred from Cicero's writings):
O Q. Does 1 grain of wheat make a heap?
O C. No.
O Q. Do 2 grains of wheat make a heap?
O C. No.
O ...
O Q. Do x (some x) grains of wheat make a heap?
O
```

O Chrysippus' response (inferred from Cicero's writings):
O Q. Does 1 grain of wheat make a heap?
O C. No.
O Q. Do 2 grains of wheat make a heap?
O C. No.
O ...
O Q. Do x (some x) grains of wheat make a heap?
O C. (Silence.)

- O Chrysippus' response (inferred from Cicero's writings):
- O Q. Does 1 grain of wheat make a heap?
- 0 C. No.
- O Q. Do 2 grains of wheat make a heap?
- 0 C. No.
- 0 ...
- **O** Q. Do x (some x) grains of wheat make a heap?
- O C. (Silence.)
- 0 ...

```
O Chrysippus' response (inferred from Cicero's writings):
O Q. Does 1 grain of wheat make a heap?
0 C. No.
O Q. Do 2 grains of wheat make a heap?
0 C. No.
0 ...
0 \ Q. \ Do \ x \ (some \ x) grains of wheat make a heap?
O C. (Silence.)
0 ...
```

You will shortly see that some modern philosophers (the *epistemicists*) arguably side with Chrysippus; many others, however, do not.

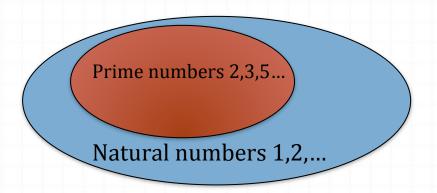
- **O** Modern response: *Theories of Vagueness.*
- O Initial problem: the monadic predicate Heap(x) is vague.
- O To explain the paradox away we need a theory of such vague predicates.
- O Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a "vague predicate" is.
- O Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.

0

- **O** Modern response: *Theories of Vagueness.*
- O Initial problem: the monadic predicate Heap(x) is vague.
- O To explain the paradox away we need a theory of such vague predicates.
- O Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a "vague predicate" is.
- O Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.
- O So there is no explanation of the Sorites Paradox that is "standard", in the sense of being most widely accepted.

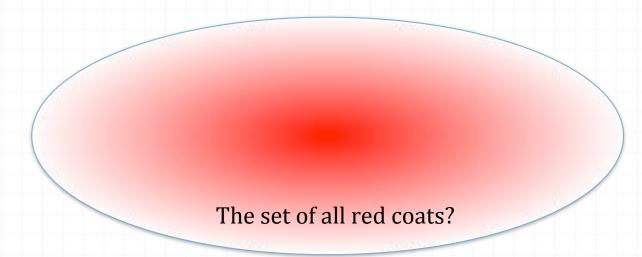
Features of a precise predicate.

The monadic predicate P(x) := "x is prime", interpreted over the set of natural numbers $x \ge 1$, is (absolutely) precise: its extension is the set of prime numbers; its anti-extension is the set of composite numbers; each number either belongs to the extension of P or to its anti-extension, but not to both; and in principle there is no issue as to whether a given number be prime or composite — though in practice it may be impossible to ascertain which is the case for an astronomic instance of x.

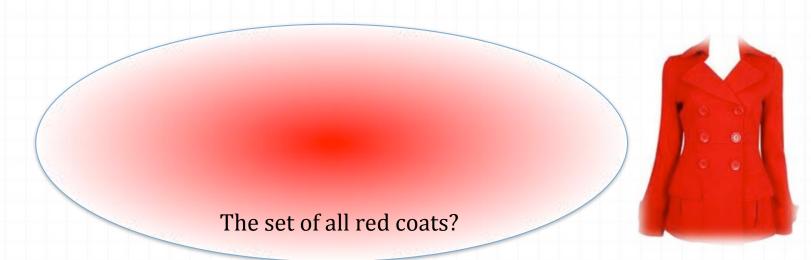


Features of a vague predicate.

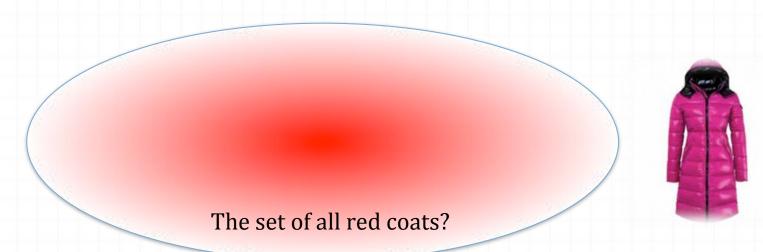
Features of a vague predicate.



Features of a vague predicate.



Features of a vague predicate.

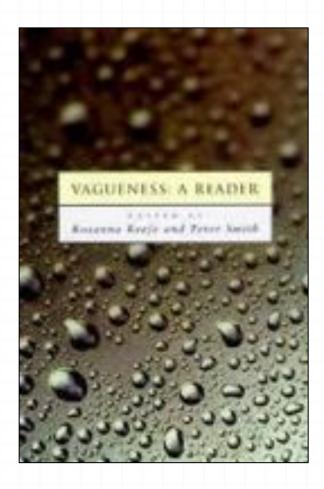


Features of a (monadic) vague predicate *R*:

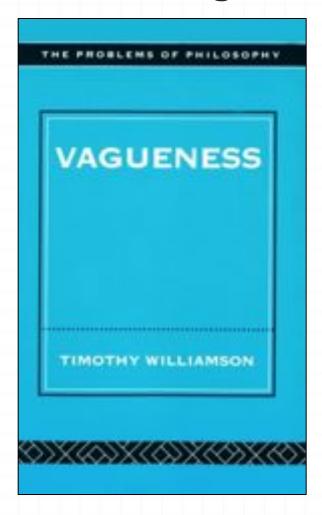
- (FV1) R admits borderline cases over the intended domain of interpretation D, i.e. there are instantiations of R(x) by (a term naming a constant) $c \in D$ such that it is unclear whether R(c) holds or its negation $\neg R(c)$ does.
- (FV2) R lacks sharp boundaries over the intended domain of interpretation D, i.e. there is no clearly defined boundary separating the extension of $R(\cdot)$ from its anti-extension.
- (FV3) R is susceptible to a *Sorites series* over the intended domain of interpretation D, *i.e.* there are instantiations of R(x) by $c_1, \ldots, c_n \in D$ such that it is clear that $R(c_1)$ holds, it is clear that $R(c_n)$ does not hold, and it seems at least plausible that if $R(c_i)$ holds then so does $R(c_{i+1})$, for each $i \in \{1, \ldots, n-1\}$.

Features of a vague proposition *p*:

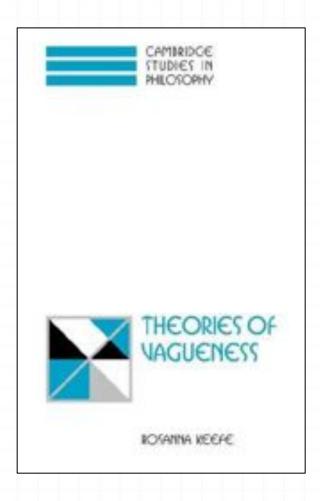
- (PV1) p admits borderline cases over the intended set of possible worlds W, i.e. there are worlds $w \in W$ such that it is unclear whether p holds in w, or its negation $\neg p$ does.
- (PV2) p lacks sharp boundaries over the intended set of possible worlds W, i.e. there is no clearly defined boundary separating the extension of p the subset of possible worlds in W at which p is true from its anti-extension.
- (PV3) p is susceptible to a *Sorites series* over the intended set of possible worlds W, *i.e.* there are possible worlds $w_1, \ldots, w_n \in W$ such that it is clear that p holds in w_1 , it is clear that p does not hold in w_n , and it seems at least plausible that if p holds in w_i then p also holds in w_{i+1} , for each $i \in \{1, \ldots, n-1\}$.



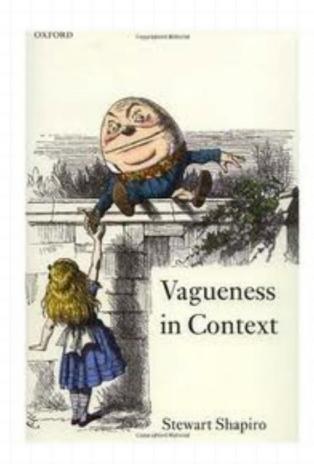
Useful Reader: R. Keefe and P. Smith, eds.



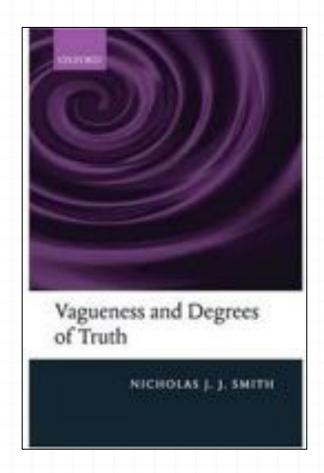
Epistemicism: Vagueness as Ignorance



Supervaluationism: Vagueness as Precisifiability



Contextualism: Vagueness as dependence from Context



Degree-Based Theories: Vagueness as Truth-in-Degrees

Degree-Based Theories of Vagueness

Main Assumption: Truth comes in degrees.

- If x is a clear case of R, then R(x) is (fully, classically) true.
- If x is a clear non-case of R, then R(x) is (fully, classically) false.
- If x is a borderline case of R, then R(x) is true (or false) to a degree.

It may seem natural to say that, in borderline cases, a certain coat is neither clearly red, nor clearly non-red, so that "This coat is red" is neither true nor false. And the further step of then saying that "This coat is red" is true (or false) to some degree may also sound appealing. (Well, does it sound appealing to you?) But we should be aware that taking this direction is a major departure from the roots of logic as we know it, both philosophically and mathematically.

Degree-Based Theories of Vagueness

Main Assumption: Truth comes in degrees.

- If x is a clear case of R, then R(x) is (fully, classically) true.
- If x is a clear non-case of R, then R(x) is (fully, classically) false.
- If x is a borderline case of R, then R(x) is true (or false) to a degree.

It may seem natural to say that, in borderline cases, a certain coat is neither clearly red, nor clearly non-red, so that "This coat is red" is neither true nor false. And the further step of then saying that "This coat is red" is true (or false) to some degree may also sound appealing. (Well, does it sound appealing to you?) But we should be aware that taking this direction is a major departure from the roots of logic as we know it, both philosophically and mathematically.

Frege on Truth

We are therefore driven into accepting the truth value [Wahrheits-wert] of a sentence as constituting its reference [Bedeutung]. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True [das Wahre], the other the False [das Falsche].

G. Frege, On Sense and Reference, 1892, p. 34.

Frege on Truth

We are therefore driven into accepting the truth value [Wahrheits-wert] of a sentence as constituting its reference [Bedeutung]. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True [das Wahre], the other the False [das Falsche].

G. Frege, On Sense and Reference, 1892, p. 34.

In other writings (notably the unpublished Logik), Frege makes the following very clear.

- Truth is a primitive notion in logic: it cannot be defined.
- True (p) is a peculiar predicate in that it does not admit comparatives: p is truer than q is a $façon\ de\ parler$ lacking genuine logical content.
- (Implicitly.) In particular, degrees of truth are non-sense, according to Fregean orthodoxy.

But pace Frege...

- Many-valued logics postulate the existence of degrees of truth. (Caution: Sometimes motivations are mathematical, not philosophical.)
- Most degree-based theories of vagueness argue that one or other system of many-valued logic **is** the logic of vague propositions (or predicates).
- By far the majority of such theories make a **far stronger** assumption, namely:

Stronger Assumption. Degrees of truth are modelled by the real unit interval $[0,1] \subseteq \mathbb{R}$.

But pace Frege...

- Many-valued logics postulate the existence of degrees of truth. (Caution: Sometimes motivations are mathematical, not philosophical.)
- Most degree-based theories of vagueness argue that one or other system of many-valued logic **is** the logic of vague propositions (or predicates).
- By far the majority of such theories make a **far stronger** assumption, namely:

Stronger Assumption. Degrees of truth are modelled by the real unit interval $[0,1] \subseteq \mathbb{R}$.

Hint: With finitely many truth values, it is impossible to deal with borderline cases of borderline cases of ... borderline cases of redness. This is called the <u>Problem of Higher-Order Vagueness</u>.

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called *Problem of Artificial Precision*.

[Fuzzy logic] imposes artificial precision [... While] one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate definitely applies to such-and-such, rather than to such-and-such other, degree (e.g. that a man 5ft 10in tall belongs to tall to degree 0.6 rather than 0.5).

S. Haack, 1979

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called Problem of Artificial Precision.

One serious objection to [the many-valued approach] is that it really replaces vagueness with the most incredible and refined precision.

M. Tye, 1989

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called *Problem of Artificial Precision*.

[T]he degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. [...] All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321?

R. Keefe, 2000

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called *Problem of Artificial Precision*.

Intuitively, it is *not* correct to say that there is one unique element of [0,1] that correctly represents the degree of truth of 'Bob is bald', with all other choices being incorrect. $[\ldots]$ we have an affront to intuition [because] [w]e cannot see what could possibly *determine* that the degree of truth of 'Bob is bald' is 0.61 rather than 0.62 or 0.6 [...]

N.J.J. Smith, 2008

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called

Problem of Artificial Precision.

How could one respond to the problem of artificial precision?

It turns out that it is not easy to say anything interesting about it, if we do not define formally the logical system involved. There are a lot of different logics that can be based on the assumption that [0,1] is the set of truth values. Whenever one of them claims that such numbers are degrees of truth, you can raise the objection of artificial precision to it.

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called

<u>Problem of Artificial Precision</u>.

How could one respond to the problem of artificial precision?

It turns out that it is not easy to say anything interesting about it, if we do not define formally the logical system involved. There are a lot of different logics that can be based on the assumption that [0,1] is the set of truth values. Whenever one of them claims that such numbers are degrees of truth, you can raise the objection of artificial precision to it.

However, it is not reasonable to expect that successful responses (if any) to the problem of artificial precision be independent of the details of the underlying logic.

There are forceful objections to the stronger assumption that degrees of truth be identifiable with real numbers. Here are some quotes on the so called

<u>Problem of Artificial Precision.</u>

How could one respond to the problem of artificial precision?

It turns out that it is not easy to say anything interesting about it, if we do not define formally the logical system involved. There are a lot of different logics that can be based on the assumption that [0,1] is the set of truth values. Whenever one of them claims that such numbers are degrees of truth, you can raise the objection of artificial precision to it.

However, it is not reasonable to expect that successful responses (if any) to the problem of artificial precision be independent of the details of the underlying logic.

Therefore we will now leave theories of vagueness and take a long excursion into formal logic, with the aim of introducing Łukasiewicz logic formally, independently of any intuitive semantics. We will eventually get back to the problem of artificial precision to see whether what we will have learnt can help us with it.