Overview

Tutorial 1. History and Basics.

- History
- Infinite games
- Baire space and its topology
- The low levels of the Borel hierarchy

Overview

Tutorial 1. History and Basics.
Tutorial 2. Proving Determinacy.

- Existence of a non-determined set
- Determinacy for open and closed games
- More Determinacy
- Graph games and their complexity

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Tutorial 3. Using Determinacy.

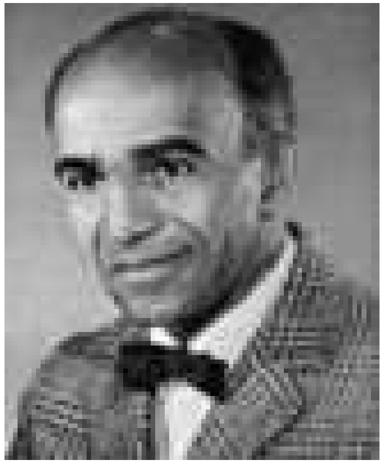
- The limits of determinacy; Projective Determinacy
- The Continuum Problem
- Uniformization

Infinite Games? (1)

What are infinite games?

We are concerned with two-player perfect information games of infinitely many rounds.

Morris H. **DeGroot**, A conversation with David Blackwell, **Statistical Science** 1 (1986), p.40-53



David H. Blackwell (born 1919)

Infinite Games? (1)

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DeGroot. What kind of things are you working on these days?

Blackwell. I am trying to understand some things about infinite games.

DeGroot. What do you mean by an infinite game?

Blackwell. A game with an infinite number of moves. Here's an example. I write down a 0 or a 1, and you write down a 0 or a 1, and we keep going indefinitely. If the sequence we produce has a limiting frequency, I win. If not, you win. That's a trivial game because I can force it to have a limiting frequency just by doing the opposite of whatever you do.

DeGroot. Fortunately, it's one in which I'll never have to pay off to you.

Blackwell. Well, we can play it in such a way that you would have to pay off.

DeGroot. How do we do that?

Infinite Games? (1)

What are infinite games?

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Blackwell. You can specify a strategy in this infinite game. For every finite sequence that you might see up to a given time as past history, you specify the next move. So you can define this function once and for all, and I can define a function, and then we can mathematically assess those functions. I can prove that there is a specific function of mine that no matter what function you specify, the set will have a limiting frequency.

DeGroot. So you could extract money from me in a finite amount of time.

Blackwell. Right.

Infinite Games? (2)

-  Ernst Zermelo (1871-1953)
  Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels (1913)
- 1920s & 1930s: Steinhaus, Banach, Mazur.
- **Ulam's Question** (1930s). *Characterize the sets A for which player I (player II) has a winning strategy in the infinite game with payoff A.*
  Kanamori 1994: "It was to take set theorists half a century to provide a fair answer to a related question: For which A does either I or II have a winning strategy in the game with payoff A."
- David **Gale** and Frank **Stewart** (1953): Theory of infinite games.
- Jan **Mycielski**, Hugo **Steinhaus** (1962): Axiom of Determinacy
- David **Blackwell** (1967): game-based proof of uniformization for co-analytic sets
- Yiannis **Moschovakis** (1967): Periodicity Phenomenon in foundations of mathematics

Infinite Games (1).

The games we are playing are of the following form: We write ω^ω for the set of all infinite sequences of natural numbers. Fix a set $A \subseteq \omega^\omega$. Then two players, player I and II pick natural numbers in turn:

Player I	x_0		x_2		x_4	$\dots$
Player II		x_1		x_3		$\dots$

We write x for the sequence $x(i) := x_i$. Then player I wins if and only if $x \in A$. We call this game $\mathrm{G}(A)$.

Ulam's Question. Characterize the sets A such that player I (player II) wins $\mathrm{G}(A)$.

Playing an infinite game.

A **strategy** is a function $\sigma : \omega^{<\omega} \to \omega$. Given a strategy σ for player I and a strategy τ for player II, they completely determine the outcome of the game:

$$\begin{aligned} (\sigma * \tau)_0 &:= \varnothing \\ (\sigma * \tau)_{2n+1} &:= (\sigma * \tau)_{2n} {}^\frown \sigma((\sigma * \tau)_{2n}) \\ (\sigma * \tau)_{2n+2} &:= (\sigma * \tau)_{2n+1} {}^\frown \tau((\sigma * \tau)_{2n+1}) \end{aligned}$$

We call a strategy σ **winning for player I in** $\mathrm{G}(A)$ if for every τ, $\sigma * \tau \in A$.
We call a strategy τ **winning for player II in** $\mathrm{G}(A)$ if for every σ, $\sigma * \tau \notin A$.

Strategic trees (1).

A strategy σ defines a **strategic tree for player I**:

$$
\begin{aligned}
T_0^{\sigma,\mathrm{I}} &:= \{\varnothing\} \\
s \in T_{2n}^{\sigma,\mathrm{I}} &\iff s^\frown\sigma(s) \in T_{2n+1}^{\sigma,\mathrm{I}} \\
s \in T_{2n+1}^{\sigma,\mathrm{I}} &\iff s^\frown x \in T_{2n+2}^{\sigma,\mathrm{I}} \\
T^{\sigma,\mathrm{I}} &:= \bigcup_{n\in\omega} T^{\sigma,\mathrm{I}}
\end{aligned}
$$

Similarly for a **strategic tree for player II**:

$$
\begin{aligned}
T_0^{\tau,\mathrm{II}} &:= \{\varnothing\} \\
s \in T_{2n}^{\tau,\mathrm{II}} &\iff s^\frown x \in T_{2n+1}^{\tau,\mathrm{II}} \\
s \in T_{2n+1}^{\tau,\mathrm{II}} &\iff s^\frown\tau(s) \in T_{2n+2}^{\tau,\mathrm{II}}
\end{aligned}
$$

Strategic Trees (2).

For a tree T, we write $[T]$ for its set of infinite branches, i.e., $x \in [T] \iff \forall n(x\restriction n \in T$.

Then for any strategic trees $T^{\sigma,\mathrm{I}}$, $T^{\tau,\mathrm{II}}$ for player I and II, we have that $[T^{\sigma,\mathrm{I}}] \cap [T^{\tau,\mathrm{II}}] = \{\sigma * \tau\}$.

Observation. A strategy σ is winning for player I in $\mathrm{G}(A)$ iff $[T^{\sigma,\mathrm{I}}] \subseteq A$; it is winning for player II in $\mathrm{G}(A)$ iff $[T^{\sigma,\mathrm{II}}] \cap A = \varnothing$.

Ulam's Question. Characterize the sets A such that player I wins $\mathrm{G}(A)$.

Lemma. If player I has a winning strategy in $\mathrm{G}(A)$, then the cardinality of A must be that of the set of all real numbers.

Ulam's Question.

Ulam's Question. Characterize the sets A such that player I wins $\mathrm{G}(A)$.

Lemma. If player I has a winning strategy in $\mathrm{G}(A)$, then the cardinality of A must be that of the set of all real numbers.

Consider $A := \{x\,;\, \exists n(x(2n) \neq x(2n+1)\}$.

The cardinality of A is that of the set of real numbers, as every sequence starting with 01 is in A. But player II has a winning strategy, the copycat strategy.

So, the lemma does not provide such a characterization, and cardinality is not a good candidate for a characterization theorem.

Kanamori 1994: "It was to take set theorists half a century to provide a fair answer to a related question: For which A does either I or II have a winning strategy in the game with payoff A."

Determinacy.

We call a set A **determined** if either player I or player II has a winning strategy in the game $\mathrm{G}(A)$.

Is the class of determined sets trivial ($=$ the class of all sets)? We come back to this question later.

I doesn't win	$\forall\sigma$	$\exists\tau$	$\sigma * \tau \notin A$
II wins	$\exists\tau$	$\forall\sigma$	$\sigma * \tau \notin A$

Gale & Stewart (1953): The right approach to prove determinacy is to consider topological properties of sets.

Topology of Baire space (1).

If x and y are different infinite sequences, there must be some n such that $x(n) \neq y(n)$. Let $n_{x,y}$ be the least such n. We define

$$\mathrm{dist}(x,y) := \begin{cases} 2^{-n_{x,y}} & \text{if } x \neq y\text{, and} \\ 0 & \text{if } x = y. \end{cases}$$

Fix x and ε. Then

$$B_\varepsilon(x) := \{y\,;\, \mathrm{dist}(x,y) < \varepsilon\} = \{y\,;\, x_0x_1...x_n \subseteq y\}$$

where $2^{-(n+1)} < \varepsilon \leq 2^{-n}$.

If s is a finite sequence of natural numbers, we write $[s] := \{y\,;\, s \subseteq y\}$ and call this a **basic open set**.

Topology of Baire space (2).

$[s] := \{y\,;\, s \subseteq y\}$

The basic open sets form a topology base. We call a set **open** if it is a union of basic open sets.

- Membership in open sets is "finitary": if $P := \bigcup_{i \in I} [s_i]$ is open and $x \in P$, then there is some n such that membership of x in P is determined by $x{\restriction}n$.
- For each x in an open set P, you can find a nonempty basic open set containing x and contained in P.
- Note that again all nonempty open sets must have the cardinality of the set of all real numbers. In particular, for any x, the set $\{x\}$ cannot be open.

Topology of Baire space (3).

The complement of an open set is called **closed**.

Observation 1. A set A is closed if and only if any sequence in A converges to a point in A.

Observation 2. A set A is closed if and only if there is a tree T such that $A = [T]$.

- All basic open sets are both open and closed ("clopen").
- All singleton sets $\{x\}$ are closed, but not open. Consequently, the sets $\omega^\omega \backslash \{x\}$ are open but not closed.

Topology of Baire space (4).

The open sets are closed under unions and under finite intersections, but not under countable intersections in general.
Let z_n be the sequence of n zeros. Then the set $A_n := [z_n]$ is open, but the intersection $\bigcap_{n\in\omega} A_n = \{z\}$ where z is the constant zero function.

Even worse:

Let Z_n be the set of all finite sequences that contain n zeros, and let $P_n := \bigcup_{s\in Z_n}[s]$. Then $P := \bigcap_{n\in\omega} P_n$ is the set of all sequences that contain infinitely many zeros.

This set is not open: arbitrarily close to each element of P there is a sequence with only finitely many zeros.

This set is not closed: If w_n is the sequence that starts with n ones and then continues with zeros, then $w_n \in P$. But the sequence (w_n) converges to the constant sequences with value one.

Topology of Baire space (5).

Complement and countable union generate a hierarchy of sets: The Borel Hierarchy.

- $\mathbf{\Sigma}^0_1$:= the open sets
- $\mathbf{\Pi}^0_1$:= the closed sets
- $\mathbf{\Sigma}^0_2$:= countable unions of closed sets (F_σ)
- $\mathbf{\Pi}^0_2$:= countable intersections of open sets (G_δ)
- $\mathbf{\Sigma}^0_3$:= countable unions of G_δ sets ($\mathrm{G}_{\delta\sigma}$)
- $\mathbf{\Pi}^0_3$:= countable intersections of F_σ sets ($\mathrm{F}_{\sigma\delta}$)

This hierarchy corresponds to the formula hierarchy: A set is $\mathbf{\Pi}^0_3$ if and only if it is definable by a ∀∃∀-formula in arithmetic with a real parameter.

Results.

Theorem (Gale-Stewart). If A is an open set, then $\mathrm{G}(A)$ is determined.

Proof next time.

The Gale-Stewart theorem is the first instance of a sequence of determinacy theorems:

Wolfe	G_δ
Davis	$\mathrm{G}_{\delta\sigma}$
Paris	$\mathbf{\Sigma}^0_4$
Martin	all Borel sets
Martin-Steel / Neeman	... even more ... (?!?)

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Is every set determined?

Is the class of determined sets trivial? It could be that every set is determined...

Theorem (Banach-Mazur; Gale-Stewart). If there is a wellordering of the set of real numbers, then there is a non-determined set.

In particular, AC implies that there is a non-determined set. The “Axiom of Determinacy” proposed by Mycielski and Steinhaus in 1962 is therefore an *alternative* to the Axiom of Choice.
The connection between non-determinacy and the axiom of choice remains intricate: we'll discuss this further when we look at the limits of determinacy (Lecture 3).

Proof (1).

Using the well-ordering of the set of real numbers, we give a well-ordered list of all strategic trees $\langle T_\alpha\,;\,\alpha < 2^{\aleph_0}\rangle$.

Remember that “player I has a winning strategy in $\mathrm{G}(A)$” means that for some σ, the tree $T^{\sigma,\mathrm{I}}$ must be contained in A; for “player II has a winning strategy”, some tree $T^{\tau,\mathrm{II}}$ must be contained in the complement of A.

We shall make sure that neither of these can be the case. Using transfinite recursion, we define two sets A and B:

$$A_0 := B_0 := \varnothing$$

$$A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

Proof (2).

$$A_0 := B_0 := \varnothing$$

$$A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

In each successor step, both A and B will gain exactly one element, thus making sure that $\mathrm{Card}(A_\alpha) = \mathrm{Card}(B_\alpha) = \mathrm{Card}(\alpha)$.

For each α, $[T_\alpha]$ has cardinality $2^{\aleph_0} > \mathrm{Card}(\alpha)$, and therefore $[T_\alpha]\backslash(A_\alpha \cup B_\alpha)$ has uncountably many elements. Pick two of them; call them a_α and b_α.

Then let $A_{\alpha+1} := A_\alpha \cup \{a_\alpha\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$.

Finally

$$A := \bigcup_{\alpha<2^{\aleph_0}} A_\alpha \text{ and } B := \bigcup_{\alpha<2^{\aleph_0}} B_\alpha.$$

Proof (3).

$$A_0 := B_0 := \varnothing \quad A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

$$A_{\alpha+1} := A_\alpha \cup \{a_\alpha\} \text{ and } B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$$

$$A := \bigcup_{\alpha<2^{\aleph_0}} A_\alpha \text{ and } B := \bigcup_{\alpha<2^{\aleph_0}} B_\alpha.$$

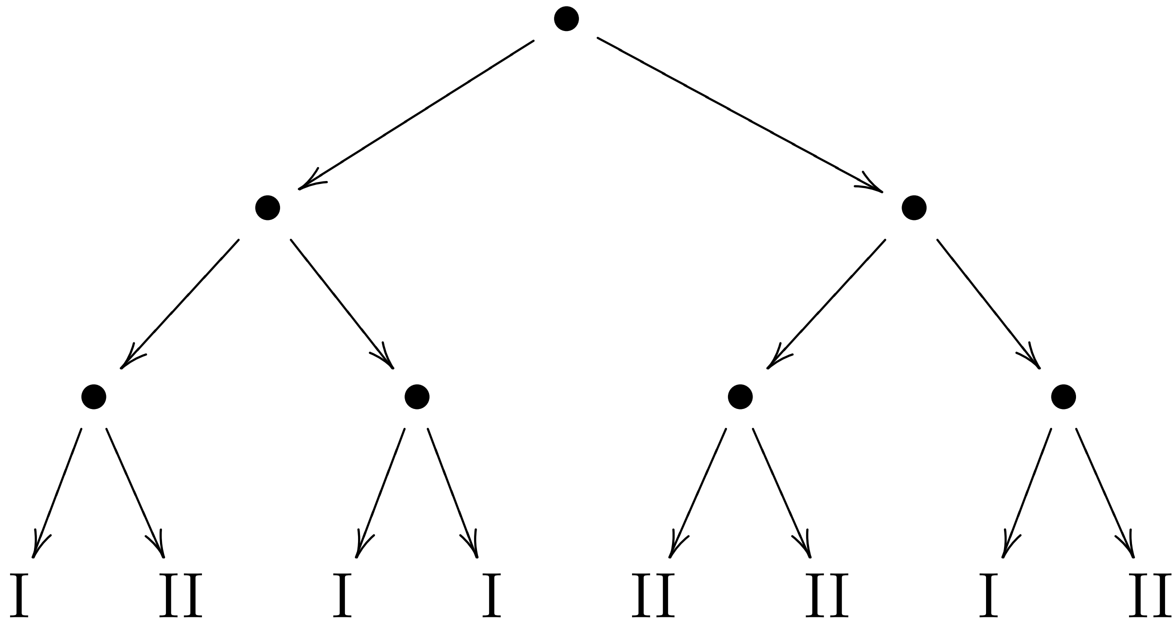
Note that $A \cap B = \varnothing$.

We **claim** that A is not determined. Suppose it was, then either there is some $T^{\sigma,\mathrm{I}} \subseteq A$ or some $T^{\tau,\mathrm{II}} \subseteq \omega^\omega \backslash A$.

In Case 1, find α such that $T^{\sigma,\mathrm{I}} = T_\alpha$. Then $b_\alpha \in [T_\alpha] \cap B$, and so $b_\alpha \notin A$. Contradiction.

In Case 2, find α such that $T^{\tau,\mathrm{II}} = T_\alpha$. Then $a_\alpha \in [T_\alpha] \cap A$, and so $[T_\alpha]$ is not disjoint from A. Contradiction. q.e.d.

Backward induction in finite games.



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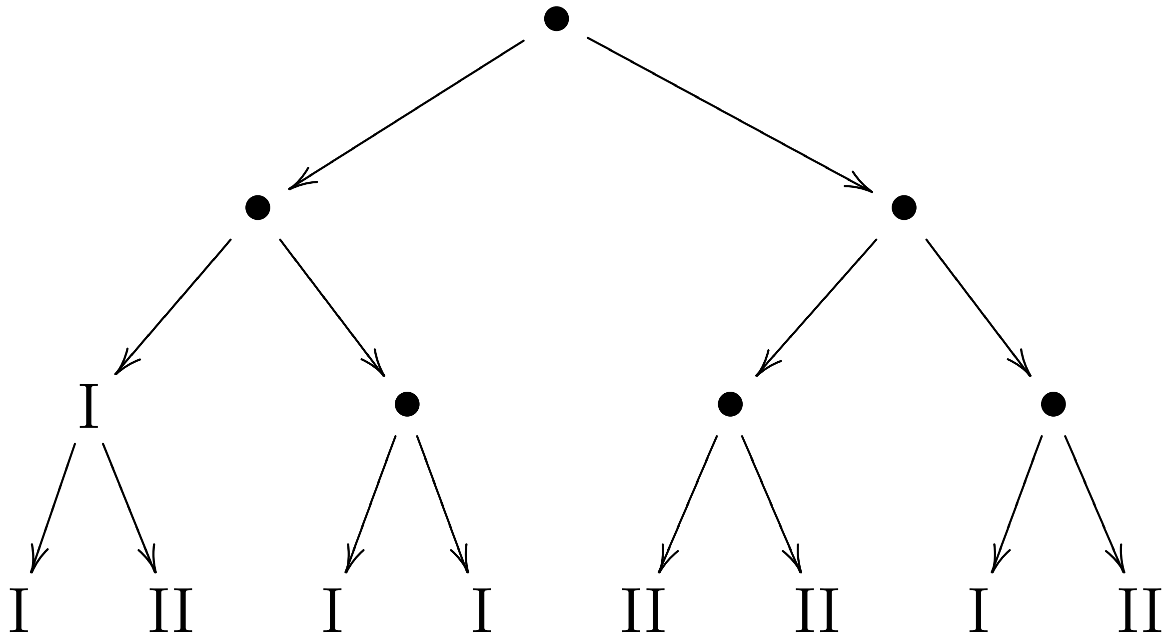

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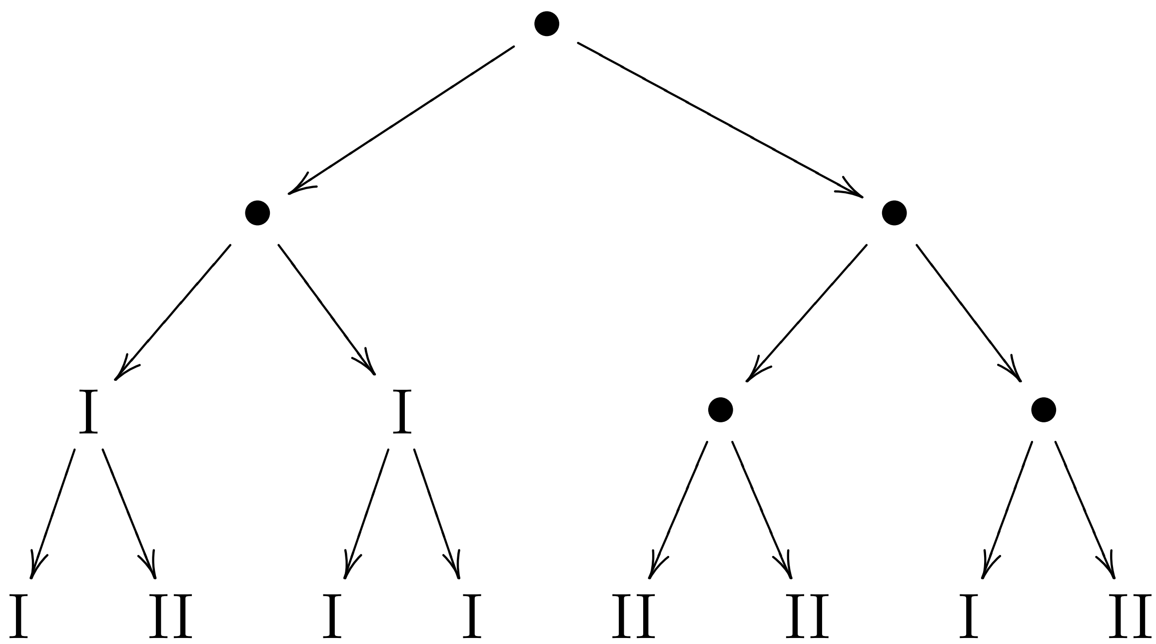

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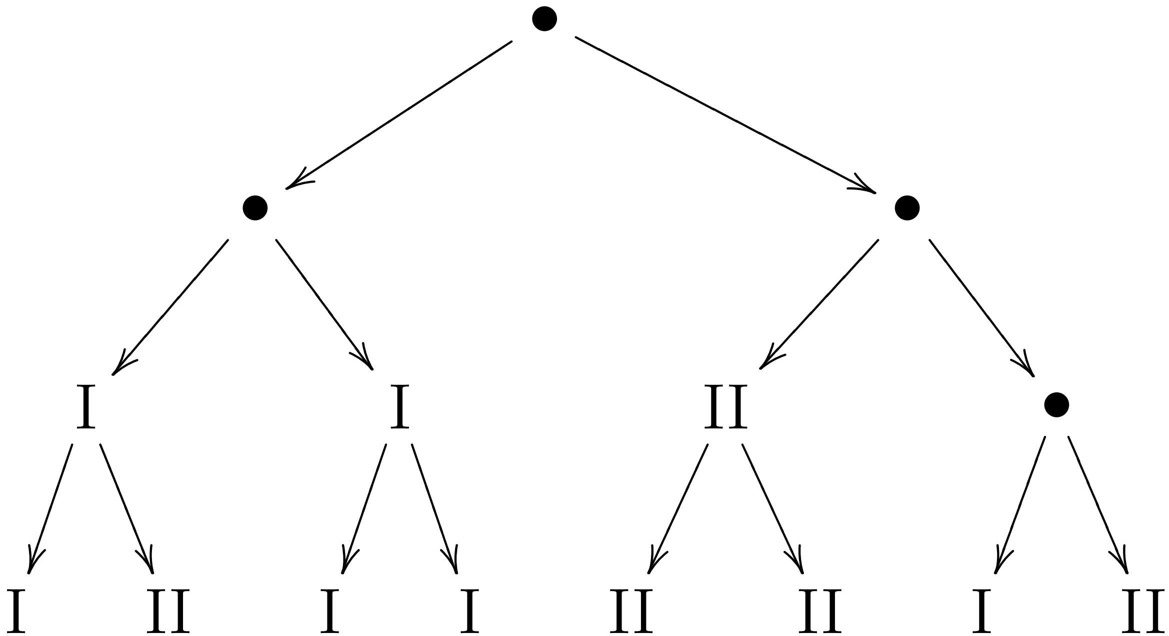

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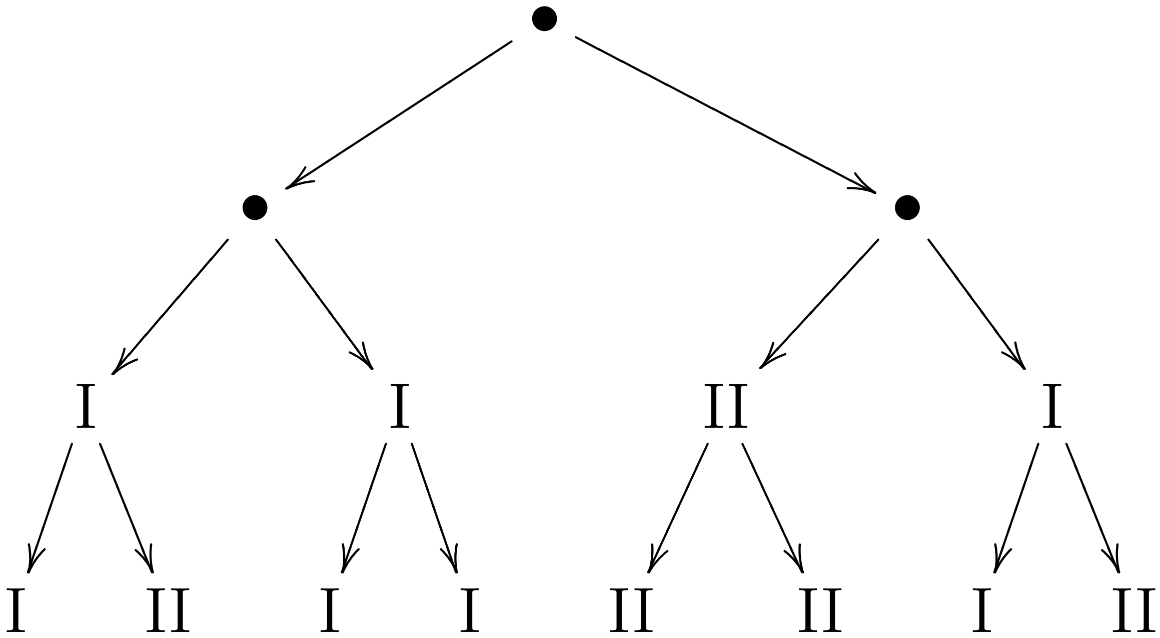

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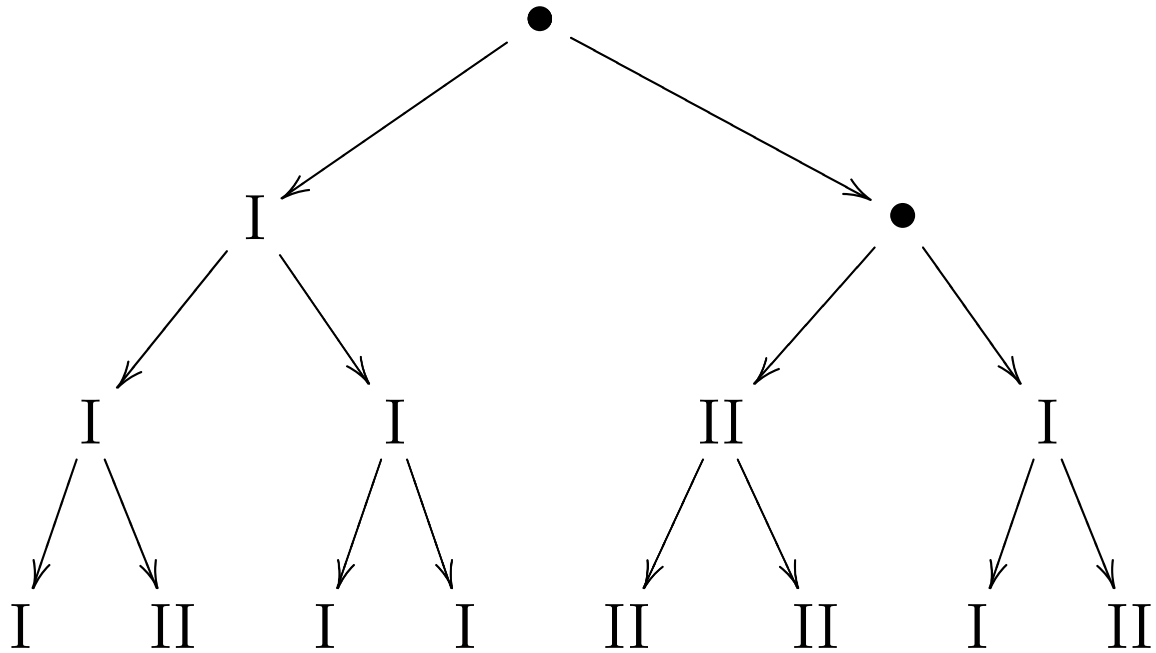

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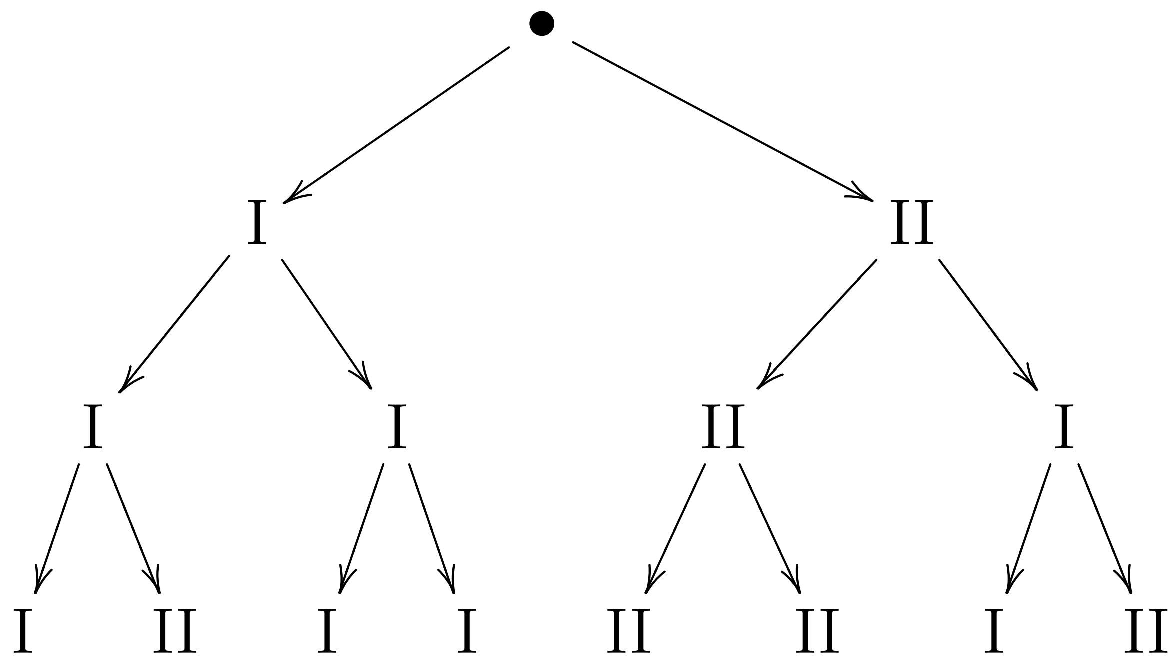

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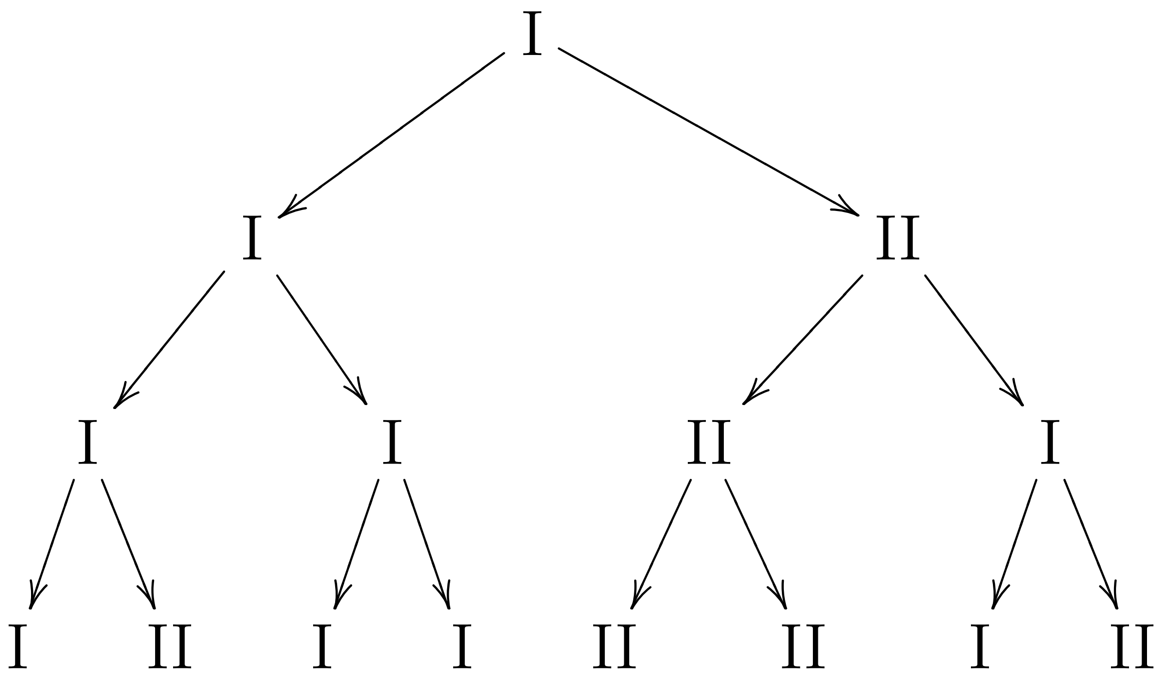


Gale-Stewart I (1).

Theorem (Gale-Stewart 1953). If A is a clopen payoff set, then $\mathrm{G}(A)$ is determined.

Proof. If A is clopen, then both A and the complement of A are unions of basic open sets. So, there are sets X and Y of finite sequences such that $A = \{x\,;\,\exists s \in X(s \subseteq x)\}$ and $\omega^\omega \backslash A = \{x\,;\,\exists s \in Y(s \subseteq x)\}$.

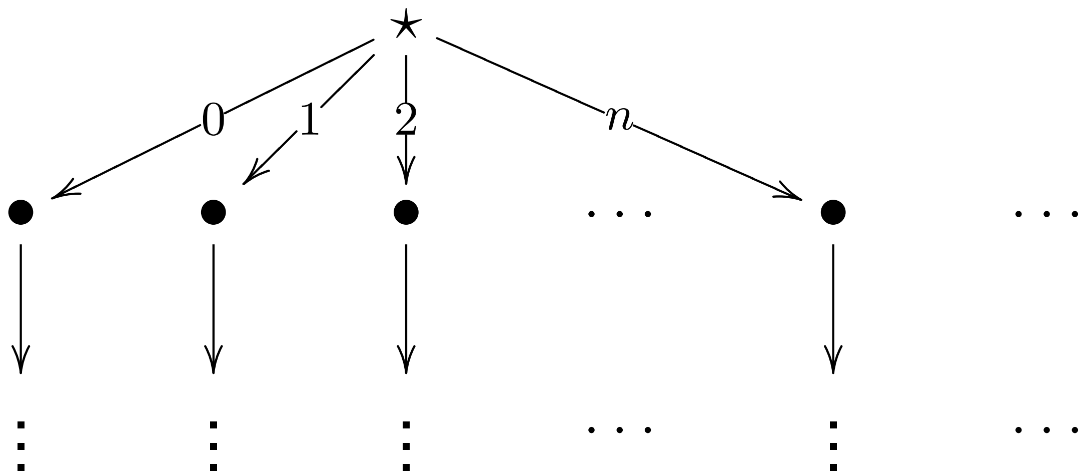
The set $X \cup Y$ has the following "barrier property": If $x \in \omega^\omega$ then there is some $s \in X \cup Y$ such that $s \subseteq x$.

We label the elements of X (and all their extensions) by I and the elements of Y (and all their extensions) by II and start our backward induction:

- If s is a move for player I (player II) and there is at least one successor labelled I (II), then we label s also with I (II).
- If s is a move for player I (player II) and all successors are labelled II (I), then we label s also with II (I).

Not quite so easy.

Consider the following game: Player I plays a natural number n, after that players I and II alternate, and player II wins if and only if he plays a 0 in his nth move. Obviously, player II has a winning strategy, but let's do the recursion for the labelling:



So, $\star$ won't get labelled in a finite amount of time as we need that all successors are labelled.

Gale-Stewart I (2).

So, we need to extend this process into the transfinite. We start with our initial labelling ℓ_0 and define a recursion as follows:

- $\ell_\lambda := \bigcup_{\alpha<\lambda} \ell_\alpha$
- $\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player I (II) and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$ (II).
- $\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player II (I) and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$ (II).

Note that the domains of the partial labellings are increasing, i.e., $\mathrm{dom}(\ell_\alpha) \subseteq \mathrm{dom}(\ell_{\alpha+1})$. As a consequence, there must be a countable ordinal ζ that is a fixed point of this procedure, i.e., $\ell_\zeta = \ell_{\zeta+1}$.

Gale-Stewart I (3).

Barrier Property: If $x \in \omega^\omega$ then there is some $s \in X \cup Y$ such that $s \subseteq x$.
$\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player I (II) and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$ (II).
$\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player II (I) and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$ (II).

Claim 1. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$.

Claim 2. ℓ_ζ is a total function.

[Suppose not, then $\ell_\zeta(s)$ is not defined. By Claim 1, there must be an infinite sequence x such that $\ell_\zeta(x{\restriction}n)$ is not defined for all $n \geq \mathrm{lh}(s)$. But by the barrier property, there must be some n such that $x{\restriction}n \in X \cup Y$. Contradiction!]

Claim 3. If $\ell_\zeta(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).

Gale-Stewart I (4).

Claim 4. Any infinite sequence whose positions are all labelled I (II) is a win for player I (II).

[Again, this is an application of the barrier property: There is some n such that $x{\restriction}n \in X \cup Y$.]

We have established in a constructive way that $\mathrm{G}(A)$ is determined. By Claim 2, $\ell_\zeta(\varnothing)$ is defined and thus is either I or II. By Claim 3, the player who owns the label has a strategy to stay on his labels; and by Claim 4, this is a winning strategy. q.e.d.

Gale-Stewart II (1).

Theorem (Gale-Stewart). If A is open, then $\mathrm{G}(A)$ is determined.

Proof. Now, $A = \bigcup_{s \in X}[s]$, but the complement may not be open. We just do the same procedure with the limited information we have at hand. We let $\ell_0(t) = \mathrm{I}$ if there is an $s \in X$ and $t \supseteq s$, and then run the Gale-Stewart procedure:

- $\ell_\lambda := \bigcup_{\alpha<\lambda} \ell_\alpha$
- $\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.
- $\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

Gale-Stewart II (2).

$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.
$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

Again, the procedure reaches a fixed point ℓ_ζ, and again, we have Claim 1.

Claim 1. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$.

Even stronger now: If player I has to move at s, and $s \notin \mathrm{dom}(\ell_\zeta)$, then no successors of s are in $\mathrm{dom}(\ell_\zeta)$.

But we cannot deduce that ℓ_ζ is total, as this relied on the barrier property. Define

$$\ell^*(s) := \begin{cases} \ell_\zeta(s) & \text{if } s \in \mathrm{dom}(\ell_\zeta)\text{, and} \\ \mathrm{II} & \text{if } s \notin \mathrm{dom}(\ell_\zeta). \end{cases}$$

Gale-Stewart II (3).

Claim 1*. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$. If player I has to move at s, then no successors of s are in $\mathrm{dom}(\ell_\zeta)$.

$$\ell^*(s) := \begin{cases} \ell_\zeta(s) & \text{if } s \in \mathrm{dom}(\ell_\zeta)\text{, and} \\ \mathrm{II} & \text{if } s \notin \mathrm{dom}(\ell_\zeta). \end{cases}$$

With this, we again have

Claim 3. If $\ell^*(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).

Are all such strategies winning? Yes for player II: If a strategy stays on label II producing x, in particular it never hits an element of X, and thus $x \notin A$, so player II wins.

Again, not quite so easy.

Consider the game $A = \{x\,;\, \exists n(x(2n) \neq 0)\}$. The set A is open. All nodes $s1$ of odd length are labelled I in the initial labelling ℓ_0. Then all nodes s of even length get labelled I in ℓ_1, and thus all nodes of odd length get labelled I in ℓ_2 which is the fixed point of the procedure.

Therefore, the strategy "play 0" for player I has the property that it stays on label I. But obviously, it is a losing strategy.

Gale-Stewart II (4).

Claim 3. If $\ell^*(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).
$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.
$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

We have to introduce the **index** of a position: this is the least α such that $\ell_\alpha(s)$ is defined (if there is such an α).

We observe that if if $\ell^*(s) = \mathrm{I}$ and player I has to play, then there is a successor of lower index with label I (unless the index of s is 0), and if player II has to play, then all successors are of lower index (unless the index of s is 0).

So, if $\ell^*(\varnothing) = \mathrm{I}$, then player I has a strategy that forces the labels to be I and that forces the sequence of indices to be a decreasing sequence of ordinals (i.e., either $\mathrm{ind}(x{\upharpoonright}n+1) < \mathrm{ind}(x{\upharpoonright}n)$ or $\mathrm{ind}(x{\upharpoonright}n+1) = \mathrm{ind}(x{\upharpoonright}n) = 0$).

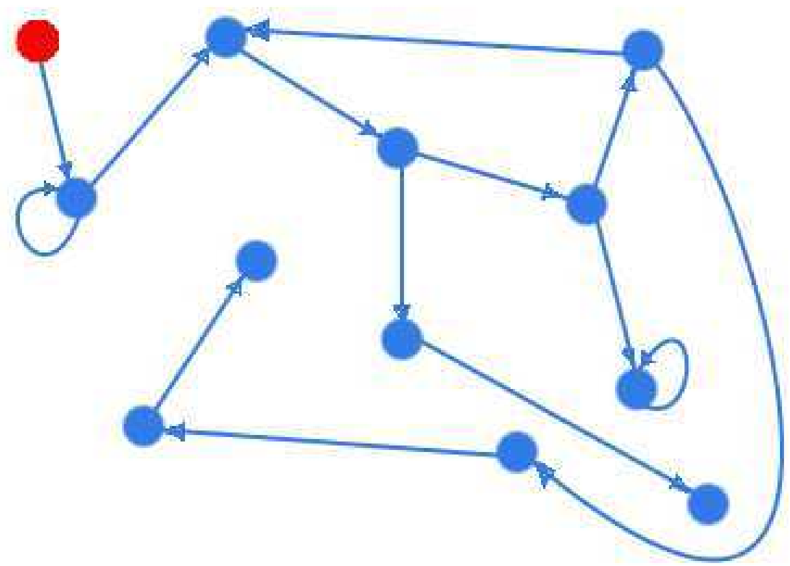
Gale-Stewart II (5).

So, if $\ell^*(\varnothing) = \mathrm{I}$, then player I has a strategy that forces the labels to be I and that forces the sequence of indices to be a decreasing sequence of ordinals (i.e., either $\mathrm{ind}(x{\restriction}n+1) < \mathrm{ind}(x{\restriction}n)$ or $\mathrm{ind}(x{\restriction}n+1) = \mathrm{ind}(x{\restriction}n) = 0$.

Now let x be a play according to that strategy. Since there is no infinite decreasing sequence of ordinals, we know that there must be some n such that the index of $x{\restriction}n$ is 0, but then $\ell_0(x{\restriction}n) = \mathrm{I}$. But that means that $x{\restriction}n \in X$, and that $x \in A$. So the strategy is a winning strategy. q.e.d.

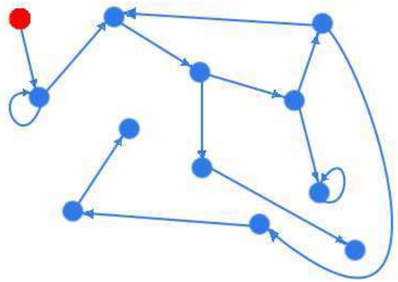
Graph games (1).

Take a directed graph, specify a vertex as the initial node and play a game in which players I and II push a token along the edges.



Such a game can easily be transferred into a game on Baire space by just labelling the vertices of the graph with natural numbers (the “tree unravelling” of a graph).

Graph games (2).

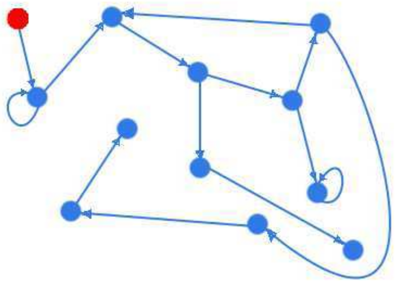


Typical winning conditions:

- “the player who makes the last (legal) move wins”.
- Player I wins if vertex v is visited.
- Player I wins if vertex v is visited n times.
- Player I wins if vertex v is visited infinitely many times.

If you unravel the trees of these games, the first three conditions give open payoffs.

Graph games (3).



"Player I wins if vertex v is visited infinitely many times."

Remember our example: $P := \bigcup_{n\in\omega} P_n$ is the set of all sequences that contain infinitely many zeros. This was a set which was neither open nor closed.

Similarly, the unravelled game for our graph game will produce a $\mathbf{\Pi}^0_2$ set which is neither closed nor open.

Thus: The Gale-Stewart theorem is not enough to deal with these games.

Extensions of Gale-Stewart.

- **Philip Wolfe** (1955). Every Σ^0_2 set is determined.
- **Morton Davis** (1963). Every Σ^0_3 set is determined.
- **Jeff Paris** (1972). Every Σ^0_4 set is determined.
- **Tony Martin** (1975). Every Borel set is determined.
- Where are the limits of determinacy?

Overview

Tutorial 1. History and Basics.
Tutorial 2. Proving Determinacy.
Tutorial 3. Using Determinacy.

- The limits of determinacy; Projective Determinacy
- The Continuum Problem
- Uniformization

The Axiom of Determinacy.

The Mycielski-Steinhaus Axiom of Determinacy.

Jan Mycielski 

AD: "All games $\mathrm{G}(A)$ are determined."

We have already proved that AC implies ¬AD, so AD is necessarily an "alternative" to AC.

Extensions of the Borel hierarchy (1).

Closure properties of the Borel sets: closed under continuous preimages, closed under countable unions and intersections, closed under complementation.

Lebesgue famously claimed that the Borel sets are also closed under continuous images. But this is false, as was shown by Suslin (1917).

The closure of the Borel sets under continuous images is called “analytic sets” or “A-sets”. You can check that this class is not closed under complements, so a new hierarchy develops on top of the Borel sets: the projective hierarchy:

- $\mathbf{\Sigma}^1_1$:= the A-sets
- $\mathbf{\Pi}^1_1$:= the complements of A-sets, the CA-sets
- $\mathbf{\Sigma}^1_2$:= continuous images of CA-sets (ACA-sets)
- $\mathbf{\Pi}^1_2$:= complements of ACA-sets (CACA-sets)

Extensions of the Borel hierarchy (2).

A class of sets of real numbers is called a pointclass. A pointclass is called boldface if it is closed under continuous preimages. If $\mathbf{\Gamma}$ is a pointclass, we write $\exists\mathbf{\Gamma}$ for the class of sets of the type $\exists y(\langle x, y\rangle \in A)$ where $A \in \mathbf{\Gamma}$, and $\forall\mathbf{\Gamma}$ for the class of sets of the type $\forall y(\langle x, y\rangle \in A)$.

Theorem (Suslin). The Borel sets are precisely those that are both analytic and co-analytic.

Fact. The smallest boldface pointclasses strictly containing the class of all Borel sets are the classes $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

So: Is every $\mathbf{\Pi}^1_1$ set determined?

Applications of determinacy.

If Γ is any boldface pointclass. Suppose that all sets in Γ are determined. Then:

- All sets in Γ are Lebesgue measurable. (Mycielski-Swierczkowski)
- All sets in Γ have the Baire property. (Banach-Mazur)
- All sets in Γ have the perfect set property. (Morton Davis)

Cantor's Continuum Problem (1).

First problem in Hilbert's list (1900).

The Continuum Hypothesis (CH). Every set of real numbers is either finite or countable or has the cardinality of the set of all real numbers.

Lemma. If the Axiom of Choice holds, then CH is equivalent to "there is a bijection between $\aleph_1$ and the set of real numbers (in short: $2^{\aleph_0} = \aleph_1$).

Proof. $\Leftarrow$ is obvious.

"$\Rightarrow$": By the Axiom of Choice, there is a bijection π between some ordinal α and $\mathbb{R}$. We only have to show that $\mathrm{Card}(\alpha) = \aleph_1$.

If $\mathrm{Card}(\alpha) < \aleph_1$, then the set of reals would be countable, contradicting Cantor's theorem.

If $\mathrm{Card}(\alpha) > \aleph_1$, then $\alpha \subseteq \aleph_1$, and we look at $X := \pi[\alpha] \subseteq \mathrm{R}$. Clearly, $\mathrm{Card}(X) = \aleph_1 < \mathrm{Card}(\mathbb{R})$. Contradiction. q.e.d.

Cantor’s Continuum Problem (2).

One approach to solving Cantor’s Continuum Problem was the perfect set property.

A tree is called perfect if any node has two incompatible extensions.

A set of real numbers has the perfect set property if it is either finite or countable or contains the branches through a perfect tree.

Cantor's Continuum Problem (3).

Observation. If a set of reals has the perfect set property, it cannot be a counterexample to the Continuum Hypothesis.

Corollary. If all sets have the perfect set property, then CH is true.

Theorem (Bernstein). The Axiom of Choice implies that there is a set without the perfect set property.

The perfect set theorem (1).

Theorem. If all sets in Γ are determined, then all sets in Γ have the perfect set property.

Weaker Theorem. If all sets are determined, then all sets have the perfect set property.

(The difference between the "Theorem" and the "Weaker Theorem" is just an analysis of the complexity of the game.)

Proof. We construct a game that encapsules the perfect set property of a set A. For technical reasons, we play on the binary branching tree (Cantor space).

PlayerI	s_0		s_1		s_2		$\ldots$
PlayerII		x_0		x_1		x_2	$\ldots$

where s_i are finite sequences of bits and x_i are binary bits. We construct $x := s_0x_0s_1x_1s_2x_2\ldots$, and say that I wins if $x \in A$.

The perfect set theorem (2).

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- This game is called $\mathrm{G}^*(A)$, the asymmetric game, due to Morton Davis.
- If all games of the type $\mathrm{G}(A)$ are determined, then all games of the type $\mathrm{G}^*(A)$ are determined.
- Rephrasing the notion of a strategic tree, we still get: If player I was a winning strategy in $\mathrm{G}^*(A)$, then A must contain a perfect set.
- If A is countable, then player II has a winning strategy.

We need to prove the converse of the last statement.

The perfect set theorem (3).

Fix a winning strategy for player II, call it τ. If $p = \langle s_0, x_0, s_1, x_1, ..., s_n, x_n \rangle$ is a position for player II, we write $p_* t$ for $s_0 x_0 s_1 x_1 ... s_n x_n t$. If $x \in 2^\omega$, we say p kills x if for all t, we have that $p_* t \tau(p_* t) \not\subseteq x$.

Observation 1. Each p kills at most one sequence x.

Observation 2. Every $x \in A$ is killed by a sequence p.

But there are only countably many sequences, so A is a countable set. q.e.d.

Non-Extensions of Determinacy.

There is a minimal model of set theory: Gödel's $\mathbf{L}$, the constructible universe. In $\mathbf{L}$, there is a wellordering of the continuum definable in a Δ^1_2 way.

Theorem (Gödel). $\mathbf{L} \models$ "There is a $\mathbf{\Pi}^1_1$ set without the perfect set property."

Corollary. In $\mathbf{L}$, there must be a non-determined $\mathbf{\Pi}^1_1$ set.

Proof. By the perfect set theorem, if every $\mathbf{\Pi}^1_1$ set was determined, then every $\mathbf{\Pi}^1_1$ set would have the perfect set property, but that contradicts Gödel's theorem. q.e.d.

Corollary. The determinacy of all coanalytic sets cannot be proved in ZFC.

Foundations of Mathematics (1).

Gödel Incompleteness phenomenon: ZFC is not complete, i.e., there are statements independent of ZFC. Even worse, there are interesting questions independent of ZFC: the Continuum Hypothesis.

Gödel's Programme: Find further axioms for set theory that are accepted by all mathematicians than resolve all interesting questions.

Gödel's main candidate for these axioms were "Axioms of Strong Infinity", also called Large Cardinal Axioms. The most famous of these is the notion of a measurable cardinal: a cardinal κ that carries a two-valued measure measuring all subsets of κ.

Foundations of Mathematics (2).

A measurable cardinal κ is a cardinal that carries a two-valued measure measuring all subsets of κ.

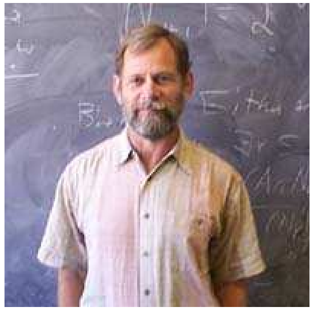
Let us denote the statement “there is a measurable cardinal” by MC.

- ZFC (unless inconsistent) doesn’t prove that there is a measurable cardinal.
- ZFC+MC proves Cons(ZFC).
- ZFC+MC proves that all co-analytic sets are determined (Martin, 1970).

Large Cardinals and Determinacy (1).

There is a level by level analysis of determinacy axioms and large cardinals:

- $\mathbf{\Pi}^1_1$ determinacy is roughly at the level of one measurable cardinal (Martin, Harrington)
- $\mathbf{\Pi}^1_{n+1}$ determinacy is roughly at the level of n Woodin cardinals (Martin-Steel, Woodin)



Donald A. Martin, John R. Steel, A proof of projective determinacy, Journal of the American Mathematical Society 2 (1989), p. 71-125

Large Cardinals and Determinacy (1).

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- $\mathbf{\Pi}^1_{n+1}$ determinacy is roughly at the level of n Woodin cardinals (Martin-Steel, Woodin)
- There is a subtle connection between the Axiom of Choice and this project, as the lower bounds are related to whether the large cardinals involved refute the existence of definable well-orderings of the reals (“Steel games”).

  John R. Steel, Determinacy in the Mitchell models, Annals of Mathematical Logic 22 (1982), p. 109-125

Large Cardinals and Determinacy (2).

- All of the statements “roughly at the level” above can be made exact.

Itay Neeman

Itay Neeman, Optimal Proofs of Determinacy, Bulletin of Symbolic Logic 1997

Large Cardinals and Determinacy (2).

- All of the statements "roughly at the level" above can be made exact.
- The strength of the axiom of determinacy is exactly that of ZFC+"there are infinitely many Woodin cardinals".

Uniformization.

Reminder: The connection between set theory and infinite games started in 1968 with a paper by Blackwell.

David Blackwell, Infinite games and analytic sets, Proceedings of the National Academy of Sciences U.S.A. 58 (1967), p. 1836-1837

Blackwell's proof triggered the development of infinite game theory in foundations of mathematics.

The First Periodicity Theorem (1).

If A is a set of real numbers, we consider relations $\leq$ on A that are prewellorderings, i.e., reflexive, symmetric, transitive, linear and well-founded relations. Prewellorderings give rise to functions from A into the ordinals, traditionally called norms.

The construction of definable norms are the first step in proving general uniformization theorems ($\rightsquigarrow$ the second periodicity theorem).

Theorem (Martin, Moschovakis). *[SIMPLIFIED!]* Suppose AD (for simplicity's sake). Suppose that $\mathbf{\Delta}$ is a class closed under complementation such that every set in $\mathbf{\Delta}$ has a prewellordering in $\mathbf{\Delta}$. Then every set in $\forall\mathbf{\Delta}$ has a prewellordering in $\forall\exists\mathbf{\Delta} \cap \exists\forall\mathbf{\Delta}$.

The First Periodicity Theorem (2).

Let $A \in \forall\mathbf{\Delta}$, i.e., there is a $B \in \mathbf{\Delta}$ such that

$$x \in A \iff \forall u(\langle x, u\rangle \in B).$$

By our assumption, there is a prewellordering $\leq$ on B that is in $\mathbf{\Delta}$.

For x and y, we define the game $\mathrm{G}_{x,y}$ as follows:

Player I	u_0		u_1		u_2		$\ldots$
Player II		v_0		v_1		v_2	$\ldots$

Player I produces u, player II produces v, and player II wins if $\langle y, v\rangle \notin B$ or $(\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle)$.

Define $x \preceq y$ if and only if player II has a winning strategy in $\mathrm{G}_{x,y}$.

The First Periodicity Theorem (3).

Define $x \preceq y$ if and only if player II has a winning strategy in $\mathrm{G}_{x,y}$.

- $\preceq$ is reflexive.
- $\preceq$ is transitive and linear.
- $\preceq$ is well-founded.

  Suppose not, then there is a sequence $\langle x_i\,;\, i \in \omega\rangle$ such that $x_0 \succ x_1 \succ x_2 \succ \ldots$. This means that player II doesn't win $\mathrm{G}_{x_i,x_{i+1}}$, and therefore (by determinacy) that player I wins $\mathrm{G}_{x_i,x_{i+1}}$. Let σ_i be a winning strategy in that game for player I.

  We use the strategies σ_i to fill up an infinite diagram.

The First Periodicity Theorem (4).

Player II wins $\mathrm{G}_{x,y}$ if $\langle y, v\rangle \notin B$ or ($\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle$).

Player I wins $\mathrm{G}_{x_i,x_{i+1}}$. Let σ_i be a winning strategy in that game for player I.

$$\begin{array}{lllll}
\mathrm{G}_{x_0,x_1} & \sigma_0(\varnothing) =: u_0^0 & \sigma_0(u_0^0 u_0^1) =: u_1^0 & \sigma_0(u_0^0 u_0^1 u_1^0 u_1^1) =: u_2^0 & \sigma_0(u_0^0 u_0^1 u_1^0 u_1^1 u_2^0 u_2^1) =: u_3^0 \\
\mathrm{G}_{x_1,x_2} & \sigma_1(\varnothing) =: u_0^1 & \sigma_1(u_0^1 u_0^2) =: u_1^1 & \sigma_1(u_0^1 u_0^2 u_1^1 u_1^2) =: u_2^1 & \cdots \\
\mathrm{G}_{x_2,x_3} & \sigma_2(\varnothing) =: u_0^2 & \sigma_2(u_0^2 u_0^3) =: u_1^2 & \cdots & \\
\mathrm{G}_{x_3,x_4} & \sigma_3(\varnothing) =: u_0^3 & \cdots & & \\
\vdots & \vdots & & &
\end{array}$$

So, we construct infinitely many sequences $u^i := u_0^i u_1^i u_2^i u_3^i \ldots$ where u^i is the result of playing σ_i against u^{i+1}. Since σ_i was winning, we know

- $\langle x_{i+1}, u_{i+1}\rangle \in B$, and
- if $\langle x_i, u_i\rangle \in B$, then $\langle x_{i+1}, u_{i+1}\rangle < \langle x_i, u_i\rangle$.

Therefore, $\langle x_1, u_1\rangle > \langle x_2, u_2\rangle > \langle x_3, u_3\rangle >$ ... is a strictly decreasing sequence in B.

The First Periodicity Theorem (5).

Player II wins $\mathrm{G}_{x,y}$ if $\langle y, v\rangle \notin B$ or ($\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle$).
Let's calculate the complexity of $\preceq$:

$$x \preceq y \quad \text{iff} \quad \exists\tau\forall\sigma(\sigma * \tau \text{ is a win for player II})$$
$$\quad \text{iff} \quad \forall\sigma\exists\tau(\sigma * \tau \text{ is a win for player II})$$

Being a win for player II is in $\boldsymbol{\Delta}$, and so $\preceq$ is in $\forall\exists\boldsymbol{\Delta} \cap \exists\forall\boldsymbol{\Delta}$.
q.e.d.

Summary.

What did we do?

- Infinite games play a role in set theory and the foundations of mathematics.
- Every Borel game is determined, but the proofs grow increasingly non-constructive as you go up the Borel hierarchy.
- There is a connection between the determinacy of infinite games and the axiom of choice: AC implies that there are non-determined games, and the definability of wellorderings of the real line is closely linked to how much determinacy is provable.
- Infinite games have plenty of applications in the general theory of the real line.