Cut: Rule for using lemmas in a proof.

Cut-Elimination:
Elimination of lemmas from proofs.
Transformation to elementary proofs.
Obtain proofs with sub-formula property.

Example:
proofs of theorems in number theory may use *topological structures.* Cut-elimination yields proofs without topology.

other applications:
extraction of bounds via Herbrand's theorem
extraction of programs from proofs

Gentzens' "Hauptsatz":
For every (**LK**-) proof of a formula A there exists a proof of A without cuts (which can be constructed effectively).

The sequent calculus:

Sequent: $\mathcal{A} \vdash \mathcal{B}$, for finite multi-sets of formulas $\mathcal{A}, \mathcal{B}$.

$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ represents

$\bigwedge A_i \rightarrow \bigvee B_j$.

$\vdash$: separation-symbol.

LK: calculus on sequents,
based on *logical* and *structural* rules.

axioms: $A \vdash A$ for atoms A.

I. The logical rules:

$\wedge$-introduction:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge : l1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge : l2$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge : r$$

$\vee$-introduction:

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \vee : l$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee : r1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \vee : r2$$

$\rightarrow$-introduction:

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \rightarrow: l$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \rightarrow: r$$

$\neg$-introduction:

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\neg : l \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\neg : r$$

$\forall$-introduction (eigenvariable cond. for $\forall : r$):

$$\frac{A(x/t), \Gamma \vdash \Delta}{(\forall x)A(x), \Gamma \vdash \Delta}\forall : l \qquad \frac{\Gamma \vdash \Delta, A(x/y)}{\Gamma \vdash \Delta, (\forall x)A(x)}\forall : r$$

$\exists$-introduction (the eigenvariable conditions for $\exists : l$ are these for $\forall : r$):

$$\frac{A(x/y), \Gamma \vdash \Delta}{(\exists x)A(x), \Gamma \vdash \Delta}\exists : l \qquad \frac{\Gamma \vdash \Delta, A(x/t)}{\Gamma \vdash \Delta, (\exists x)A(x)}\exists : r$$

II. The structural rules:

weakening:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} w:r \qquad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} w:l$$

contraction:

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} c:l \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} c:r$$

cut:

$$\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} cut(A)$$

Let A be a formula s.t. A occurs in Δ and in Π. Then the *mix* is defined as:

$$\frac{\Gamma \vdash \Delta \quad \Pi \vdash \Lambda}{\Gamma, \Pi^* \vdash \Delta^*, \Lambda} mix(A)$$

where $\Pi^* = \Pi$ after elimination of A, similar for Δ.

LK-proof without cut:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(y) \vdash P(y)}{P(y) \vdash P(b), P(y)}\, w:r}{P(y), \neg P(y) \vdash P(b)}\, \neg:l \qquad \dfrac{P(b) \vdash P(b)}{P(y), P(b) \vdash P(b)}\, w:l}{P(y), \neg P(y) \vee P(b) \vdash P(b)}\, \vee:l}{P(y), (\forall x)(\neg P(x) \vee P(b)) \vdash P(b)}\, \forall:l}{(\exists x)P(x), (\forall x)(\neg P(x) \vee P(b)) \vdash P(b)}\, \exists:l}{(\forall x)(\neg P(x) \vee P(b)) \vdash (\exists x)P(x) \rightarrow P(b)}\, \rightarrow:r}{\vdash (\forall x)(\neg P(x) \vee P(b)) \rightarrow ((\exists x)P(x) \rightarrow P(b))}\, \rightarrow:r
$$

LK-proof with cut:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{(\forall x)(\neg P(x) \vee P(b)) \vdash A} \qquad \overset{(\varphi_2)}{A \vdash (\exists x)P(x) \to P(b)}}{(\forall x)(\neg P(x) \vee P(b)) \vdash (\exists x)P(x) \to P(b)}\ cut}{\vdash (\forall x)(\neg P(x) \vee P(b)) \to ((\exists x)P(x) \to P(b))}\ \to: r$$

for $A = (\forall x)\neg P(x) \vee P(b)$ and

$\varphi_2 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(y) \vdash P(y)}{P(y) \vdash P(b), P(y)}\ w:r}{P(y), \neg P(y) \vdash P(b)}\ \neg : l}{P(y), (\forall x)\neg P(x) \vdash P(b)}\ \forall : l \qquad \dfrac{P(b) \vdash P(b)}{P(y), P(b) \vdash P(b)}\ w:l}{P(y), (\forall x)\neg P(x) \vee P(b) \vdash P(b)}\ \vee : l}{(\exists x)P(x), (\forall x)\neg P(x) \vee P(b) \vdash P(b)}\ \exists : l}{(\forall x)\neg P(x) \vee P(b) \vdash (\exists x)P(x) \to P(b)}\ \to: r$$

Gentzen's method of cut-elimination:

- reduction of *rank* and *grade*.
- "peeling" the cut-formulas from outside.
- elimination of an uppermost cut.

The method can be described as a

normal form computation

based on a set of rules $\mathcal{R}$.

Computational features:

- very slow
- weak in detecting redundancy.

Example of a Gentzen reduction:

$$\dfrac{\dfrac{\dfrac{P(a)\vdash P(a)}{(\forall x)P(x)\vdash P(a)}\;\forall:l \qquad \dfrac{P(b)\vdash P(b)}{(\forall x)P(x)\vdash P(b)}\;\forall:l}{(\forall x)P(x)\vdash P(a)\wedge P(b)}\;\wedge:r \qquad \dfrac{\dfrac{P(a)\vdash P(a)}{P(a)\wedge P(b)\vdash P(a)}\;\wedge:l}{P(a)\wedge P(b)\vdash(\exists x)P(x)}\;\exists:r}{(\forall x)P(x)\vdash(\exists x)P(x)}\;cut$$

rank = 3, grade = 1.

reduce to rank = 2, grade = 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a)\vdash P(a)}{(\forall x)P(x)\vdash P(a)}\;\forall:l \qquad \dfrac{P(b)\vdash P(b)}{(\forall x)P(x)\vdash P(b)}\;\forall:l}{(\forall x)P(x)\vdash P(a)\wedge P(b)}\;\wedge:r \qquad \dfrac{P(a)\vdash P(a)}{P(a)\wedge P(b)\vdash P(a)}\;\wedge:l}{(\forall x)P(x)\vdash P(a)}\;cut}{(\forall x)P(x)\vdash(\exists x)P(x)}\;\exists:r$$

$$\dfrac{\dfrac{\dfrac{P(a) \vdash P(a)}{(\forall x)P(x) \vdash P(a)}\,\forall : l \quad \dfrac{P(b) \vdash P(b)}{(\forall x)P(x) \vdash P(b)}\,\forall : l}{(\forall x)P(x) \vdash P(a) \land P(b)}\,\land : r \quad \dfrac{P(a) \vdash P(a)}{P(a) \land P(b) \vdash P(a)}\,\land : l}{\dfrac{(\forall x)P(x) \vdash P(a)}{(\forall x)P(x) \vdash (\exists x)P(x)}\,\exists : r}\,cut$$

rank = 2, grade = 1.
reduce to grade = 0, rank = 3:

$$\dfrac{\dfrac{P(a) \vdash P(a)}{(\forall x)P(x) \vdash P(a)}\,\forall : l \quad P(a) \vdash P(a)}{\dfrac{(\forall x)P(x) \vdash P(a)}{(\forall x)P(x) \vdash (\exists x)P(x)}\,\exists : r}\,cut$$

eliminate cut with axiom:

$$\dfrac{\dfrac{P(a) \vdash P(a)}{(\forall x)P(x) \vdash P(a)}\,\forall : l}{(\forall x)P(x) \vdash (\exists x)P(x)}\,\exists : r$$

Cut-elimination by Resolution (CERES):

based on a structural analysis of **LK**-proofs.

sub-derivation into cuts

φ

sub-derivation into end sequent

CL(φ): *characteristic clause set*, carries substantial information on derivations of cut formulas.

clause = atomic sequent.

sequent = $\Gamma \vdash \Delta$. Γ, Δ multisets of formulas

cut-elimination = reduction to *atomic cuts*.

The Method CERES:

Example:

$$\cfrac{\varphi_1 \qquad\qquad \varphi_2}{(\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y))}\ cut$$

$\varphi_1 =$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(u)^\star \vdash P(u) \qquad Q(u) \vdash Q(u)^\star}{P(u)^\star, P(u) \to Q(u) \vdash Q(u)^\star}\ \to: l}{P(u) \to Q(u) \vdash (P(u) \to Q(u))^\star}\ \to: r}{P(u) \to Q(u) \vdash (\exists y)(P(u) \to Q(y))^\star}\ \exists : r}{(\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(u) \to Q(y))^\star}\ \forall : l}{(\forall x)(P(x) \to Q(x)) \vdash (\forall x)(\exists y)(P(x) \to Q(y))^\star}\ \forall : r$$

$\varphi_2 =$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(a) \vdash P(a)^\star \qquad Q(v)^\star \vdash Q(v)}{P(a), (P(a) \to Q(v))^\star \vdash Q(v)}\ \to: l}{(P(a) \to Q(v))^\star \vdash P(a) \to Q(v)}\ \to: r}{(P(a) \to Q(v))^\star \vdash (\exists y)(P(a) \to Q(y))}\ \exists : r}{(\exists y)(P(a) \to Q(y))^\star \vdash (\exists y)(P(a) \to Q(y))}\ \exists : l}{(\forall x)(\exists y)(P(x) \to Q(y))^\star \vdash (\exists y)(P(a) \to Q(y))}\ \forall : l$$

$$S_1 = \{P(u) \vdash\},\ \ S_2 = \{\vdash Q(u)\},$$
$$S_3 = \{\vdash P(a)\},\ \ S_4 = \{Q(v) \vdash\}.$$

$$S = S_1 \times S_2 = \{P(u), Q(u)\}.$$

$$S' = S_3 \cup S_4 = \{\vdash P(a);\ \ Q(v) \vdash\}.$$

$$\mathsf{CL}(\varphi) = S \cup S' =$$
$$\{P(u) \vdash Q(u);\ \vdash P(a);\ \ Q(v) \vdash\}.$$

Projection to CL(φ)**:**

- *Skip inferences leading to cuts.*

- Obtain cut-free proof of end-sequent
  $+$ a clause in CL(φ).

Let φ be the proof of the sequent

S: $(\forall x)(P(x) \rightarrow Q(x)) \vdash (\exists y)(P(a) \rightarrow Q(y))$
shown above.

$$\mathsf{CL}(\varphi) = \{P(u) \vdash Q(u); \quad \vdash P(a); \quad Q(v) \vdash\}.$$

Skip inferences in φ_1 leading to cuts:

$$\dfrac{\dfrac{P(u) \vdash P(u) \qquad Q(u) \vdash Q(u)}{P(u), P(u) \to Q(u) \vdash Q(u)} \to: l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash Q(u)} \,\forall : l$$

$\varphi(C_1) =$

$$\dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \qquad Q(u) \vdash Q(u)}{P(u), P(u) \to Q(u) \vdash Q(u)} \to: l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash Q(u)} \,\forall : l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y)), Q(u)} \,w : r$$

For $C_2 = \vdash P(a)$ we obtain the projection $\varphi(C_2)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(v)} \,w : r}{\vdash P(a) \to Q(v), P(a)} \to: r}{\vdash (\exists y)(P(a) \to Q(y)), P(a)} \,\exists : l}{(\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y)), P(a)} \,w : l$$

next step:

- Construct an R-refutation γ of CL(φ),
- insert the projections of φ into γ.

Let φ be the proof of

S: $(\forall x)(P(x) \rightarrow Q(x)) \vdash (\exists y)(P(a) \rightarrow Q(y))$

as defined above. Then

CL(φ) =

$\{C_1 : P(u) \vdash Q(u),\ C_2 : \vdash P(a),\ C_3 : Q(u) \vdash\}$.

First we define a resolution refutation δ of $\mathrm{CL}(\varphi)$:

$$\cfrac{\cfrac{\vdash P(a) \qquad P(u) \vdash Q(u)}{\vdash Q(a)}\,R \qquad Q(v) \vdash}{\vdash}\,R$$

$R =$ atomic mix $+$ most general unification.

ground projection γ of δ:

$$\cfrac{\cfrac{\vdash P(a) \qquad P(a) \vdash Q(a)}{\vdash Q(a)}\,R \qquad Q(a) \vdash}{\vdash}\,R$$

The ground substitution defining the ground projection is

$\sigma : \{u \leftarrow a, v \leftarrow a\}$.

Let $\chi_1 = \varphi(C_1)\sigma$,
$\chi_2 = \varphi(C_2)\sigma$ and
$\chi_3 = \varphi(C_3)\sigma$.

$B = (\forall x)(P(x) \rightarrow Q(x))$,

$C = (\exists y)(P(a) \rightarrow Q(y))$.

Then $\varphi(\gamma) =$

$$\dfrac{\dfrac{\dfrac{\overset{(\chi_2)}{B \vdash C, P(a)} \qquad \overset{(\chi_1)}{P(a), B \vdash C, Q(a)}}{B, B \vdash C, C, Q(a)}\,cut \qquad \overset{(\chi_3)}{Q(a), B \vdash C}}{B, B, B \vdash C, C, C}\,cut}{B \vdash C}\,\text{contractions}$$

Definition 1

- $\mathcal{SK}$ = set of all **LK**-derivations with skolemized end-sequents.

- $\mathcal{SK}_\emptyset$ = set of all cut-free proofs in $\mathcal{SK}$.

- $\mathcal{SK}^i$ = derivations in $\mathcal{SK}$ with cut-formulas of formula complexity $\leq i$. $\sharp$

Goal: reduction to derivations with only atomic cuts, i.e.

transform $\varphi \in \mathcal{SK}$ into $\psi \in \mathcal{SK}^0$.

first step: construction of the

characteristic clause set

Characteristic Clause Set:

Let φ be an **LK**-derivation of S and let Ω be the set of all occurrences of cut formulas in φ. We define the set of clauses CL(φ) inductively:

Let ν be the occurrence of an initial sequent in φ and seq_ν the corresponding sequent. Then

$$S/\nu = \{seq(\nu, \Omega)\}$$

where $seq(\nu, \Omega)$ is the subsequent of seq_ν containing the ancestors of Ω.

Assume:
S/ν already constructed for depth(ν) $\leq k$.

depth(ν) $= k + 1$:

(a) ν is the consequent of μ:
$S/\nu = S/\mu$.

(b) ν is the consequent of μ_1 and μ_2:

(b1) The auxiliary formulas of ν are *ancestors* of Ω, i.e. the formulas occur in $seq(\mu_1, \Omega), seq(\mu_2, \Omega)$:

(+) $S/\nu = S/\mu_1 \cup S/\mu_2$.

(b2) The auxiliary formulas of ν are *not ancestors* of Ω:

($\times$) $S/\nu = S/\mu_1 \times S/\mu_2$.

CL(φ) $= S/\nu_0$ where ν_0 is the occurrence of the end-sequent.

Remark: If φ is a cut-free proof then there are no occurrences of cut formulas in φ and $\mathrm{CL}(\varphi) = \emptyset$.

Proposition 1
Let φ be an **LK***-derivation. Then* $\mathrm{CL}(\varphi)$ *is unsatisfiable.*

Projection:

Lemma 1
Let φ *be a deduction in* $\mathcal{SK}$ *of a sequent* S : $\Gamma \vdash \Delta$. *Let* $C\colon \bar{P} \vdash \bar{Q}$ *be a clause in* $\mathrm{CL}(\varphi)$. *Then there exists a deduction*

$\varphi(C)$ *of* $\quad \bar{P}, \Gamma \vdash \Delta, \bar{Q}$

s.t.

$$\varphi(C) \in \mathcal{SK}_{\emptyset} \ \text{ and } \ l(\varphi(C)) \leq l(\varphi).$$

Projection of φ to C: construct $\varphi(C)$.

the remaining steps:

• Construct an R-refutation γ of CL(φ),

• insert the projections of φ into γ.

• add some contractions and obtain a proof with (only) atomic cuts.

(• eliminate the atomic cuts)

CERES does *not only* work for **LK**.

- any sound sequent calculus for classical logic (with cut) does the job.

- unary rules do not "count".

- *necessary:* auxiliary formulas, principal formulas, ancestor relation

Example: LKDe
LK + equality rules + definition introduction.
Important to *formalization of mathematical proofs.*

Corresponding clausal calculus:
resolution + paramodulation.

Example:

If a divides b then it divides b^2.

D stands for "divides" and is defined by

$$D(x, y) \leftrightarrow \exists z x * z = y.$$

The active equations are written in boldface.

$$\dfrac{\vdash (\mathbf{a} * \mathbf{z_0}) * \mathbf{b} = \mathbf{a} * (\mathbf{z_0} * \mathbf{b}) \qquad \dfrac{a * z_0 = b \vdash \mathbf{a} * \mathbf{z_0} = \mathbf{b} \qquad \vdash b * b = b * b}{a * z_0 = b \vdash (a * z_0) * b = b * b} =: r}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a * z_0 = b \vdash a * (z_0 * b) = b * b}{a * z_0 = b \vdash \exists z.a * z = b * b} \exists : r}{\exists z.a * z = b \vdash \exists z.a * z = b * b} \exists : l}{\exists z.a * z = b \vdash D(a, b * b)} \text{def}_D : r}{D(a, b) \vdash D(a, b * b)} \text{def}_D : l}{\vdash D(a, b) \rightarrow D(a, b * b)} \rightarrow : r}{} =: r$$

Axioms:
(1) an instance of associativity,
(2) $\vdash b * b = b * b$ (instance of reflexivity),
(3) $a * z_0 = b \vdash a * z_0 = b$.

Experiments with CERES:

- underlying theorem prover: OTTER.

- very large proofs can be handled.

- Analysis of an example from C. Urban.
  mathematically different proofs from CERES.

- work in progress:
  analysis of a proof from the BOOK.
  elimination of topological arguments
  from a proof in number theory.

- system ceres available at
  `http://www.logic.at/ceres/`

Complexity:

complexity of cut-elimination is *nonelementary*.

Orevkov, Statman (1979):
There exists a sequence of **LK**-proofs φ_n of sequents S_n s.t.

- $\|\varphi_n\| \leq 2^{k*n}$ and
- for all cut-free proofs ψ of φ_n:
  $\|\psi\| > s(n)$ where
  $s(0) = 1,\ s(n+1) = 2^{s(n)}$.

There exists no cheap way of cut-elimination *in principle!*

CERES:

main point of complexity: resolution proof.

φ: **LK**-proof of S.
Let γ be a resolution refutation of CL(φ).
Then there exists a proof ψ of S with (only) atomic cuts s.t.

$\|\psi\| \leq 2 * \|\gamma\| * \|\varphi\|$.

If all axioms are standard ($A \vdash A$) then there exists a cut-free proof ψ' of S s.t.

$\|\psi'\| \leq 2^{d*\|\gamma\|*\|\varphi\|}$.

CERES is superior to Gentzen:

nonelementary speed-up of Gentzen by CERES:

- There exists a sequence of LK-proofs φ_n s.t. $\|\varphi_n\| \leq 2^{k*n}$ and all Gentzen-eliminations are of size $> s(n)$.

  CERES produces $\leq 2^{m*n}$ symbols.

- There is no nonelementary speed-up of CERES by Gentzen!

Characteristic Clause Sets and Cut-Reduction

Lemma 2
Let φ, φ' be **LK***-derivations with $\varphi > \varphi'$ for a cut reduction relation $>$ based on $\mathcal{R}$. Then $\mathsf{CL}(\varphi) \leq_{ss} \mathsf{CL}(\varphi')$.*

proof:
by cases according to the definitions of $>$ and $\mathcal{R}$. $\diamond$

$\mathcal{R} =$ set of cut-reduction rules extracted from Gentzen's proof.

$\leq_{ss}$: subsumption relation on clause sets.

Theorem 1
Let φ be an **LK***-deduction and ψ be a normal form of φ under a cut reduction relation $>$ based on $\mathcal{R}$. Then*

$$\mathrm{CL}(\varphi) \ \leq_{ss} \ \mathrm{CL}(\psi).$$

Theorem 2
Let φ be an **LK***-derivation and ψ be a normal form of φ under a cut reduction relation $>_{\mathcal{R}}$ based on $\mathcal{R}$. Then there exists a resolution refutation γ of $\mathrm{CL}(\varphi)$ s.t.*

$$\gamma \leq_{ss} \mathrm{RES}(\psi).$$

$\mathrm{RES}(\psi)$ = (standard) resolution refutation of $\mathrm{CL}(\psi)$.

Corollary 1
Let φ be an **LK**-*derivation and ψ be a normal form of φ under a cut reduction relation $>_{\mathcal{R}}$ based on $\mathcal{R}$. Then there exists a resolution refutation γ of* CL(φ) *s.t.*

$$l(\gamma) \leq l(\mathrm{RES}(\psi)) \leq l(\psi) * 2^{2*l(\psi)}.$$

Corollary 2
Let φ be an **LK**-*derivation and ψ be a normal form of φ under a cut reduction relation $>_{\mathcal{R}}$ based on $\mathcal{R}$. Then there exists a proof χ obtained from φ by CERES s.t.*

$$l(\chi) \leq l(\varphi) * l(\psi) * 2^{2*l(\psi)}.$$

Proof: χ is defined by inserting the projections of φ into a refutation γ of CL(φ). $\diamond$

Corollary 3

*Let φ be an **LK**-derivation and ψ be a normal form of φ under Gentzen's or Tait's method. Then there exists an proof χ obtained from φ by CERES s.t.*

$$l(\chi) \leq l(\varphi) * l(\psi) * 2^{2*l(\psi)}.$$

Proof: Gentzens and Tait's methods are based on $\mathcal{R}$. $\diamond$

Cut Reduction Rules:

If a cut-derivation ψ is transformed to ψ' then we define

$$\psi > \psi'$$

where $\psi =$

$$\frac{\begin{matrix}(\rho) \\ \Gamma \vdash \Delta\end{matrix} \qquad \begin{matrix}(\sigma) \\ \Pi \vdash \Lambda\end{matrix}}{\Gamma, \Pi^* \vdash \Delta^*, \Lambda}\, cut$$

3.11. rank $= 2$.

The last inferences in ρ, σ are logical ones and the cut-formula is the principal formula of these inferences:

3.113.31.

$$\dfrac{\dfrac{\begin{array}{c}(\rho_1)\\ \Gamma \vdash \Delta, A\end{array} \quad \begin{array}{c}(\rho_2)\\ \Gamma \vdash \Delta, B\end{array}}{\Gamma \vdash \Delta, A \wedge B}\wedge : r \qquad \dfrac{\begin{array}{c}(\sigma')\\ A, \Pi \vdash \Lambda\end{array}}{A \wedge B, \Pi \vdash \Lambda}\wedge : l}{\Gamma, \Pi \vdash \Delta, \Lambda}\, cut(A \wedge B)$$

transforms to

$$\dfrac{\dfrac{\begin{array}{c}(\rho_1)\\ \Gamma \vdash \Delta, A\end{array} \quad \begin{array}{c}(\sigma')\\ A, \Pi \vdash \Lambda\end{array}}{\Gamma, \Pi^* \vdash \Delta^*, \Lambda}\, cut(A)}{\Gamma, \Pi \vdash \Delta, \Lambda}\, w :^*$$

For the other form of $\wedge : l$ the transformation is straightforward.

3.113.33.

$$\dfrac{\dfrac{\begin{array}{c}(\rho'[\alpha])\\ \Gamma \vdash \Delta, B^x_\alpha\end{array}}{\Gamma \vdash \Delta, (\forall x)B}\,\forall : r \qquad \dfrac{\begin{array}{c}(\sigma')\\ B^x_t, \Pi \vdash \Lambda\end{array}}{(\forall x)B, \Pi \vdash \Lambda}\,\forall : l}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut((\forall x)B)$$

transforms to

$$\dfrac{\dfrac{\begin{array}{c}(\rho'[t])\\ \Gamma \vdash \Delta, B^x_t\end{array} \qquad \begin{array}{c}(\sigma')\\ B^x_t, \Pi \vdash \Lambda\end{array}}{\Gamma, \Pi^* \vdash \Delta^*, \Lambda}\,cut(B^x_t)}{\Gamma, \Pi \vdash \Delta, \Lambda}\,w :^*$$

3.113.34. The last inferences in ρ, σ are $\exists : r, \exists : l$: symmetric to 3.113.33.

3.12. rank > 2:

3.121. right-rank > 1:

3.121.2. The cut formula does not occur in the antecedent of the end-sequent of ρ.

3.121.23. The last inference in σ is binary:

3.121.231. The case $\wedge : r$. Here

$$\dfrac{\begin{array}{c}(\rho)\\ \Pi \vdash \Lambda\end{array} \qquad \dfrac{\begin{array}{c}(\sigma_1)\\ \Gamma \vdash \Delta, B\end{array} \quad \begin{array}{c}(\sigma_2)\\ \Gamma \vdash \Delta, C\end{array}}{\Gamma \vdash \Delta, B \wedge C}\wedge : r}{\Pi, \Gamma^* \vdash \Lambda^*, \Delta, B \wedge C}\, cut(A)$$

transforms to

$$\dfrac{\dfrac{\begin{array}{c}(\rho)\\ \Pi \vdash \Lambda\end{array} \quad \begin{array}{c}(\sigma_1)\\ \Gamma \vdash \Delta, B\end{array}}{\Pi, \Gamma^* \vdash \Lambda^*, \Delta, B}\, cut(A) \qquad \dfrac{\begin{array}{c}(\rho)\\ \Pi \vdash \Lambda\end{array} \quad \begin{array}{c}(\sigma_2)\\ \Gamma \vdash \Delta, C\end{array}}{\Pi, \Gamma^* \vdash \Lambda^*, \Delta, C}\, cut(A)}{\Pi, \Gamma^*, \vdash \Lambda^*, \Delta, B \wedge C}\wedge : r$$

3.121.232. The case $\vee : l$. Then ψ is of the form

$$\dfrac{\dfrac{(\rho)}{\Pi \vdash \Lambda} \quad \dfrac{\dfrac{(\sigma_1)}{B, \Gamma \vdash \Delta} \quad \dfrac{(\sigma_2)}{C, \Gamma \vdash \Delta}}{B \vee C, \Gamma \vdash \Delta}\, \vee : l}{\Pi, (B \vee C)^*, \Gamma^* \vdash \Lambda^*, \Delta}\, cut(A)$$

$(B \vee C)^*$ is empty if $A = B \vee C$ and $B \vee C$ otherwise.

We first define the proof τ:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\rho)}{\Pi \vdash \Lambda} \quad \dfrac{(\sigma_1)}{B, \Gamma \vdash \Delta}}{B^*, \Pi, \Gamma^* \vdash \Lambda^*, \Delta}\, cut(A)}{B, \Pi, \Gamma^* \vdash \Lambda^*, \Delta}\, x \qquad \dfrac{\dfrac{\dfrac{(\rho)}{\Pi \vdash \Lambda} \quad \dfrac{(\sigma_2)}{C, \Gamma \vdash \Delta}}{C^*, \Pi, \Gamma^* \vdash \Lambda^*, \Delta}\, cut(A)}{C, \Pi, \Gamma^* \vdash \Lambda^*, \Delta}\, x}{B \vee C, \Pi, \Gamma^* \vdash \Lambda^*, \Delta}\, \vee : l$$

Note that, in case $A = B$ or $A = C$, the inference x is $w : l$; otherwise x is the identical transformation and can be dropped.

If $(B \vee C)^* = B \vee C$ then ψ transforms to τ.

If, on the other hand, $(B \vee C)^*$ is empty (i.e. $B \vee C = A$) then we transform ψ to

$$\dfrac{\dfrac{\begin{array}{c}(\rho)\\ \Pi \vdash \Lambda\end{array} \quad \tau}{\Pi, \Pi^*, \Gamma^* \vdash \Lambda^*, \Lambda^*, \Delta}\ cut(A)}{\Pi, \Gamma^* \vdash \Lambda^*, \Delta}\ c:^*$$

3.121.233. The last inference in ψ_2 is $\rightarrow: l$. Then ψ is of the form:

$$\dfrac{\begin{matrix}(\psi_1)\\ \Pi \vdash \Sigma\end{matrix} \qquad \dfrac{\begin{matrix}(\chi_1)\\ \Gamma \vdash \Theta, B\end{matrix} \qquad \begin{matrix}(\chi_2)\\ C, \Delta \vdash \Lambda\end{matrix}}{B \rightarrow C, \Gamma, \Delta \vdash \Theta, \Lambda}\rightarrow: l}{\Pi, (B \rightarrow C)^*, \Gamma^*, \Delta^* \vdash \Sigma^*, \Theta, \Lambda}\, cut(A)$$

As in 3.121.232 $(B \rightarrow C)^* = B \rightarrow C$ for $B \rightarrow C \neq A$ and $(B \rightarrow C)^*$ empty otherwise.

3.121.233.1. A occurs in Γ and in Δ. Again we define a proof τ:

$$\dfrac{\dfrac{\begin{matrix}(\psi_1)\\ \Pi \vdash \Sigma\end{matrix} \qquad \begin{matrix}(\chi_1)\\ \Gamma \vdash \Theta, B\end{matrix}}{\Pi, \Gamma^* \vdash \Sigma^*, \Theta, B}\, cut(A) \qquad \dfrac{\dfrac{\begin{matrix}(\psi_1)\\ \Pi \vdash \Sigma\end{matrix} \qquad \begin{matrix}(\chi_2)\\ C, \Delta \vdash \Lambda\end{matrix}}{C^*, \Pi, \Delta^* \vdash \Sigma^*, \Lambda}\, cut(A)}{C, \Pi, \Delta^* \vdash \Sigma^*, \Lambda}\, x}{B \rightarrow C, \Pi, \Gamma^*, \Pi, \Delta^* \vdash \Sigma^*, \Theta, \Sigma^*, \Lambda}\rightarrow: l$$

If $(B \rightarrow C)^* = B \rightarrow C$ then, as in 3.121.232, ψ is transformed to τ + some additional contractions. Otherwise an additional cut with cut formula A is appended.

3.121.233.2 A occurs in Δ, but not in Γ. As in 3.121.233.1 we define a proof τ:

$$\dfrac{\begin{array}{c}(\chi_1)\\ \Gamma \vdash \Theta, B\end{array} \quad \dfrac{\dfrac{\begin{array}{cc}(\psi_1) & (\chi_2)\\ \Pi \vdash \Sigma & C, \Delta \vdash \Lambda\end{array}}{C^*, \Pi, \Delta^* \vdash \Sigma^*, \Lambda}\, cut(A)}{C, \Pi, \Delta^* \vdash \Sigma^*, \Lambda}\, x}{B \to C, \Gamma, \Pi, \Delta^* \vdash \Theta, \Sigma^*, \Lambda}\, \to: l$$

Again we distinguish the cases $B \to C = A$ and $B \to C \neq A$ and define the transformation of ψ exactly like in 3.121.233.1.

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