

Kripke Coalgebras over Set^2

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$Set = \text{sets} + \text{functions}.$

$$P : Set \rightarrow Set$$

$$P(X) = \{A \mid A \subseteq X\}$$

Definition. *A Kripe coalgebra is a pair (X, τ) where X is a set and $\tau : X \rightarrow P(X)$.*

Example. $X = \{x, y, z\}$

$$x \longrightarrow \{y, z\}$$

$$y \longrightarrow \{y\}$$

$$z \longrightarrow \emptyset$$

Definition. A coalgebraic kripke model is a pair (X, τ) where X is a set and

$$\tau : X \rightarrow P(X) \times P(Prop).$$

Example. $X = \{x, y, z\}$

$$x \longrightarrow (\{y, z\}, \{p, q\})$$

$$y \longrightarrow (\{y\}, \{r, q\})$$

$$z \longrightarrow (\emptyset, \{p, q, r\})$$

KR - all Kripke frames and bounded morphisms.
 Set_P - all Kripke coalgebras and coalgebra homomorphisms.

Fact. $KR \equiv Set_P$.

Also we have the same isomorphism for kripke models (Kripke frames with valuations) and coalgebraic Kripke models.

$$\tau(x) = (R[x], V(x)).$$

Basic modal language ML

Alphabet $p, q, r, \dots, \neg, \wedge, \Diamond$

$\forall \equiv \neg \wedge \neg$ and $\Box \equiv \neg \Diamond \neg$

Formulas

$p \in Form,$
 $\alpha, \beta \in Form \Rightarrow \alpha \wedge \beta \in Form,$
 $\alpha \in Form \Rightarrow \neg \alpha \in Form,$
 $\alpha \in Form \Rightarrow \Diamond \alpha \in Form.$

Interpretation

$x \Vdash p$ iff $p \in \pi_2 \tau(x),$
...
 $x \Vdash \Diamond(\alpha)$ iff $\exists y \in \pi_1 \tau(x)$ such that $y \Vdash \alpha.$

$Set^2 =$ Pairs of sets + pairs of functions.

$$\mathbb{A} = (A_1, A_2)$$

$f : \mathbb{A} \rightarrow \mathbb{B}$ are given by the pairs of functions
 $f_1 : A_1 \rightarrow B_1$ and $f_2 : A_2 \rightarrow B_2$.

The power object $\mathbb{P}(\mathbb{A})$ for every $\mathbb{A} \in Ob(Set^2)$
is represented as a pair of power sets $(P(A_1), P(A_2))$.

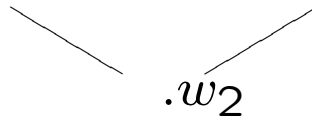
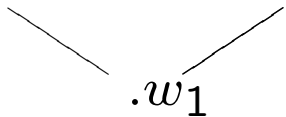
The object of propositional letters Pr is the
pair of sets $(Prop, Prop)$.

An element of $\mathbb{A} = (A_1, A_2)$ is a pair of elements (w_1, w_2) where $w_i \in A_i$.

Definition. A Set^2 Kripke coalgebra is a pair $(\mathbb{X}, \tau)$ where $\mathbb{X} \in Ob(Set^2)$ and $\tau : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{X})$.

Theorem. $Set_{\mathbb{P}}^2 = Set_P \times Set_P$.

- $\tau : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{X})$.
- $\tau_1 : X_1 \rightarrow P(X_1)$ and $\tau_2 : X_2 \rightarrow P(X_2)$.
- (X_1, R_1) and (X_2, R_2) .



Definition. A Set^2 Kripke coalgebraic model is a pair $(\mathbb{X}, \tau)$ where $\mathbb{X} \in Ob(Set^2)$ and $\tau : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{X}) \times \mathbb{P}(Pr)$.

Theorem. $Set_{\mathbb{P}}^2 = Set_P \times Set_P$.

- $\tau : \mathbb{X} \rightarrow \mathbb{P}(\mathbb{X}) \times \mathbb{P}(Pr)$.
- $\tau_1 : X_1 \rightarrow P(X_1) \times P(Prop)$ and $\tau_2 : X_2 \rightarrow P(X_2) \times P(Prop)$.
- (X_1, R_1, V_1) and (X_2, R_2, V_2) .

$w_1 \Vdash p$

$w_2 \Vdash \neg p$

Interpretation

For a given Set^2 coalgebraic model $(\mathbb{X}, \tau)$ and a point $w = (w_1, w_2)$ we inductively define satisfaction ($\Vdash_2$) of a formula in the following way:

$w \Vdash_2 p$ iff $(p, p) \in \pi_2 \tau(w)$ iff $w_1 \Vdash p$ and $w_2 \Vdash p$,
 $w \Vdash_2 \alpha \vee \beta$ iff $(w \Vdash_2 \alpha)$ or $(w \Vdash_2 \beta)$ or $(w_1 \Vdash \alpha$ and $w_2 \Vdash \beta)$ or $(w_2 \Vdash \alpha$ and $w_1 \Vdash \beta)$,
 $w \Vdash_2 \alpha \wedge \beta$ iff $w \Vdash_2 \alpha$ and $w \Vdash_2 \beta$,
 $w \Vdash_2 \neg \alpha$ iff $w_1 \not\Vdash \alpha$ and $w_2 \not\Vdash \alpha$,
 $w \Vdash_2 \Diamond \alpha$ iff $\exists w' = (w'_1, w'_2) \in \pi_1 \tau(w)$ such that $w' \Vdash \alpha$,
 $w \Vdash_2 \Box \alpha$ iff $\forall w' = (w'_1, w'_2) \in \pi_1 \tau(w)$ holds $w' \Vdash \alpha$.

Definition. *The modal logic K is given with the following axioms:*

1. *All classical tautologies,*
2. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta),$

And the rules of inference are:

$$\frac{\vdash\alpha, \vdash\alpha\rightarrow\beta}{\vdash\beta}, \quad \frac{\vdash\alpha}{\vdash\Box\alpha}.$$

Let C denote the class of all Set^2 Kripke coalgebras.

Theorem. $C \Vdash_2 \alpha$ *iff* $\vdash \alpha$ *in the modal logic K .*

Products of modal logics were designed to talk about combination of modalities such as 'time and space', 'knowledge and action' etc. The semantics for products of modal logics is formed by the product frames.

Definition. *The frame (U, R_h, R_v) is called a product frame if there exist frames (W_1, R_1) and (W_2, R_2) such that $U = W_1 \times W_2$ and*

$$(w_1, w_2)R_h(w'_1, w'_2) \iff w_1R_1w'_1 \ \& \ w_2 = w'_2$$

$$(w_1, w_2)R_v(w'_1, w'_2) \iff w_1 = w'_1 \ \& \ w_2R_2w'_2.$$

Definition. *A bounded morphism f between two product frames (U, R_h, R_v) and (V, R'_h, R'_v) is called a hover map if*

$$\forall u_1, u'_1 \in U_1 \ \forall u_2 \in U_2 \quad \pi_2 f(u_1, u_2) = \pi_2 f(u'_1, u_2)$$

and

$$\forall u_2, u'_2 \in U_2 \ \forall u_1 \in U_1 \quad \pi_1 f(u_1, u_2) = \pi_1 f(u_1, u'_2).$$

where π_i denote i -th projection.

Definition. Let Kr_{hv}^2 denote the category of all product frames and hover maps.

Theorem. The category Kr_{hv}^2 is isomorphic to $Set_{\mathbb{P}}^2$.
