SATISFACTION AND FRIENDLINESS RELATIONS WITHIN CLASSICAL LOGIC: PROOF-THEORETIC APPROACH

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PROPOSITIONAL LANGUAGE

Formulas:

Formula ≔ $\Downarrow$|(*true*)|*variable*|$(\alpha\wedge\beta)$|$(\alpha\vee\beta)$|$(\alpha\to\beta)$

|$\neg\alpha$

- ***Var*** is a (countable) set of variables.
- ***Fm*** is the set of formulas.
- Γ, Δ are unspecified multisets of formulas.

Substitution:

$\boldsymbol{s}(\alpha)=\alpha[p_1\backslash\beta_1, p_2\backslash\beta_2,\ldots, p_n\backslash\beta_n]$, where $p_1, p_2,\ldots, p_n$ are variables (not necessarily all), occurring in α.

SEMANTICS: *Partial Valuations*

Notation:

$E(\alpha)$ is the set of variables which occur in α.

$E(\Gamma) = \cup\{E(\alpha) \mid \alpha \in \Gamma\}$.

A *valuation* is a mapping v: $\boldsymbol{Var} \to \{0,1\}$, extended up to a homomorphism to $\boldsymbol{Fm} \to \{0,1\}$.

If v is a valuation, its restriction $v \restriction E(\Gamma)$ is called a *partial valuation* w.r.t. $E(\Gamma)$.

Partial valuations will play a key role in the sequel.

A VARIATION ON CONSEQUENCE RELATION: *Logical Friendliness*

A valuation v_1 is an *extension* of v_0, in symbols $v_1 \geq v_0$, if the domain of v_0 is included in the domain of v_1 and for each variable p in the domain of v_0, $v_1(p) = v_0(p)$.

Main Definition: A muliset Γ is *friendly* to a formula α, in symbols $\Gamma \square \alpha$, if for any partial valuation w.r.t. $E(\Gamma)$, which validates Γ, there is its extension w.r.t. $E(\Gamma,\alpha)$, which validates α. (Cf. [Makinson 2005].)

Comparisons: Consequence Relation (⬀) *vs.* Logical Friendliness (□)

<u>Observations</u> [Makinson 2005]:I

- If Γ⬀α then $\Gamma\square\alpha$.
- Both relations are *compact.* Namely, [Logical Friendliness]: *If for nonvoid* Γ, $\Gamma\square\alpha$, *then there is a finite nonvoid* $\Gamma_0\subseteq\Gamma$, *such that* $\Gamma_0\square\alpha$. [Makinson 2005].
- Logical Friendliness relation □ is not monotone. Indeed, it is clear that $p\square q$ holds but $p,\neg q\ \square q$ does not.

Comparisons: Consequence Relation (⬀) *vs.* Logical Friendliness (▫) (*continued*)

Main result:

- There is a deduction relation ⬃ (defined below) such that Γ▫α is equivalent to Γ⬃α. (Theorem 5 below)

RESTRICTED CASE FOR □: SATISFACTION

If $\Gamma=\varnothing$ in Γ□α, then □α just means that there is a valuation which validates α; or in other words, α is a *satisfiable* formula.

The satisfiable formulas form a *recursive*, and hence *recursively enumerable*, set of propositional formulas. However, it has not been known any deduction for generating this set.

DEDUCTION FOR SATISFACTION PROPERTY: ***System S***

System ***S***:

Axioms:

- ⬃⇩;
- ⬃*p*, where *p* is a variable.

Rules of Inference:

- ⇨$\alpha \rightarrow \beta$ and ⬃α (*Soundness w.r.t. classical deduction*)

  ⬃β

- ⬃***s***(α) (*Reverse substitution*)

  ⬃α

SYSTEM *S* (*continued*)

Theorem 1 (*soundness and completeness*)
For any formula α, α is satisfiable, i.e. $\models \alpha$, if and only if the sequent $\Rightarrow \alpha$ is derivable in ***S***.

DEDUCTION FOR LOGICAL FRIENDLINESS: *System F*

Now we will extend the unary relation $\vdash\alpha$ to the binary relation $\Gamma\vdash\alpha$ (defined below).

<u>Remark</u>:

To prevent a possible confusion, we note that, although derivability in ***S*** is used in the definition of ***F*** and, hence, derivability in ***F*** depends on derivability in ***S***, as it will be proven, a sequent of the form $\vdash\alpha$ is provable in ***F*** if and only if it is provable in ***S***. (Proposition 2 below)

SYSTEM *F*

Axioms:

- Γ⬃⇩;
- Γ⬃α, when $E(\Gamma)\cap E(\alpha)=\varnothing$ and the sequent ⬃α is derivable in ***S***;
- Γ⬃∧Δ, where $\Delta\subseteq\Gamma$ and Δ is finite. Here ∧Δ is conjunction of the formulas in Δ; if $\Delta=\varnothing$ then ∧Δ=⇩.

SYSTEM *F* (*continued*)

Rules of Inference:

1* $\dfrac{\Gamma \Rightarrow \alpha}{\Gamma \swarrow \alpha}$ (*soundness w.r.t. classical deduction*)

2* $\dfrac{\Gamma,\alpha \swarrow \gamma \text{ and } \Gamma,\beta \swarrow \gamma}{\Gamma \alpha \vee \beta}$ ($\vee$-*intro. in antecedent*) providing that $E(\alpha)=E(\beta)$.

3* $\dfrac{\Gamma \swarrow \alpha}{\Gamma \swarrow \alpha \vee \beta}$ ($\vee$-*intro. in consequent*)

SYSTEM *F* (*continued*)

<u>Rules of Inference (*continued*)</u>:

4* $\dfrac{\Gamma \vdash \boldsymbol{s}(\alpha)}{\Gamma \vdash \alpha}$ (*Reverse substitution*)

where $\boldsymbol{s}(\alpha)=\alpha[p_1\backslash\beta_1, p_2\backslash\beta_2, \ldots, p_n\backslash\beta_n]$

and $E(\Gamma)\cap\{p_1, p_2, \ldots, p_n\}=\emptyset$.

5* $\dfrac{\Gamma \vdash \alpha \text{ and } \alpha \vdash \beta}{\Gamma \vdash \beta}$ (*Cut*)

providing that either $E(\Gamma)\subseteq E(\underline{\alpha})$, or $E(\underline{\alpha})\subseteq E(\Gamma)$

and $E(\Gamma)\cap E(\beta)\subseteq E(\underline{\alpha})$.

SYSTEM *F* (*continued*)

<u>Rules of Inference (continued)</u>:

6* $\underline{\Gamma,\alpha ⇙ \beta \text{ and } \gamma ⇨ \alpha}$ (*deduct. replacem. in antecedent*)

$$\Gamma\gamma ⇙ \beta$$

providing that $E(\gamma) \subseteq E(\Gamma,\alpha)$.

7* $\underline{\Gamma\alpha ⇙ \beta \text{ and } \beta ⇨ \gamma}$ (*deduct. replacem. in consequent*)

$$\Gamma,\alpha ⇙ \gamma$$

SOUNDNESS FOR SYSTEM *F*

Proposition 2 A sequent $\Rightarrow \alpha$ is derivable in ***S*** if and only if the sequent $\varnothing \Rightarrow \alpha$ is derivable in ***F***.

Theorem 3 (*soundness*) If a sequent $\Gamma \Rightarrow \alpha$ is derivable in ***F*** then $\Gamma \models \alpha$ holds.

FINITE COMPLETENESS FOR SYSTEM *F*

Theorem 4 (*finite completeness*) For any finite Γ, if $\Gamma \models \alpha$ holds, then the sequent $\Gamma \Rightarrow \alpha$ is derivable in ***F***.

What is a problem in proving completeness for ⬃ for any Γ?

The consequence relation ⬀ is *monotone*,

which means that if Δ⬀α and $\Delta \subseteq \Gamma$ then Γ⬀α.

Thus to prove Completeness Theorem for classical

derivation ⇨ one can use monotonicity and

compactness of the relation ⬀.

It is not the case for Logical Friendliness ▫.

FULL COMPLTENESS FOR SYSTEM *F*

Theorem 5 (*completeness*) For any finite Γ, if $\Gamma \vDash \alpha$ holds, then the sequent $\Gamma \Rightarrow \alpha$ is derivable in ***F***.

Thank you