SYSMICS2019

Syntax Meets Semantics

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Institute for Logic, Language and Computation
University of Amsterdam

SYSMICS 2019

The international conference "Syntax meet Semantics 2019" (SYSMICS 2019) will take place from 21 to 25 January 2019 at the University of Amsterdam, The Netherlands. This is the closing conference of the European Marie Skłodowska-Curie Rise project *Syntax meets Semantics – Methods, Interactions, and Connections in Substructural logics*, which unites more than twenty universities from Europe, USA, Brazil, Argentina, South Africa, Australia, Japan, and Singapore.

Substructural logics are formal reasoning systems that refine classical logic by weakening structural rules in a Gentzen-style sequent calculus. Traditionally, non-classical logics have been investigated using proof-theoretic and algebraic methods. In recent years, combined approaches have started to emerge, thus establishing new links with various branches of non-classical logic. The program of the SYSMICS conference focuses on interactions between syntactic and semantic methods in substructural and other non-classical logics. The scope of the conference includes but is not limited to algebraic, proof-theoretic and relational approaches towards the study of non-classical logics.

This booklet consists of the abstracts of SYSMICS 2019 invited lectures and contributed talks. In addition, it also features the abstract of the SYSMICS 2019 Public Lecture, on the interaction of logic and artificial intelligence. We thank all authors, members of the Programme and Organising Committees and reviewers of SYSMICS 2019 for their contribution.

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Nick Bezhanishvili (Organizing Committee Chair)
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Invited Talks

An infinitary treatment of full mu-calculus

Bahareh Afshari

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We explore the proof theory of the modal μ-calculus with converse, aka the 'full μ-calculus'. Building on nested sequent calculi for tense logics [2] and infinitary proof theory of fixed point logics [1], a cut-free sound and complete proof system for full μ-calculus is proposed. As a corollary of our framework, we also obtain a direct proof of the regular model property for the logic [4]: every satisfiable formula has a tree model with finitely many distinct subtrees. To obtain this result we appeal to the basic theory of well-quasi-orderings in the spirit of Kozen's proof of the finite model property for μ-calculus [3].

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Algebraic Functions

Miguel Campercholi

National University of Cordoba

Let **A** be an algebraic structure, and consider the system of equations

$$\alpha(\bar{x},\bar{z}) := \begin{cases} t_1(x_1,\ldots,x_n,z_1,\ldots,z_m) = s_1(x_1,\ldots,x_n,z_1,\ldots,z_m) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ t_k(x_1,\ldots,x_n,z_1,\ldots,z_m) = s_k(x_1,\ldots,x_n,z_1,\ldots,z_m) \end{cases}$$

where t_j and s_j are a terms for $j \in \{1,,\ldots,k\}$. Suppose that for all $\bar{a} \in A^n$ there is exactly one $\bar{a} \in A^m$ such that $\alpha(\bar{a},\bar{b})$ holds. Then the system $\alpha(\bar{x},\bar{z})$ defines m functions $f_1,\ldots,f_m : A^n \to A$ by

$$(f_1(\bar{a}),\ldots,f_m(\bar{a})) := \text{unique } \bar{b} \text{ such that } \alpha(\bar{a},\bar{b}).$$

A function is called *algebraic* on **A** if it is one of the functions defined by a system of equations in the manner just described. For example, the complement function is algebraic on the two-element bounded lattice, as witnessed by the system

$$\alpha(x,z) := \begin{cases} x \wedge z = 0 \\ x \vee z = 1. \end{cases}$$

Given an algebraic structure **A**, it is easy to see that every term-function of **A** is algebraic on **A**, and that algebraic functions on **A** are closed under composition; that is, they a form a clone on A. Algebraic functions can be seen as a natural generalization of term-functions, and share some of their basic properties (e.g., they are preserved by endomorphisms and direct products).

Algebraic functions have been characterized for algebras in several well-known classes such as: Boolean Algebras, Distributive Lattices, Vector Spaces and Abelian Groups, among others. In our talk we will review these characterizations and discuss the main tools used to obtain them. We will also show how algebraic functions can be used in the study of epimorphisms and to describe intervals in the lattice of clones over a finite set.

Free algebras endomorphisms: Ruitenburg's Theorem and Beyond

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Ruitenburg's Theorem says that every endomorphism f of a finitely generated free Heyting algebra is ultimately periodic if f fixes all the generators but one. More precisely, there is $N \geq 0$ such that $f^{N+2} = f^N$, thus the period equals 2. We give a semantic proof of this theorem, using duality techniques and bounded bisimulations ranks. By the same techniques, we tackle investigation of arbitrary endomorphisms between free algebras. We show that they are not, in general, ultimately periodic. Yet, when they are (e.g. in the case of locally finite subvarieties), the period can be explicitly bounded as function of the cardinality of the set of generators.

Keywords. Ruitenburg's Theorem, Sheaf Duality, Bounded Bisimulations, Endomorphisms

Pro-aperiodic monoids via Stone duality

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This talk is about joint work with B. Steinberg, in which we apply Stone duality and model theory to study free pro-aperiodic monoids.

The class of aperiodic monoids has long played a fundamental role in finite semigroup theory and automata theory. The famous Schützenberger theorem proved that the aperiodic monoids recognize precisely the star-free languages, which also coincides with the class of languages recognizable by counter-free automata. The connection with logic comes from a later result, which shows that the class is also exactly the class of languages definable in first order logic.

Algorithmic questions about aperiodic languages lead to challenges that the algebraic approach can often help resolve. Within this algebraic approach to aperiodic languages, free pro-aperiodic monoids are a useful tool. The structure of free pro-aperiodic monoids has been studied recently by several authors, but many difficult questions remain open. Existing results about free pro-aperiodic monoids are often about the submonoid of elements definable by ω-terms and rely on an ingenuous normal form algorithm due to McCammond, which solves the word problem for ω-terms.

Stone duality and Schützenberger's theorem together imply that elements of the free pro-aperiodic monoid may be viewed as elementary equivalence classes of pseudofinite words. Concretely, this means that one may 'compute' with elements of the free pro-aperiodic monoid as if they were finite words, in a way reminiscent of the methods of non-standard analysis. In particular, model theory provides us with saturated words in each class of pseudofinite words, i.e., words in which all possible factorizations are realized. We prove that such saturated words are stable under algebraic operations. We give several applications of this new approach, including a solution to the word problem for ω-terms that avoids using McCammonds normal forms, as well as new proofs and extensions of other structural results concerning free pro-aperiodic monoids.

Inquisitive Intuitionistic Logic: An Application of Nuclear Semantics

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Inquisitive logic is a research program seeking to expand the purview of logic beyond declarative sentences to include the logic of questions. To this end, the system of inquisitive propositional logic extends classical propositional logic for declarative sentences with principles governing a new binary connective of inquisitive disjunction, which allows the formation of questions. Recently inquisitive logicians have considered what happens if the logic of declarative sentences is assumed to be intuitionistic rather than classical. In short, what should inquisitive intuitionistic logic be? This talk, based on joint work with Guram Bezhanishvili, will provide an answer to the question from the perspective of nuclear semantics, an approach to classical and intuitionistic semantics pursued in our previous work (A Semantic Hierarchy for Intuitionistic Logic, forthcoming in Indagationes Mathematicae). In particular, we show how Beth semantics for intuitionistic logic naturally extends to a semantics for inquisitive intuitionistic logic. In addition, we show how an explicit view of inquisitive intuitionistic logic comes via a translation into the system of propositional lax logic, whose completeness we prove with respect to Beth semantics.

Positive Subreducts in Finitely Generated Varieties of MV-algebras

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Abstract

Positive MV-algebras are negation-free and implication-free subreducts of MV-algebras. In this contribution we show that a finite axiomatic basis exists for the quasivariety of positive MV-algebras coming from any finitely generated variety of MV-algebras.

1 Positive subreducts of MV-algebras

Let $\mathcal{MV}$ be the variety of MV-algebras [1] in the language containing all the usual definable operations and constants. Using this signature we denote an MV-algebra $\mathbf{M} \in \mathcal{MV}$ as

$$\mathbf{M} = \langle M, \oplus, \odot, \vee, \wedge, \rightarrow, \neg, 0, 1\rangle.$$

An algebra

$$\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1\rangle$$

is a *positive subreduct* of $\mathbf{M}$ if $\mathbf{A}$ is a subreduct of $\mathbf{M}$.

Definition 1. Let $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$ be a set of function symbols, where $\oplus, \odot, \vee, \wedge$ are interpreted as binary operations and $0, 1$ as constants. An algebra $\mathbf{P}$ of type F is a *positive MV-algebra* if it is isomorphic to a positive subreduct of some MV-algebra.

Clearly, every MV-algebra gives rise to a positive MV-algebra and every bounded distributive lattice is a positive MV-algebra. In fact, positive MV-algebras are to MV-algebras as distributive lattices are to Boolean algebras.

Example 1 (Lower Chang algebra)**.** Let $\mathbf{C}$ be Chang algebra and

$$\operatorname{Rad}\mathbf{C} = \{0, \varepsilon, 2\varepsilon, \dots\}$$

be its radical, where the symbol ε denotes the least positive infinitesimal. Then the algebra $\mathbf{C}_l$ having the universe $\operatorname{Rad}\mathbf{C} \cup \{1\}$ is a positive subreduct of $\mathbf{C}$.

Example 2 (Non-decreasing McNaughton functions)**.** For each natural number n, the free n-generated MV-algebra is isomorphic to the algebra $\mathbf{F}_n$ of McNaughton functions $[0,1]^n \to [0,1]$. Then the algebra $\mathbf{F}_n^{\leq}$ of nondecreasing McNaughton functions is a positive subreduct of $\mathbf{F}_n$.

The class of all positive MV-algebras is denoted by $\mathcal{P}$. Since $\mathcal{P}$ is a class of algebras containing the trivial algebra and closed under isomorphisms, subalgebras, direct products and ultraproducts, it is a quasivariety. The following example shows that $\mathcal{P}$ is not a variety.

Example 3. Let θ be an equivalence relation on the algebra $\mathbf{C}_l$ from Example 1 with classes $\{0\}$, $\{\varepsilon, 2\varepsilon, \dots\}$, and $\{1\}$. Then θ is a $\mathcal{P}$-congruence on $\mathbf{C}_l$. The quotient $\mathbf{C}_l/\theta$ is isomorphic to the three-element algebra $\{\bar{0}, \bar{\varepsilon}, \bar{1}\}$ that satisfies the identities $\bar{\varepsilon} \oplus \bar{\varepsilon} = \bar{\varepsilon}$ and $\bar{\varepsilon} \odot \bar{\varepsilon} = \bar{0}$. However, the two equations cannot hold simultaneously in any MV-algebra. Hence, $\mathbf{C}_l/\theta$ is not a positive MV-algebra.

It can be shown that the quasivariety $\mathcal{P}$ is generated by the positive reduct of the standard MV-algebra $[0, 1]$. Moreover, the free n-generated positive MV-algebra is isomorphic to the positive subreduct $\mathbf{F}_n^{\leq}$ from Example 2.

2 Axiomatization

We define a class $\mathcal{Q}$ of algebras of type $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$. Specifically, an algebra $\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1\rangle$ belongs to $\mathcal{Q}$ if $\mathbf{A}$ satisfies the following identities and quasi-identities:

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice
2. $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids
3. $x \oplus 1 = 1$ and $x \odot 0 = 0$
4. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$ and $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$
5. $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$ and $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$
6. $x \oplus y = (x \oplus y) \oplus (x \odot y)$
7. $x \oplus y = (x \vee y) \oplus (x \wedge y)$
8. $x \odot y = (x \odot y) \odot (x \oplus y)$
9. If $x \oplus y = x$, then $z \oplus y \leq z \vee x$
10. If $x \oplus y = x \oplus z$ and $x \odot y = x \odot z$, then $y = z$

Every positive MV-algebra is a member of $\mathcal{Q}$ since 1.–10. are valid for any MV-algebra. The main open problem is to prove the opposite, that is, to show that any $\mathbf{A} \in \mathcal{Q}$ is a positive MV-algebra. We solve this problem for those $\mathbf{A} \in \mathcal{Q}$ satisfying additional identities of a special form. Namely let $\mathcal{V}$ be any finitely generated variety of MV-algebras. Di Nola and Lettieri proved in [3] that there exists a finite set S of identities axiomatizing the variety $\mathcal{V}$ within $\mathcal{MV}$, and every identity in S uses only terms of the language $\{\oplus, \odot, \vee, \wedge, 0, 1\}$.

Theorem 1. *The quasivariety of positive subreducts of $\mathcal{V}$ is axiomatized by the quasi-identities 1.–10. and the identities from S.*

The essential ingredient of the proof of Theorem 1 is a certain non-trivial generalization of the technique of good sequences, which was introduced by Mundici [2]. It remains an open problem to extend this result beyond finitely generated varieties of MV-algebras, possibly using an axiomatization different from 1.–10.

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A Dialectica Model of Relevant Type Theory

Valeria de Paiva

Nuance Communications

Relevant logics are a family of substructural logics, developed by Anderson-Belnap and collaborators, whose basic tenant is that in logical implications antecedents and consequents should be relevantly connected. Dialectica models are sophisticated categorical models of Girard's Linear Logic, conceived as an internal description of Gödel's Dialectica Interpretation. Dialectica models (also called Dialectica spaces) have proved themselves precise (capable of distinguishing all the connectives proposed in the logic) and versatile (have been used in diverse applications such as modelling Petri nets, modelling the Lambek calculus, explaining proofs between cardinalities of the continuum, explaining compiler refinements, etc). In this talk we want to show that Dialectica spaces can be used to model a version of relevant type theory and its logic and discuss how well this modelling works.

On the axiomatizability of modal many-valued logics

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Modal logic is one of the most developed and studied non-classical logics, yielding a beautiful equilibrium between complexity and expressibility. The idea of enriching a Kripke frame with an evaluation over an arbitrary algebra offers a generalization of the concepts of necessity and possibility offer a rich setting to model and study graded and resource-sensitive notions from many different areas, including proof-theory, temporal and epistemic concepts, workflow in software applications, etc. While the first publications on modal many-valued logics can be traced back to the 90s [5, 6], it has been only in the latter years when a more systematic work has been developed, addressing the axiomatizability question over certain algebras of evaluation, characterization and study of model-theoretic notions analogous to the ones from the classical case, decidability and applicability issues, etc (see eg. [7], [3, 4], [1], [9], [8], [2]...).

An open problem was that of the axiomatization of the finitary companion of those deductive systems, starting from their definition based on Kripke models evaluated over FL_{ew}-algebras. In particular, they were not known axiomatizations for the modal logics arising from models with a crisp accessibility and using both $\Box$ and $\Diamond$ modalities, and evaluated locally at the standard Gödel, MV and product algebras. In this talk we will see that the global deduction over those classes of Kripke models is not recursively enumerable, and so, they are not R.E axiomatizable.

Moreover, it was also a question whether for any of the above logics, the global deduction arises from the local one extended with the (unrestricted) necessity rule $N_\Box : \varphi \vdash \Box\varphi$. We will also see that this is not the case for a large family of algebras of evaluation, including the modal expansions of Łukasiewicz and Product Logics, in contrast to the modal logics studied up to now in the literature.

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Public Lecture

AI? That's logical!

Frank van Harmelen

Vrije Universiteit Amsterdam

The history of AI has been a continuous swing of the pendulum between the extremes of logical reasoning and statistical learning; or, as Judea Pearl has it: between Greek philosophers and Babylonian curve fitters. In recent years, the pendulum has swung strongly towards the statistical methods. We'll take a close look at the history of AI, and we'll identify the strong and weak points of both schools of thought. This will lead to a set of challenges to be taken up by logicians if they are interested in contributing to one of the most exciting intellectual endeavours of our time.

Contributed Talks

Combinatorial Proofs for the Modal Logic K

Matteo Acclavio and Lutz Straßburger

Inria

Proof theory is the area of theoretical computer science which studies proofs as mathematical objects. However, unlike many other mathematical fields, proof theory is lacking a representation for its basic objects able to capture the notion of identity. We are used to consider proofs as expressions generated by sets of production rules we call *proof systems*; and the main obstacle to understand when two proofs are the same is this syntactic representation itself. Thus, depending on the chosen formalism, a proof can be represented by different syntactic expressions. Moreover, even in the same proof system, there can not be a "natural way" to identify a *canonical representative*. This condition makes it difficult to understand when two proofs are the same object. As an example we show in Figure 1 a semantic tableau, a resolution proof and a sequent calculus derivation for the same formula.

The standard approach to the question of proof identity is based on rule permutations. Two proofs in the same proof system are considered the to be equal if they can be transformed into each other by a series of simple rule permutation steps. However this can not be considered as a solution since it relies on each specific syntax and, it is not suitable to compare proofs presented in two different proof systems for the same logic.

Combinatorial proofs [6, 7] have been introduced by Hughes to address this problem in classical logic. A combinatorial proof of a formula F consist of a *skew fibration* $f : \mathfrak{C} \mapsto \mathfrak{G}(F)$ between a RB-cograph $\mathfrak{C}$ [9] and the cograph $\mathfrak{G}(F)$ representing the formula F. The notion of cograph [4] and skew fibration [6, 10] are independent from the syntactic restrictions of proof formalisms and are described by graph condition only. Moreover, the correctness of combinatorial proofs can be checked in polynomial time on the size of a proof, i.e. they form a proof system in the sense of Cook and Reckhow [3].

It has been shown in [7, 11, 1] how syntactic proofs in Gentzen sequent calculus, the deep inference system SKS, semantic tableaux, and resolution can be translated into combinatorial proofs. Figure 2 shows the combinatorial proof corresponding to the syntactic proofs in Figure 1.

In this talk we want to address the question whether the theory of combinatorial proofs can be extended to modal logics.

In the literature, proof systems of various kinds have been defined for different modal logics [2, 8, 12, 5]. However, the notion of proof equivalence in modal logic has never been studied. Part of the problem of defining this notion is inheritance of the problem for proof equivalence in classical logic.

We are presently working on the definition of the notion of proof equivalence for different modal logics by means of combinatorial proof. The first step in this investigation is to give a representation of proofs for the modal logic K, for which we show the sequent system LK-K in Figure 3 and the deep infernce system KS-K in Figure 4.

We define a class of cograph, called RG-cograph, suitables to represent formulas with modalities and similarly we extend the notion of RB-cograph which represent the *linear* part of a classical proof, to the one of RGB-cographs.

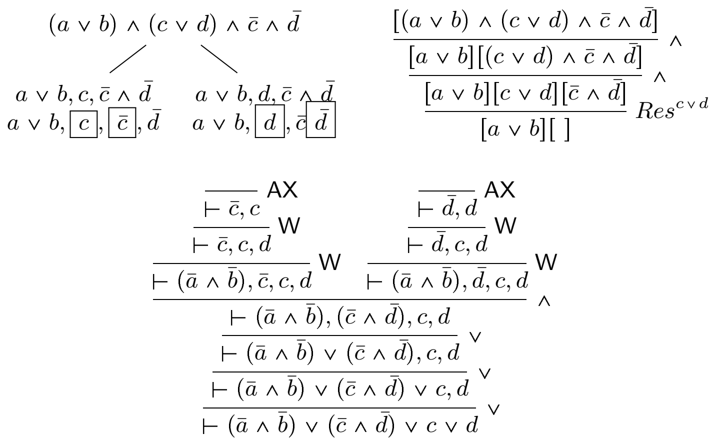


Figure 1: A semantic tableau, a resolution proof and a sequent calculus derivation of $F = (\bar a \wedge \bar b) \vee (\bar c \wedge \bar d) \vee c \vee d$

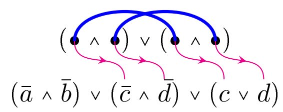


Figure 2: The combinatorial proof corresponing to the proof in Figure 1

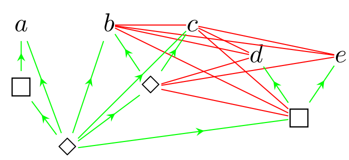
$$\frac{}{A,\bar A}\,\mathsf{AX}\quad \frac{\Gamma,A,B}{\Gamma,A\vee B}\,\vee \quad \frac{\Gamma,A\quad B,\Delta}{\Gamma,A\wedge B,\Delta}\,\wedge\quad \frac{A,B_1,\dots,B_n}{\Box A,\Diamond B_1,\dots,\Diamond B_n}\,\mathsf{K}\quad \frac{}{\mathbf{t}}\,\mathsf{t}\;\Big|\; \frac{\Gamma}{\Gamma,A}\,\mathsf{W}\quad\frac{\Gamma,A,A}{\Gamma,A}\,\mathsf{C}$$

Figure 3: Sequent system LK-K (cut free) for modal logic K. The first six rules on the left form the sequent system MLLK

$$\mathsf{i}\,\frac{\mathbf t}{a,\bar a}\qquad \mathsf{k}\,\frac{\Box(A\vee B)}{\Box A\vee\Diamond B}\qquad \mathsf{s}\,\frac{(A\vee B)\wedge C}{A\vee(B\wedge C)}\qquad \mathsf{e}\,\frac{\mathbf t}{\Box\mathbf t}\qquad \mathsf{w}\,\frac{\mathbf f}{A}\qquad \mathsf{c}\,\frac{A\vee A}{A}$$

Figure 4: Deep sequent system KS-K

Below is the RG-cograph of $\Diamond(\Box a \vee (\Diamond(b \wedge c) \wedge \Box(d \vee e)))$:



(1)

For these graphs, we recover a correctness criterion similar to the one given for RB-cographs [9] by means of æ-connectedness and æ-acyclicity (acyclic with respect of alternating paths).

In fact, given a RGB-cograph $\mathfrak{G}(F)$ we are able to define for each $\Box$-node m a set P_m of modality-nodes by means of paths between "same-depth" nodes. Intuitively, each set P_m corresponds to an application of a K-rule. Then we define a RB-cograph $\partial(\mathfrak{G}(F))$ from $\mathfrak{G}(F)$ by transforming each set P_m into a RB-cograph $\partial(P_m)$ and opportunely updating the edges interacting with the nodes with P_m. Thus, a RGB-cograph $\mathfrak{G}(F)$ corresponds to a correct

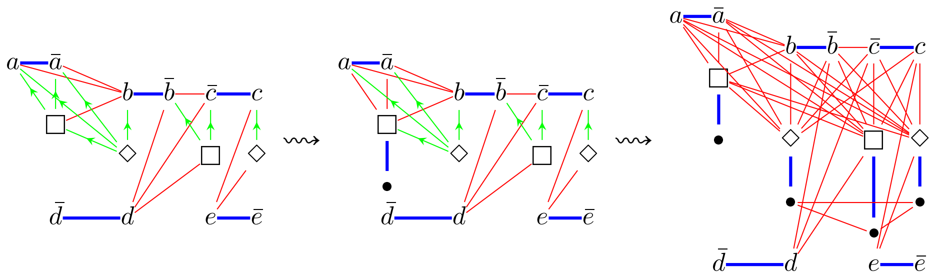


Figure 5: The RGB-cograph $\mathfrak{G}(F)$ of $F = \bar{d} \vee \bar{e} \vee (d \wedge (\bar{b} \wedge \bar{c})) \vee (e \wedge \Diamond c) \vee \Diamond(b \wedge \Box(a \vee \bar{a}))$ and its associated RB-cograph $\partial(\mathfrak{G}(F))$.

derivation if the $\Box$-nodes induce a partition over all modality-nodes and if the RB-cograph $\partial(\mathfrak{G}(F))$ is æ-connected and æ-acyclic.

Using some features of the calculus of structures, we are able to represent K proofs in the deep sequent system $\mathsf{KS\text{-}K}$ pushing all weakening and contraction rules at the end of a derivation. This allows us to define combinatorial proof by means of axiom-preserving RG-skew fibrations $f : \mathfrak{C} \mapsto \mathfrak{G}(F)$ from a RGB-cograph $\mathfrak{C}$ to the RG-cograph of F.

These results allow us to define a notion of equivalence for proofs in K and give a direct translation of the classical sequent calculus $\mathsf{LK\text{-}K}$ into combinatorial proofs and vice versa.

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Intermediate Logic Proofs as Concurrent Programs

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Building on ideas of Haskell Curry, in 1969 William Howard showed that constructing an intuitionistic proof is not at all different from writing a program in λ-calculus [8]. He also showed that the reduction of the proof to its normal form exactly corresponds to the evaluation of the associated program. This relation between intuitionistic natural deduction and simply typed λ-calculus is now called the Curry–Howard correspondence. In 1990 Griffin showed that such a correspondence is not limited to intuitionistic logic but a similar relation holds between classical logic and sequential extensions of simply typed λ-calculus featuring control operators [7]. One year later, in 1991, Avron noticed a connection between concurrent computation and hypersequent calculus – a proof calculus well suited for capturing logics intermediate between intuituionistic and classical logic, see [3]. He envisaged, in particular, the possibility of using the intermediate logics that can be captured by hypersequent calculi "as bases for parallel λ-calculi" [4].

The translation in [5] from hypersequent rules into higher-level natural deduction rules [9] made it possible to define natural deduction calculi matching the parallel structure of hypersequents. Building on this, we establish modular Curry–Howard correspondences for a family of natural deduction calculi and we prove their normalization. These correspondences provide a concurrent computational interpretation for intermediate logics that are naturally formalized as hypersequent calculi. The calculi resulting from this computational interpretation are extensions of the simply typed λ-calculus by a parallelism operator and communication channel variables. We thus confirm Avron's 1991 thesis for a rather general class of intermediate logics and present some specific instances of particular proof-theoretical interest.

In particular, we first introduce the typed concurrent λ-calculi λ_{Cl} [2] and λ_{G} [1]. These calculi are defined extending simply typed λ-calculus by the type assignment rules

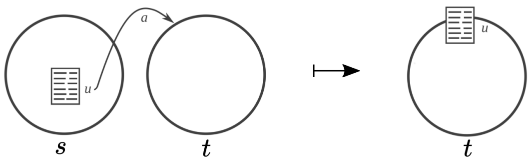
$$\dfrac{\begin{matrix}[a:\neg A]\\ \vdots\\ s:B\end{matrix} \qquad \begin{matrix}[a:A]\\ \vdots\\ t:B\end{matrix}}{s \parallel_a t : B} \qquad \text{and} \qquad \dfrac{\begin{matrix}[a:A\to B]\\ \vdots\\ s:C\end{matrix} \qquad \begin{matrix}[a:B\to A]\\ \vdots\\ t:C\end{matrix}}{s \parallel_a t : C}$$

respectively. These rules logically correspond to the excluded middle law $\neg A \vee A$ and to the linearity axiom $(A \to B) \vee (B \to A)$, respectively, and hence allow us to provide a concurrent interpretation of classical logic and Gödel–Dummett logic. The computational rôle of these rules is to introduce the parallelism operator $\parallel_a$. The parallelism operator, in turn, acts as a binder for the communication variables a occurring in s and in t. Thus, in the calculus λ_{Cl} we can compose processes in parallel and establish communication channels of the form

between them. The communication reduction rule (*basic cross reduction*) of λ_{Cl} is

$$\mathcal{S}[a^{\neg A}\, v] \parallel_a t \;\mapsto\; t[v/a] \qquad\qquad \text{for } s = \mathcal{S}[a^{\neg A} v] \text{ and } v \text{ closed term}$$

and can be intuitively represented as

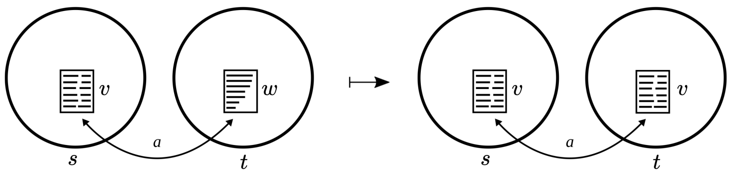


This reduction rule enables us to use channels in one direction only: we can only transmit the argument v of $a^{\neg A}$ from s to t. On the other hand, in λ_{G} we can establish channels of the form

The corresponding basic reduction rules are two, one for transmitting messages from left to right:

$$\mathcal{S}[a\,v] \parallel_a \mathcal{T}[a\,w] \;\mapsto\; \mathcal{S}[a\,v] \parallel_a \mathcal{T}[v] \qquad \text{for } s = \mathcal{S}[a\,v], t = \mathcal{T}[a\,w] \text{ and } v \text{ closed term}$$

which we can represent as



and one for transmitting messages from right to left:

$$\mathcal{S}[a\,v] \parallel_a \mathcal{T}[a\,w] \;\mapsto\; \mathcal{S}[w] \parallel_a \mathcal{T}[a\,w] \qquad \text{for } s = \mathcal{S}[a\,v], t = \mathcal{T}[a\,w] \text{ and } w \text{ closed term}$$

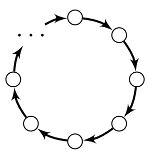
Thus in λ_{G} we can encode dialogues between processes during which messages are exchanged in both directions.

Generalising the ideas used for λ_{Cl} and λ_{G}, we then present a family of concurrent λ-calculi which provide concurrent computational interpretations for all the intermediate logics that can be defined extending intuitionistic logic by axioms of the form

$$(F_1 \to G_1) \vee \ldots \vee (F_n \to G_n)$$

where for $i \in \{1, \ldots, n\}$ no F_i is repeated and if $F_i \neq \top$ then $F_i = G_j$ for some $j \in \{1, \ldots, n\}$.

The corresponding minimal communication topologies generalize those of λ_{Cl} and λ_{G} and include, for instance, cyclic graphs such as



Even though the rather simple communication reductions shown above – the *basic cross reductions* – seem to cover in practice most of the expressiveness needs of concurrent programming, the normalisation of the proof-systems on which the discussed concurrent λ-calculi are based induces much more general forms of communications. In order to obtain analytic proof-terms, we need also to be able to transmit open processes that have bonds with their original environment. We need thus, more importantly, to be able to restore the required dependencies after the communication. The corresponding computational problem is often called the problem of the *transmission of closures*, see for example [6], and is very well known in the context of *code mobility*, which is the field of study precisely concerned with the issues related to the transmission of functions between programs. Fortunately, our proof systems do not only require very general reductions, but also provides a solution to the problems arising from them. This solution is realized in the presented λ-calculi as the *full cross reduction* rules, which implement the required term communication and establish a new communication channel on the fly in order to handle the dependencies, or *closure*, of the transmitted term.

We prove a general normalization result for the introduced calculi, we show that they are strictly more expressive than simply typed λ-calculus and discuss their computational features.

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Preservation theorems in graded model theory*

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Graded model theory is the generalized study, in mathematical fuzzy logic (MFL), of the construction and classification of graded structures. The field was properly started in [8] and it has received renewed attention in recent years [1,3–7]. Part of the programme of graded model theory is to find non-classical analogues of results from classical model theory (e.g., [2,9,10]). This will not only provide generalizations of classical theorems but will also provide insight into what avenues of research are particular to classical first-order logic and do not make sense in a broader setting.

On the other hand, classical model theory was developed together with the analysis of some very relevant mathematical structures. In consequence, its principal results provided a logical interpretation of such structures. Thus, if we want the model theory's idiosyncratic interaction with other disciplines to be preserved, the redefinition of the fundamental notions of graded model theory cannot be obtained from directly fuzzifying every classical concept. Quite the contrary, the experience acquired in the study of different structures, the results obtained using specific classes of structures, and the potential overlaps with other areas should determine the light the main concepts of graded model theory have to be defined in. It is in this way that several fundamental concepts of the model theory of mathematical fuzzy logic have already appeared in the literature.

The goal of this talk is to give syntactic characterizations of classes of graded structures; more precisely, we want to study which kind of formulas can be used to axiomatize certain classes of structures based on finite MTL-chains. Traditional examples of such sort of results are preservation theorems in classical model theory, which, in general, can be obtained as consequences of certain amalgamation properties (cf. [9]). We provide some amalgamation results using the technique of diagrams which will allow us to establish analogues of the Łoś–Tarski preservation theorem [9, Theorem 6.5.4] and the Chang–Łoś–Suszko theorem [9, Theorem 6.5.9].

The formalism of first-order fuzzy logics uses classical syntax with a signature $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ (predicate and functional symbols with their arities) and a many-valued semantics as in Mostowski–Rasiowa–Hájek tradition in which *models* are pairs $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where:

- $\boldsymbol{A}$ is an algebra of truth-values (for the propositional language)

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- $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where
  - M is a set
  - $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$
  - $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$.
- An $\mathfrak{M}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to M$

$$\begin{aligned}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} &= \mathrm{v}(x), \\
\|F(t_1, \ldots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|P(t_1, \ldots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|\circ(\varphi_1, \ldots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \ldots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}, \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}.
\end{aligned}$$

In this talk, we will assume that the algebra $\boldsymbol{A}$ of truth values is an MTL-algebra. MTL-algebras are algebraic structures of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \,\&^{\boldsymbol{A}}\, b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \qquad \text{(residuation)}$$

$$(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelinearity)}$$

$\boldsymbol{A}$ is called an MTL-chain if its underlying lattice is linearly ordered.

Let us fix a finite non-trivial MTL-chain $\boldsymbol{A}$. Finiteness ensures that the infima and suprema used in the interpretation of quantifiers always exist. We will consider the expansion of a signature, denoted $\mathcal{P}^{\boldsymbol{A}}$, in which we add a propositional constant $\overline{a}$ for each element a of $\boldsymbol{A}$. Also, we assume signatures to have crisp equality. We write $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \varphi[e]$ if $\varphi(x)$ has a free variable x and $\|\varphi\|^{\langle \boldsymbol{A}, \mathbf{M} \rangle}_{\mathrm{v}} = \overline{1}^{\boldsymbol{A}}$ for any evaluation v that maps x to e.

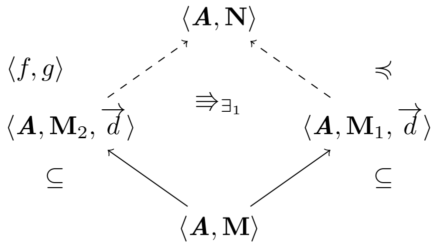
We will write $\langle \mathbf{A}, \mathbf{M}_2, \overrightarrow{d} \rangle \Rrightarrow_{\exists_n} \langle \mathbf{A}, \mathbf{M}_1, \overrightarrow{d} \rangle$ if for any $\exists_n$ formula φ, $\langle \mathbf{A}, \mathbf{M}_2 \rangle \models \varphi[\overrightarrow{d}]$ only if $\langle \mathbf{A}, \mathbf{M}_1 \rangle \models \varphi[\overrightarrow{d}]$. Also, we need to speak about embeddability of one model into another; see the usual definitions in e.g. [6].

In classical model theory amalgamation properties are often related in elegant ways to preservation theorems (see e.g. [9]). We will try an analogous approach to obtain our desired preservation result. The importance of this idea is that the problem of proving a preservation result reduces then to finding a suitable amalgamation counterpart. This provides us with proofs that have a neat common structure.

Proposition 1. (Existential amalgamation) *Let $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ be two structures for $\mathcal{P}^{\boldsymbol{A}}$ with a common part $\langle \boldsymbol{A}, \mathbf{M} \rangle$ with domain generated by a sequence of elements $\overrightarrow{d}$. Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d} \rangle \Rrightarrow_{\exists_1} \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d} \rangle.$$

Then there is a structure $\langle \boldsymbol{A}, \mathbf{N} \rangle$ *into which* $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ *can be strongly embedded by* $\langle f, g \rangle$ *while* $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *is* $\mathcal{P}^{\boldsymbol{A}}$*-elementarily strongly embedded (taking isomorphic copies, we may assume that* $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *is just a* $\mathcal{P}^{\boldsymbol{A}}$*-elementary substructure). The situation is described by the following picture:*

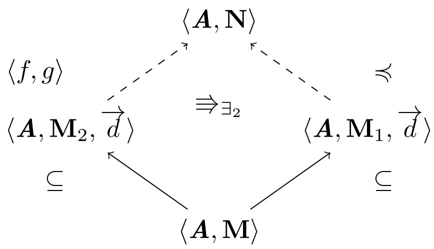


Moreover, the result is also true when $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *and* $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ *have no common part.*

Proposition 2. ($\exists_2$ amalgamation) *Let* $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *and* $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ *be two structures for* $\mathcal{P}^{\boldsymbol{A}}$ *with a common part* $\langle \boldsymbol{A}, \mathbf{M} \rangle$ *with domain generated by a sequence of elements* $\overrightarrow{d}$. *Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d} \rangle \Rrightarrow_{\exists_2} \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d} \rangle.$$

Then there is a structure $\langle \boldsymbol{A}, \mathbf{N} \rangle$ *into which* $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ *can be strongly embedded by* $\langle f, g \rangle$ *preserving all* $\forall_1$ *formulas, while* $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *is* $\mathcal{P}^{\boldsymbol{A}}$*-elementarily strongly embedded (taking isomorphic copies, we may assume that* $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *is just a* $\mathcal{P}^{\boldsymbol{A}}$*-elementary substructure). The situation is described by the following picture:*



Moreover, the result is also true when $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ *and* $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ *have no common part.*

These amalgamation properties let us prove analogues of the classical preservation theorems listed below. For their formulation we use the following notation: given a theory T and two sets of formulas Φ and Ψ, we denote by $T \vdash \Phi \Rightarrow \Psi$ the fact that for every model $\langle \boldsymbol{A}, \mathbf{M} \rangle$ of T, if $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is a model of Φ, then $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is also a model of Ψ.

Theorem 3. (Łoś–Tarski preservation theorem) *Let* T *be a theory and* $\Phi(\overrightarrow{x})$ *a set of formulas in* $\mathcal{P}^{\boldsymbol{A}}$*. Then the following are equivalent:*

(i) For any models of T, $\langle \boldsymbol{A}, \mathbf{M} \rangle \subseteq \langle \boldsymbol{A}, \mathbf{N} \rangle$*, we have:*
if $\langle \boldsymbol{A}, \mathbf{N} \rangle \models \Phi$*, then* $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \Phi$.

(ii) There is a set of $\forall_1$*-formulas* $\Theta(\overrightarrow{x})$ *such that:*
$T \vdash \Phi \Rightarrow \Theta$ *and* $T \vdash \Theta \Rightarrow \Phi$.

Theorem 4. *Let $\mathbb{K}$ be a class of structures. Then the following are equivalent:*

(i) *$\mathbb{K}$ is closed under isomorphisms, substructures, and ultraproducts.*

(ii) *$\mathbb{K}$ is axiomatized by a set of universal $\mathcal{P}^{\boldsymbol{A}}$-sentences.*

By a $\forall_2^*$-formula we will mean any formula which is either $\forall_2$ or of the form

$$(\exists \overrightarrow{x})(\forall \overrightarrow{y})\varphi(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) \rightarrow \overline{a},$$

for φ quantifier-free and a the immediate predecessor of $\overline{1}^{\boldsymbol{A}}$.

Theorem 5. (Chang–Łoś–Suszko preservation theorem) *Let T be a theory and $\Phi(\overrightarrow{x})$ a set of formulas in $\mathcal{P}^{\boldsymbol{A}}$. Then the following are equivalent:*

(i) *$\Phi(\overrightarrow{x})$ is preserved under unions of chains of models of T.*

(ii) *There is a set of $\forall_2^*$-formulas $\Theta(\overrightarrow{x})$ such that: $T \vdash \Phi \Rightarrow \Theta$ and $T \vdash \Theta \Rightarrow \Phi$.*

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A (Co)algebraic Approach to Hennessy-Milner Theorems for Weakly Expressive Logics

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1 Introduction

Coalgebraic modal logic, as in [9, 6], is a framework in which modal logics for specifying coalgebras can be developed parametric in the signature of the modal language and the coalgebra type functor T. Given a base logic (usually classical propositional logic), modalities are interpreted via so-called predicate liftings for the functor T. These are natural transformations that turn a predicate over the state space X into a predicate over TX. Given that T-coalgebras come with general notions of T-bisimilarity [11] and behavioral equivalence [7], coalgebraic modal logics are designed to respect those. In particular, if two states are behaviourally equivalent then they satisfy the same formulas. If the converse holds, then the logic is said to be expressive. and we have a generalisation of the classic Hennessy-Milner theorem [5] which states that over the class of image-finite Kripke models, two states are Kripke bisimilar if and only if they satisfy the same formulas in Hennessy-Milner logic.

General conditions for when an expressive coalgebraic modal logic for T-coalgebras exists have been identified in [10, 2, 12]. A condition that ensures that a coalgebraic logic is expressive is when the set of predicate liftings chosen to interpret the modalities is *separating* [10]. Informally, a collection of predicate liftings is separating if they are able to distinguish non-identical elements from TX. This line of research in coalgebraic modal logic has thus taken as starting point the semantic equivalence notion of behavioral equivalence (or T-bisimilarity), and provided results for how to obtain an expressive logic. However, for some applications, modal logics that are not expressive are of independent interest. Such an example is given by *contingency logic* (see e.g. [3, 8]). We can now turn the question of expressiveness around and ask, given a modal language, what is a suitable notion of semantic equivalence?

This abstract is a modest extension of [1] in which the first two authors proposed a notion of Λ-bisimulation which is parametric in a collection Λ of predicate liftings, and therefore tailored to the expressiveness of a given coalgebraic modal logic. The main result was a finitary Hennessy-Milner theorem (which does not assume Λ is separating): If T is finitary, then two states are Λ-bisimilar if and only if they satisfy the same modal Λ-formulas. The definition of Λ-bisimulation was formulated in terms of so-called Z-coherent pairs, where Z is the Λ-bisimulation relation. It was later observed by the third author that Λ-bisimulations can be characterised as the relations Z between T-coalgebras for which the dual relation (consisting of so-called Z-coherent pairs) is a congruence between the complex algebras. Here we collect those results.

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2 Syntax and semantics of coalgebraic modal logic.

Due to lack of space, we assume the reader is familiar with the basic theory of coalgebras and algebras for a functor, and with coalgebraic modal logic. Here we only introduce a few basic concepts and fix notation. We refer to [6, 11] for more details.

A *similarity type* Λ is a set of modal operators with finite arities. Given such a Λ, the set $\mathcal{L}_\Lambda$ of modal formulas is defined in the usual inductive manner.

We denote by Q the contravariant powerset functor on Set. A T-coalgebraic semantics of $\mathcal{L}_\Lambda$-formulas is given by providing a Λ-*structure* $(T, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$ where T is a functor on Set, and for each n-ary $\heartsuit \in \Lambda$, $\llbracket\heartsuit\rrbracket$ is an n-ary predicate lifting, i.e., $\llbracket\heartsuit\rrbracket : Q^n \Rightarrow QT$ is a natural transformation. Different choices of predicate liftings yield different Λ-structures and consequently different logics.

Given a Λ-structure $(T, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$, and a T-coalgebra $\mathbb{X} = (X, \gamma\colon X \to TX)$, the truth of $\mathcal{L}_\Lambda$-formulas in $\mathbb{X}$ is defined inductively in the usual manner for atoms (i.e., $\top$ and $\bot$) and Boolean connectives, and for modalities: $(\mathbb{X}, v), x \models \heartsuit(\varphi_1, \ldots, \varphi_n)$ iff $\gamma(x) \in \llbracket\heartsuit\rrbracket_X(\llbracket\varphi_1\rrbracket_\mathbb{X}, \ldots, \llbracket\varphi_n\rrbracket_\mathbb{X})$. (Atomic propositions can be included in the usual way via a valuation.)

In the remainder, we let T be a fixed but arbitrary endofunctor on the category Set of sets and functions, and $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ are T-coalgebras. We write $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$, if $\mathbb{X}, x$ and $\mathbb{Y}, y$ satisfy the same $\mathcal{L}_\Lambda$-formulas,

3 Λ-bisimulations

Let $R \subseteq X \times Y$ be a relation with projections $\pi_l\colon R \to X$ and $\pi_r\colon R \to Y$, and let $U \subseteq X$ and $V \subseteq Y$. The pair (U, V) is *R-coherent* if $R[U] \subseteq V$ and $R^{-1}[V] \subseteq U$. One easily verifies that (U, V) is R-coherent iff (U, V) is in the pullback of $Q\pi_l$ and $Q\pi_r$.

Definition 3.1 (Λ-bisimulation)
A relation $Z \subseteq X \times Y$ is a Λ-bisimulation *between $\mathbb{X}$ and $\mathbb{Y}$, if whenever $(x, y) \in Z$, then for all $\heartsuit \in \Lambda$, n-ary, and all Z-coherent pairs $(U_1, V_1), \ldots, (U_n, V_n)$, we have that*

$$\gamma(x) \in \llbracket\heartsuit\rrbracket_X(U_1, \ldots, U_n) \quad \textit{iff} \quad \delta(y) \in \llbracket\heartsuit\rrbracket_Y(V_1, \ldots, V_n). \qquad (\textbf{Coherence})$$

We write $\mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$, if there is a Λ-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ that contains (x, y). A Λ-bisimulation on a T-coalgebra $\mathbb{X}$ is a Λ-bisimulation between $\mathbb{X}$ and $\mathbb{X}$.

We have the following basic properties.

Lemma 3.2

1. *The set of Λ-bisimulations between two T-coalgebras forms a complete lattice.*
2. *On a single T-coalgebra, the largest Λ-bisimulation is an equivalence relation.*
3. *Λ-bisimulations are closed under converse, but not composition.*

The following proposition compares Λ-bisimulations with the coalgebraic notions of T-bisimulations [11] and the weaker notion of precocongruences [4]. Briefly stated, a relation is a precocongruence of its pushout is a behavioural equivalence [7]).

Proposition 3.3 *Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be T-coalgebras, and Z be a relation between X and Y.*

1. *If Z is a T-bisimulation then Z is a Λ-bisimulation.*

2. *If Z is a precocongruence then Z is a Λ-bisimulation.*
3. *If Λ is separating then Z is a Λ-bisimulation iff Z is a precocongruence.*

It was shown in [4, Proposition 3.10] that, in general, T-bisimilarity implies precocongruence equivalence which in turn implies behavioural equivalence [7]. This fact together with Proposition 3.3 tells us that Λ-bisimilarity implies behavioural equivalence, whenever Λ is separating. Moreover, it is well known [11] that if T preserves weak pulbacks, then T-bisimilarity coincides with behavioural equivalence. Hence in this case, by Proposition 3.3, it follows that Λ-bisimilarity coincides with T-bisimilarity and behavioural equivalence.

The main result in [1] is the following.

Theorem 3.4 (Finitary Hennessy-Milner theorem) *If T is a finitary functor, then*

1. *For all states $x, x' \in X$:* $\mathbb{X}, x \equiv_\Lambda \mathbb{X}, x'$ *iff* $\mathbb{X}, x \sim_\Lambda \mathbb{X}, x.'$
2. *For all $x \in X$ and $y \in Y$:* $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$ *iff* $\mathbb{X} + \mathbb{Y}, \mathsf{in}_l(x) \sim_\Lambda \mathbb{X} + \mathbb{Y}, \mathsf{in}_r(y)$.

where $\mathsf{in}_l, \mathsf{in}_r$ are the injections into the coproduct/disjoint union.

4 Λ-Bisimulations as duals of congruences

We now use the fact that the contravariant powerset functor Q can be viewed as one part of the duality between Set and CABA, the category of complete atomic Boolean algebras and their homomorphisms. By duality, Q turns a pushout in Set into a pullback in CABA. So given a relation $Z \subseteq X \times Y$ with projections π_l, π_r (forming a span in Set), and letting (P, p_l, p_r) be its pushout, we have that $(QP, Qp_l, Qp_r) \cong (pb(Q\pi_l, Q\pi_r), Q\pi_l, Q\pi_r)$.

In the context of coalgebraic modal logic, we define complex algebras as follows. This definition coincides with the classic one.

Definition 4.1 (Complex algebras)

- *Let $L\colon \mathsf{CABA} \to \mathsf{CABA}$ be the functor $L(A) = \coprod_{\heartsuit \in \Lambda} A^{ar(\heartsuit)}$, and let $\sigma\colon LQ \Longrightarrow QT$ be the bundling up of $[\![\Lambda]\!]$ into one natural transformation. For example, if Λ consists of one unary modality and one binary modality, then $L(A) = A + A^2$ and $\sigma_X : QX + (QX)^2 \Longrightarrow QTX$.*
- *The complex algebra of $\mathbb{X} = (X, \gamma\colon X \to TX)$ is the L-algebra $\mathbb{X}^* = (QX, \gamma^*)$ where $\gamma^* = LQX \xrightarrow{\sigma_X} QTX \xrightarrow{Q\gamma} QX$.*

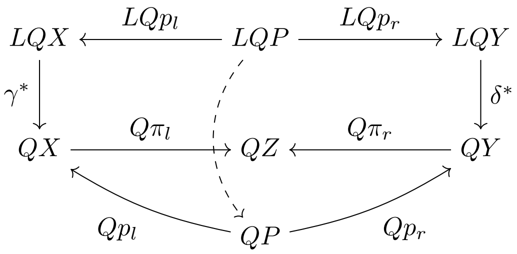
We can now reformulate the definition of Λ-bisimilarity in terms of the complex algebras associated with the coalgebras (by using $(QP, Qp_l, Qp_r) \cong (pb(Q\pi_l, Q\pi_r), Q\pi_l, Q\pi_r)$).

Lemma 4.2 *Z is Λ-bisimulation if and only if the following diagram commutes:*

$$\begin{array}{ccccc} LQX & \xleftarrow{LQp_l} & LQP & \xrightarrow{LQp_r} & LQY \\ \downarrow{\scriptstyle \gamma^*} & & & & \downarrow{\scriptstyle \delta^*} \\ QX & \xrightarrow{Q\pi_l} & QZ & \xleftarrow{Q\pi_r} & QY \end{array}$$

Proposition 4.3 *Z is Λ-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ iff the dual of its pushout is a congruence between the complex algebras $\mathbb{X}^*$ and $\mathbb{Y}^*$ (i.e. a span in the category of L-algebras and L-algebra homomorphisms).*

Proof. ($\Rightarrow$) Since (QP, Qp_l, Qp_r) is a pullback of $(QZ, Q\pi_l, Q\pi_r)$, we get a map $h : LQP \to QP$ such that (QP, Qp_l, Qp_r) is a congruence:



($\Leftarrow$) Follows from commutativity of pullback square. □

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Multi-Agent Topological Evidence Logics*

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1 Introduction

In [BBÖS16] a topological semantics for evidence-based belief and knowledge is introduced, where epistemic sentences are built in a language $\mathcal{L}_{\forall KB\Box\Box_0}$, which includes modalities allowing us to talk about defeasible knowledge (K), infallible knowledge ($[\forall]$), belief (B), basic evidence ($\Box_0$) and combined evidence ($\Box$).

Definition 1 (The dense interior semantics)**.** Sentences of $\mathcal{L}_{\forall KB\Box\Box_0}$ are read on *topological evidence models* (topo-e-models), which are tuples (X, τ, E_0, V) where (X, τ) is a topological space, E_0 is a subbasis of τ and $V : \mathsf{Prop} \to 2^X$ is a valuation.

The semantics of a formula ϕ is as follows: $\|p\| = V(p)$; $\|\phi\wedge\psi\| = \|\phi\|\cap\|\psi\|$; $\|\neg\phi\| = X\setminus\|\phi\|$; $\|\Box\phi\| = \mathrm{Int}\,\|\phi\|$; $x \in \|K\phi\|$ iff $x \in \mathrm{Int}\,\|\phi\|$ and $\mathrm{Int}\,\|\phi\|$ is dense[1]; $x \in \|B\phi\|$ iff $\mathrm{Int}\,\|\phi\|$ is dense; $x \in \|[\forall]\phi\|$ iff $\|\phi\| = X$; $x \in \|\Box_0\phi\|$ iff there is $e \in E_0$ with $x \in e \subseteq \|\phi\|$; $x \in \|\Box\phi\|$ iff $x \in \mathrm{Int}\,\|\phi\|$.

Crucially, using topological spaces to model epistemic sentences grants us an *evidential* perspective of knowledge and belief. Indeed, we can see the opens in the topology as the pieces of evidence the agent has (and thus our modality $\Box$, which encodes "having evidence", becomes the topological interior operator). For some proposition ϕ to constitute (defeasible) knowledge, we demand that the agent has a factive *justification* for ϕ, i.e. a piece of evidence that cannot be contradicted by any other evidence the agent has. In topological terms, a justification amounts to a *dense* piece of evidence. Having a (not necessarily factive) justification constitues belief. The set X encodes all the possible worlds which are consistent with the agent's information, thus for the agent to know ϕ infallibly ($[\forall]\phi$), ϕ needs to hold throughout X.

The fragment of this language that only contains the Booleans and the K modality, $\mathcal{L}_K$, has $\mathsf{S4.2}$ as its logic.

The framework introduced in [BBÖS16] is single-agent. A multi-agent generalisation is presented in this text, along with some "generic models" and a notion of group knowledge. Our proposal differs conceptually from previous multi-agent approaches to the dense interior semantics [Ö17, Ram15].

2 Going Multi-Agent

For clarity of presentation we work in a two-agent system.[2] Our language now contains modalities $K_i, B_i, [\forall]_i, \Box_i, \Box_i^0$ for $i = 1, 2$, each encoding the same notion as in the single-agent system.

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*This paper compiles the results contained in Chapters 3 to 5 of Saúl Fernández González's Master's thesis [FG18]. The authors wish to thank Guram Bezhanishvili for his input.

[1] A set $U \subseteq X$ is dense whenever $\mathrm{Cl}\,U = X$, or equivalently when it has nonempty intersection with every nonempty open set.

[2] Extending these results to $n \geq 2$ agents is straightforward, see [FG18, Section 6.1].

The Problem of Density. The first issue one comes across when defining a multi-agent semantics is that of accounting for the notion of defeasibility, which, as we have seen, is closely tied to density. A first (naive) approach would be to consider two topologies and a valuation defined on a common space, (X, τ_1, τ_2, V) and simply have: $x \in \|K_i\phi\|$ iff there exists some τ_i-dense open set such that $x \in U \subseteq \|\phi\|$. This does not work, neither conceptually (for we are assuming that the set of worlds compatible with each agent's information is the same for both agents) nor logically (adopting this semantics gives us highly undesirable theorems such as $\neg K_1\neg K_1 p \to K_2\neg K_1\neg K_1 p$). Seeing as each agent's knowledge is an $\mathsf{S4.2}$ modality and no interaction between the agents is being assumed, one would expect the two-agent logic to simply combine the $\mathsf{S4.2}$ axioms for each of the agents.

Simply defining two topologies on the whole space is not the right move. Instead, we want to make explicit, at each world $x \in X$, which subsets of worlds in X are compatible with each agent's information. A straightforward way to do this is via the use of partitions.

Topological-partitional models.

Definition 2. A *topological-partitional model* is a tuple $(X, \tau_1, \tau_2, \Pi_1, \Pi_2, V)$ where X is a set, τ_1 and τ_2 are topologies defined on X, Π_1 and Π_2 are partitions and V is a valuation.

For $U \subseteq X$ we write $\Pi_i[U] := \{\pi \in \Pi_i : U \cap \pi \neq \varnothing\}$. For $i = 1, 2$ and $\pi \in \Pi_i[U]$ we say U is *i-locally dense in π* whenever $U \cap \pi$ is dense in the subspace topology $(\pi, \tau_i|_\pi)$; we simply say U is *i-locally dense* if it is locally dense in every $\pi \in \Pi_i[U]$.

For the remainder of this text, we limit ourselves to the fragment of the language including the K_1 and K_2 modalities.

Definition 3 (Semantics)**.** We read $x \in \|K_i\phi\|$ iff there exists an i-locally dense τ_i-open set U with $x \in U \subseteq \|\phi\|$.

This definition generalises one-agent models, appears to hold water conceptually and, moreover, gives us the logic one would expectedly extrapolate from the one-agent case.

Lemma 4. *If $(X, \leq_1, \leq_2)$ is a birelational frame where each $\leq_i$ is reflexive, transitive and weakly directed (i.e. $x \leq_i y, z$ implies there exists some $t \geq_i y, z$), then the collection τ_i of $\leq_i$-upsets and the set Π_i of $\leq_i$-connected components give us a topological-partitional model $(X, \tau_1, \tau_2, \Pi_1, \Pi_2)$ in which the semantics of Def. 2 and the Kripke semantics coincide.*

Now, the Kripke logic of such frames is the fusion $\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$, i.e. the least normal modal logic containing the $\mathsf{S4.2}$ axioms for each K_i. As an immediate consequence:

Corollary 5. *$\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$ is the $\mathcal{L}_{K_1K_2}$-logic of topological-partitional models.*

3 Generic Models

[FG18] is partially concerned with finding *generic models* for topological evidence logics, i.e. single topological spaces whose logic (relative to a certain fragment $\mathcal{L}$) is precisely the sound and complete $\mathcal{L}$-logic of topo-e-models. Let us showcase two examples of two-agent generic models for the $\mathcal{L}_{K_1K_2}$ fragment. These are particular topological-partitional models whose logic is precisely $\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$.

The Quaternary Tree $\mathcal{T}_{2,2}$. The *quaternary tree* $\mathcal{T}_{2,2}$ is the full infinite tree with two relations R_1 and R_2 where every node has exactly four successors: a left R_i-successor and a right R_i-successor for $i = 1, 2$. Let $\leq_i$ be the reflexive and transitive closure of R_i.

We can, as we did before, define two topologies τ_i and two partitions Π_i on $\mathcal{T}_{2,2}$ in a very natural way, namely by taking, respectively, the set of $\leq_i$-upsets and the set of $\leq_i$-connected components. And we get:

Theorem 6. *$\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$ is sound and complete with respect to $(\mathcal{T}_{2,2}, \tau_{1,2}, \Pi_{1,2})$.*

The completeness proof uses the fact that $\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$ is complete with respect to finite rooted birelational Kripke frames in which both relations are reflexive, transitive and weakly directed, plus the fact proven in [vBBtCS06] that, given a preordered birelational finite frame W, there is an onto map $f : \mathcal{T}_{2,2} \to W$ which is continuous and open in both topologies.

The result then follows immediately from:

Lemma 7. *Given an $\mathsf{S4.2} + \mathsf{S4.2}$ frame W and a map f as described above, plus a valuation V on W, we have that $W, V, fx \vDash \phi$ under the Kripke semantics if and only if $\mathcal{T}_{2,2}, V^f, x \vDash \phi$ under the semantics of Def. 2, where $V^f(p) = \{x \in \mathcal{T}_{2,2} : fx \in V(p)\}$.*

Proof sketch. The proof of this lemma is an induction on formulas. The right to left direction for the induction step corresponding to K_i uses the fact that, if U is a connected i-upset in W with $fx \in U$, then $U' = \{z : z \geq_i y \text{ for some } y \in [x]_{\Pi_i} \text{ with } fy \in U\}$ is an i-locally dense open set in $\mathcal{T}_{2,2}$ with $x \in U \subseteq [x]_{\Pi_i}$. ■

The rational plane $\mathbb{Q} \times \mathbb{Q}$. We can define two topologies on $\mathbb{Q}$ by "lifting" the open sets in the rational line horizontally or vertically. Formally, the *horizontal topology* τ_H is the topology generated by $\{U \times \{y\} : U \text{ is open}, y \in \mathbb{Q}\}$. Similarly, the *vertical topology* τ_V is generated by the sets $\{y\} \times U$. We have the following result:

Proposition 8. *There exist partitions Π_H and Π_V such that $(\mathbb{Q} \times \mathbb{Q}, \tau_{H,V}, \Pi_{H,V})$ is a topological-partitional model whose logic is $\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_2}$.*

Proof sketch. It is shown in [vBBtCS06] that there exists a surjective map $g : \mathbb{Q} \times \mathbb{Q} \to \mathcal{T}_{2,2}$ which is open and continuous in both topologies. Given such a map and a valuation V on $\mathcal{T}_{2,2}$, we can define a valuation V^g on $\mathbb{Q} \times \mathbb{Q}$ as above and two equivalence relations: $x \sim_H y$ iff $[gx]_{\Pi_1} = [gy]_{\Pi_1}$, and $x \sim_V y$ iff $[gx]_{\Pi_2} = [gy]_{\Pi_2}$. As we did before, we can prove that $(\mathbb{Q} \times \mathbb{Q}, \tau_H, \tau_V, \Pi_H, \Pi_V), V^g, x \vDash \phi$ iff $\mathcal{T}_{2,2}, V, gx \vDash \phi$, whence completeness follows. ■

4 Distributed Knowledge

Once a multi-agent framework is defined, the obvious next step is to account for some notion of *knowledge of the group.* We will focus on *distributed* or *implicit* knowledge, i.e., a modality that accounts for that which the group of agents knows implicitly, or what would become known if the agents were to share their information.

One way to do this is to follow the evidence-based spirit inherent to the dense interior semantics. On this account, we would code distributed knowledge as the knowledge modality which corresponds to a fictional agent who has all the pieces of evidence the agents have (we can code this via the *join* topology $\tau_1 \vee \tau_2$, which is the smallest topology containing τ_1 and τ_2), and only considers a world compatible with x when all agents in the group do (the partition of this agent being $\{\pi_1 \cap \pi_2 : \pi_i \in \Pi_i\}$). Coding distributed knowledge like this gives us some rather strange results: unlike more standard notions, it can obtain that an agent knows a proposition but, due to the density condition on this new topology, the group does not (for an example, see [FG18, Example 5.2.3]).

Our proposal differs from this. Here we follow [HM92] when they refer to this notion as "that which a fictitious 'wise man' (one who knows exactly which each individual agent knows) would know". Instead of conglomerating the evidence of all the agents, we account exclusively for what they know, and we treat this information as indefeasible. Thus, our account of distributed knowledge, which is not strictly evidence-based, interacts with the K_i modalities in a more standard way, much like in relational semantics.

Definition 9 (Semantics for distributed knowledge)**.** Our language includes the operators K_1, K_2 and an operator D for distributed knowledge. In a topological-partitional model $(X, \tau_1, \tau_2, \Pi_1, \Pi_2, V)$, we read $x \in \|D\phi\|$ iff for $i = 1, 2$ there exist i-locally dense sets $U_i \in \tau_i$ such that $x \in U_1 \cap U_2 \subseteq \|\phi\|$.

That is to say, ϕ constitutes distributed knowledge whenever the agents have indefeasible pieces of evidence which, when put together, entail ϕ.

As mentioned above, the logic of distributed knowledge is unsurprising:

Definition 10. $\mathsf{Logic}_{K_1K_2D}$ is the least set of formulas containing the $\mathsf{S4.2}$ axioms and rules for K_1 and K_2, the $\mathsf{S4}$ axioms and rules for D plus the axiom $K_i\phi \to D\phi$ for $i = 1, 2$.

Theorem 11. $\mathsf{Logic}_{K_1K_2D}$ *is sound and complete with respect to topological-partitional models.*

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The McKinsey-Tarski Theorem for Topological Evidence Models*

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1 Introduction

Epistemic logics (i.e. the family of modal logics concerned with that which an epistemic agent *believes* or *knows*) found a modelisation in [Hin62] in the form of Kripke frames. [Hin62] reasonably claims that the accessibility relation encoding knowledge must be minimally reflexive and transitive, which on the syntactic level translates to the corresponding logic of knowledge containing the axioms of S4. This, paired with the fact (proven by [MT44]) that S4 is the logic of topological spaces under the *interior semantics*, lays the ground for a topological treatment of knowledge. Moreover, treating the K modality as the topological interior operator, and the open sets as "pieces of evidence" adds an evidential dimension to the notion of knowledge that one cannot get within the framework of Kripke frames.

Reading epistemic sentences using the interior semantics might be too simplistic: it equates "knowing" and "having evidence", plus attempts to bring a notion of belief into this framework have not been very felicitous.

Following the precepts of [Sta06], a logic that allows us to talk about knowledge, belief and the relation thereof, about evidence (both basic and combined) and justification is introduced in [BBÖS16]. This is the framework of *topological evidence models* and this paper builds on it.

1.1 The Interior Semantics: the McKinsey-Tarski Theorem

Let Prop be a countable set of propositional variables and let us consider a modal language $\mathcal{L}_\Box$ defined as follows: $\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi$, with $p \in \mathsf{Prop}$.

A *topological model* is a topological space (X, τ) along with a valuation $V : \mathsf{Prop} \to 2^X$. The semantics of a formula ϕ is defined recursively as follows: $\|p\| = V(p)$; $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$, $\|\neg\phi\| = X \backslash \|\phi\|$, $\|\Box\phi\| = \mathrm{Int}\,\|\phi\|$.

Theorem 1 ([MT44])**.** *The logic of topological spaces under the interior semantics is* S4*.*

As mentioned above, reading epistemic sentences via the interior semantics has some issues For details, see Section 1.2 of [FG18], and Chapters 3 and 4 of [Ö17]. A new semantics devoid of these issues is proposed in [BBÖS16]: the *dense interior semantics*.

1.2 The Dense Interior Semantics

Our language is now $\mathcal{L}_{\forall KB\Box\Box_0}$, which includes the modalities K (knowledge), B (belief), $[\forall]$ (infallible knowledge), $\Box_0$ (basic evidence), $\Box$ (combined evidence).

*This paper compiles the results contained in the first two chapters of Saúl Fernández González's Master's thesis [FG18]. The authors wish to thank Guram Bezhanishvili for his input.

Definition 2 (The dense interior semantics)**.** We read sentences on *topological evidence models* (i.e. tuples (X, τ, E_0, V) where (X, τ, V) is a topological model and E_0 is a designated subbasis) as follows: $x \in \|K\phi\|$ iff $x \in \mathrm{Int}\,\|\phi\|$ and $\mathrm{Int}\,\|\phi\|$ is *dense*[1]; $x \in \|B\phi\|$ iff $\mathrm{Int}\,\|\phi\|$ is dense; $x \in \|[\forall]\phi\|$ iff $\|\phi\| = X$; $x \in \|\Box_0\phi\|$ iff there is $e \in E_0$ with $x \in e \subseteq \|\phi\|$; $x \in \|\Box\phi\|$ iff $x \in \mathrm{Int}\,\|\phi\|$. Validity is defined in the standard way.

Fragments of the logic. The following logics are obtained by considering certain fragments of the language (i.e. certain subsets of the modalities above).

"K-only", $\mathcal{L}_K$	S4.2.
"Knowledge", $\mathcal{L}_{\forall K}$	S5 axioms and rules for $[\forall]$, plus S4.2 for K, plus axioms $[\forall]\phi \to K\phi$ and $\neg[\forall]\neg K\phi \to [\forall]\neg K\neg\phi$.
"Combined evidence", $\mathcal{L}_{\forall\Box}$	S5 for $[\forall]$, S4 for $\Box$, plus $[\forall]\phi \to \Box\phi$.
"Evidence", $\mathcal{L}_{\forall\Box\Box_0}$	S5 for $[\forall]$, S4 for $\Box$, plus the axioms $\Box_0\phi \to \Box_0\Box_0\phi$, $[\forall]\phi \to \Box_0\phi$, $\Box_0\phi \to \Box\phi$, $(\Box_0\phi \wedge [\forall]\psi) \to \Box_0(\phi \wedge [\forall]\psi)$.

K and B are definable in the evidence fragments, thus we can think of the logic of $\mathcal{L}_{\forall\Box\Box_0}$ as the "full logic".

2 Generic Models

McKinsey and Tarski also proved the following:

Theorem 3 ([MT44])**.** *The logic of a single dense-in-itself metrisable space*[2] *under the interior semantics is* S4*.*

Within the framework of the interior semantics, this tells us that there exist "natural" spaces, such as the real line, which are "generic" enough to capture the logic of the whole class of topological spaces. The main aim of this paper is to translate this idea to the framework of topological evidence models, i.e., finding topo-e-models which are "generic". Formally:

Definition 4 (Generic models)**.** Let $\mathcal{L}$ be a language and (X, τ) a topological space. We will say that (X, τ) is a *generic model for* $\mathcal{L}$ if the sound and complete $\mathcal{L}$-logic over the class of all topological evidence models is sound and complete with respect to the family

$$\{(X, \tau, E_0) : E_0 \text{ is a subbasis of } \tau\}.$$

If $\Box_0$ is not in the language, then a generic model is simply a topological space for which the corresponding $\mathcal{L}$-logic is sound and complete.

2.1 The K-only Fragment

Recall that the logic of the "K-only" fragment of our language is S4.2. The following is our main result:

Theorem 5. S4.2 *is the logic of any d-i-i metrisable space under the dense interior semantics.*

[1] A set $U \subseteq X$ is dense whenever $\mathrm{Cl}\,U = X$.

[2] A space is *dense-in-itself* (d-i-i) if it has no open singletons and *metrisable* if there is a metric d generating the topology. The real line $\mathbb{R}$, the rational line $\mathbb{Q}$, and the Cantor space are examples of d-i-i metrisable spaces.

Proof sketch. Let (X, τ) be such a space. The proof of completeness relies on the following:
Lemma. S4.2 is sound and complete with respect to finite cofinal rooted preorders. Each of these can be written as a disjoint union $W = A \cup B$, where B is a finite rooted preorder and A is a final cluster (i.e. $x \leq y$ for all $x \in W, y \in A$).
Partition lemma [BBLBvM18]. Any d-i-i metrisable space admits a partition $\{G, U_1, ..., U_n\}$, where G is a d-i-i subspace with dense complement and each U_i is open, for every $n \geq 1$.
Theorem [BBLBvM18]. Given a rooted preorder B, and a d-i-i metrisable space G, there exists a continuous, open and surjective map $f : G \to B$.

Now, let $W = A \cup B$ be a finite cofinal rooted preorder, with $A = \{a_1, ..., a_n\}$ its final cluster. We partition X into $\{G, U_1, ..., U_n\}$ as per the partition lemma and we extend the open, continuous and surjective map $f : G \to B$ to a map $\bar{f} : X \to W$ by mapping each $x \in U_i$ to a_i. We can see that under $\bar{f}$: (i) the image of a dense open set is an upset ($\bar{f}$ is *dense-open*); (ii) the preimage of an upset is a dense open set ($\bar{f}$ is *dense-continuous*).

Moreover, we have:
Lemma. Given a dense-open and dense-continous onto map $\bar{f} : X \to W$, and given a formula ϕ and a valuation V such that $W, V, \bar{f}x \nvDash \phi$ under the Kripke semantics, we have that $X, V^f, x \nvDash \phi$ under the dense interior semantics, where $V^f(p) = \{x \in X : \bar{f}x \in V(p)\}$.

Completeness follows. ■

Corollary 6. *$\mathbb{R}$, $\mathbb{Q}$ and the Cantor space are generic models for the knowledge fragment $\mathcal{L}_K$.*

2.2 Universal Modality and the Logic of $\mathbb{Q}$

As a connected space, $\mathbb{R}$ is not a generic model for the fragments $\mathcal{L}_{\forall K}$, $\mathcal{L}_{\forall\Box}$ and $\mathcal{L}_{\forall\Box\Box_0}$. We can however see that there are d-i-i, metrisable yet disconnected spaces (such as $\mathbb{Q}$) which are generic models for these fragments.

Theorem 7. *$\mathbb{Q}$ is a generic model for $\mathcal{L}_{\forall K}$ and $\mathcal{L}_{\forall\Box}$.*

Proof sketch for $\mathcal{L}_{\forall K}$. We use: (i) the logic of the $\mathcal{L}_{\forall K}$ fragment is sound and complete with respect to finite cofinal preorders under the Kripke semantics; (ii) any finite cofinal preorder W is a p-morphic image via a dense-open dense-continuous p-morphism of a disjoint finite union of finite *rooted* cofinal preorders, $p : W_1 \uplus ... \uplus W_n \to W$.

Take $a_1 < ... < a_{n-1} \in \mathbb{R}\backslash\mathbb{Q}$ and let $A_1 = (-\infty, a_1)$, $A_n = (a_{n-1}, \infty)$ and $A_i = (a_{i-1}, a_i)$ for $1 < i < n$. We have that $\{A_1, ..., A_n\}$ partitions $\mathbb{Q}$ in n open sets each isomorphic to $\mathbb{Q}$. As per Theorem 5, there exists a dense-open dense-continuous onto map $f_i : A_i \to W_i$. By taking $f = f_1 \cup ... \cup f_n$ and composing it with p above we obtain a dense-open, dense-continuous onto map $\mathbb{Q} \to W$. Completeness then follows as in Theorem 5. ■

Theorem 8. *$\mathbb{Q}$ is a generic model for $\mathcal{L}_{\forall\Box\Box_0}$.*

Proof sketch. This proof uses the fact that the logic is complete with respect to quasi-models of the form $(X, \leq, E_0, V)$, where $\leq$ is a preorder and E_0 is a collection of $\leq$-upsets. Given a continuous, open and surjective map $f : \mathbb{Q} \to (X, \leq)$, we can define a valuation $V^f(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ and a subbasis of $\mathbb{Q}$, $E_0^f = \{e \subseteq \mathbb{Q} : f[e] \in E_0\}$ such that

$$(\mathbb{Q}, E_0^f, V^f), x \vDash \phi \text{ iff } (X, \leq, E_0, V), fx \vDash \phi,$$

whence the result follows. ■

Completeness with respect to a single topo-e-model. The logic of the fragment $\mathcal{L}_{\forall\Box\Box_0}$ is sound and complete with respect to the class of topo-e-models based on $\mathbb{Q}$ with arbitrary subbases. Could we get completeness with respect to a designated subbasis? An obvious candidate would be perhaps the most paradigmatic case of subbasis-which-isn't-a-basis, namely $\mathcal{S} = \{(-\infty, a), (b, \infty) : a, b \in \mathbb{Q}\}$. As it turns out, the logic is not complete with respect to the class of topo-e-models based on $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S})$. Let $\mathsf{Prop} = \{p_1, p_2, p_3\}$ and consider the formula

$$\gamma = \bigwedge_{i=1,2,3} (\Box_0 p_i \wedge \neg[\forall]\neg\Box_0\neg p_i) \bigwedge_{i\neq j\in\{1,2,3\}} \neg[\forall]\neg(\Box_0 p_i \wedge \neg\Box_0 p_j).$$

as it turns out, γ is consistent in the logic yet $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S}) \vDash \neg\gamma$.

Generalising the results. We finish by outlining a class of generic models for all the fragments we are working with. The only part in the previous proofs that makes $\mathbb{Q}$ a generic model for these fragments but not other d-i-i metrisable spaces like $\mathbb{R}$ is the possibility to partition $\mathbb{Q}$ in n open sets which are homeomorphic to $\mathbb{Q}$ itself. A topological space can be partitioned in this way if and only if it is *idempotent*.

Definition 9. A topological space (X, τ) is *idempotent* if it is homeomorphic to the disjoint union $(X, \tau) \oplus (X, \tau)$.

And thus:

Theorem 10. *Any dense-in-itself, metrisable and idempotent space (such as $\mathbb{Q}$ or the Cantor space) is a generic model for the fragments $\mathcal{L}_K, \mathcal{L}_{KB}, \mathcal{L}_{\forall K}, \mathcal{L}_{\forall\Box}$ and $\mathcal{L}_{\forall\Box\Box_0}$.*

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Trees and Topological Semantics of Modal Logic

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1 Introduction

Topological semantics of modal logic has a long history. It was shown by McKinsey and Tarski [11] that if we interpret $\Box$ as interior and hence $\Diamond$ as closure, then S4 is the modal logic of all topological spaces. Many topological completeness results have been obtained since the inception of topological semantics. We list some relevant results: (1) S4 is the logic of any crowded metric space [11, 13] (this result is often referred to as the *McKinsey-Tarski theorem*); (2) Grz is the logic of any ordinal space $\alpha \geq \omega^\omega$ [1, 8]; (3) Grz_n (for nonzero $n \in \omega$) is the logic of any ordinal space α satisfying $\omega^{n-1} + 1 \leq \alpha \leq \omega^n$ [1] (see also [7, Sec. 6]); (4) S4.1 is the logic of the Pełczyński compactification of the discrete space ω (that is, the compactification of ω whose remainder is homeomorphic to the Cantor space) [6, Cor. 3.19]. If in (2) we restrict to a countable α, then all the above completeness results concern metric spaces. In fact, as was shown in [3], the above logics are the only logics arising from metric spaces.

The McKinsey-Tarski theorem yields that S4 is the logic of the Cantor space. An alternative proof of this result was given in [12] (see also [2]), where the infinite binary tree was utilized. Kremer [10] used the infinite binary tree with limits to prove that S4 is strongly complete for any crowded metric space. Further utility of trees with limits is demonstrated in [4].

Herein we summarize a general technique of topologizing trees which allows us to provide a uniform approach to topological completeness results for zero-dimensional Hausdorff spaces. It also allows us to obtain new topological completeness results with respect to non-metrizable spaces. Embedding these spaces into well-known extremally disconnected spaces (ED-spaces for short) then yields new completeness results for the logics above S4.2 indicated in Figure 1.

It was proved in [5] that S4.1.2 is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space ω, and this result was utilized in [6] to show that S4.2 is the logic of the Gleason cover of the real unit interval $[0, 1]$. However, these results require a set-theoretic axiom beyond ZFC, and it remains an open problem whether these results are true in ZFC. In contrast, all our results are obtained within ZFC.

We briefly outline some of the techniques employed to obtain the indicated completeness results. A unified way of obtaining a zero-dimensional topology on an infinite tree with limits, say T, is by designating a particular Boolean algebra of subsets of T as a basis. If T has countable branching, then the topology ends up being metrizable. If the branching is 1, then the obtained space is homeomorphic to the ordinal space $\omega + 1$; if the branching is ≥ 2 but

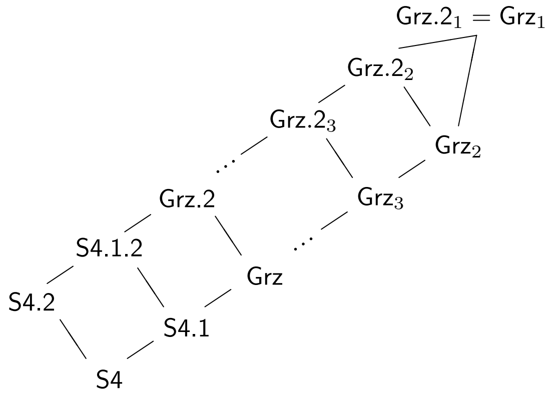

Figure 1: Some well-known extensions of $\mathsf{S4}$.

finite, then it is homeomorphic to the Pełczyński compactification of ω; and if the branching is countably infinite, then there are subspaces homeomorphic to the space of rational numbers, the Baire space, as well as to the ordinal spaces $\omega^n + 1$.

For uncountable branching, it is required to designate a Boolean σ-algebra as a basis for the topology. This leads to topological completeness results for $\mathsf{S4}, \mathsf{S4.1}, \mathsf{Grz}$, and Grz_n with respect to non-metrizable zero-dimensional Hausdorff spaces.

To obtain topological completeness results for logics extending $\mathsf{S4.2}$, we select a dense subspace of either the Čech-Stone compactification βD of a discrete space D with large cardinality or the Gleason cover E of a large enough power of $[0, 1]$. This selection is realized by embedding a subspace of an uncountable branching tree with limits into either βD or E. The latter gives rise to $\mathsf{S4.2}$, while the former yields the other logics of interest extending $\mathsf{S4.2}$. We point out that these constructions can be done in ZFC.

2 Topologizing trees and topological completeness results

Let κ be a nonzero cardinal. The *κ-ary tree with limits* is $\mathcal{T}_\kappa = (T_\kappa, \leq)$ where T_κ is the set of all sequences, both finite and infinite, in κ and $\leq$ is the initial segment partial ordering of T_κ. For any $\sigma \in T_\kappa$, let $\uparrow\sigma = \{\varsigma \in T_\kappa \mid \sigma \leq \varsigma\}$. The following table presents some topologies on T_κ; τ is a spectral topology, π is the patch topology of τ, and we introduce the *σ-patch* topology Π of τ.

Topology	Generated by
τ	the set $\mathscr{S} := \{\uparrow\sigma \mid \sigma \in T_\kappa$ is a finite sequence$\}$
π	the least Boolean algebra $\mathscr{B}$ containing $\mathscr{S}$
Π	the least Boolean σ-algebra $\mathscr{A}$ containing $\mathscr{S}$

2.1 The patch topology π

Here we are concerned with the space $\mathfrak{T}_\kappa := (T_\kappa, \pi)$ and its subspaces $\mathfrak{T}_\kappa^\infty$, $\mathfrak{T}_\kappa^\omega$, and $\mathfrak{T}_\kappa^n$ ($n \in \omega$) whose underlying sets are $T_\kappa^\infty = \{\sigma \in T_\kappa \mid \sigma$ is an infinite sequence$\}$, $T_\kappa^\omega = \{\sigma \in T_\kappa \mid$

σ is a finite sequence$\}$, and $T_\kappa^n = \{\sigma \in T_\kappa \mid \sigma$ is a finite sequence of length $n\}$, respectively. It ends up that $\mathfrak{T}_\kappa$ is metrizable iff κ is countable. The following table for $1 \le \kappa \le \omega$ indicates a subspace X of $\mathfrak{T}_\kappa$ and a well known space Y that are homeomorphic.

X	Y
$\mathfrak{T}_1$	the ordinal space $\omega + 1$
$\mathfrak{T}_\kappa^\infty$ $(2 \le \kappa < \omega)$	the Cantor discontinuum
$\mathfrak{T}_\kappa$ $(2 \le \kappa < \omega)$	the Pełczyński compactification of the countable discrete space ω
$\mathfrak{T}_\omega^\infty$	the Baire space
$\mathfrak{T}_\omega^\infty$	the space of irrational numbers
$\mathfrak{T}_\omega^\omega$	the space of rational numbers
$\mathfrak{T}_\omega^n$ $(n \in \omega)$	the ordinal space $\omega^n + 1$

Assuming familiarity with $\mathsf{S4}$, for $n \ge 1$, we recall the formulas $\mathsf{bd}_1 := \Diamond\Box p_1 \to p_1$ and $\mathsf{bd}_{n+1} := \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1}$ as well as the logics $\mathsf{S4.1} := \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p$, $\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$, and $\mathsf{Grz}_n := \mathsf{Grz} + \mathsf{bd}_n$. In the following table, the indicated subspace X of $\mathfrak{T}_\kappa$ satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X, giving the logic of X is L. In conjunction with the above table, this yields new proofs for many known topological completeness results.

L	X
$\mathsf{S4}$	$\mathfrak{T}_\omega^\omega$, $\mathfrak{T}_\omega^\infty$, and $\mathfrak{T}_\kappa^\infty$ $(2 \le \kappa < \omega)$
$\mathsf{S4.1}$	$\mathfrak{T}_\kappa$ $(2 \le \kappa < \omega)$
Grz	$\bigoplus_{n\in\omega} \mathfrak{T}_\omega^n$
Grz_{n+1}	$\mathfrak{T}_\omega^n$ $(n \in \omega)$

2.2 The σ-patch topology Π

We now fucus on the space $\mathbb{T}_\kappa := (T_\kappa, \Pi)$ and its subspaces $\mathbb{T}_\kappa^\infty$, $\mathbb{T}_\kappa^\omega$, and $\mathbb{T}_\kappa^n$ $(n \in \omega)$ whose underlying sets are T_κ^∞, T_κ^ω, and T_κ^n, respectively. It turns out that $\mathbb{T}_\kappa$ is a P-space; that is, a Tychonoff space such that every G_δ-set is open, and $\mathbb{T}_\kappa$ is discrete iff κ is countable. Thus, we consider only uncountable κ. In the following table, just as we had for the patch topology, the logic of the indicated subspace X of $\mathbb{T}_\kappa$ is L since X satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X. Hence, we obtain completeness for the same logics as in the previous section but for non-metrizable spaces.

L	X
$\mathsf{S4}$	$\mathbb{T}_\kappa^\omega$
$\mathsf{S4.1}$	$\mathbb{T}_\kappa$
Grz	$\bigoplus_{n\in\omega} \mathbb{T}_\kappa^n$
Grz_{n+1}	$\mathbb{T}_\kappa^n$ $(n \in \omega)$

2.3 Moving to the ED setting

Finally, we transfer these results into the setting of ED-spaces. By an unpublished result of van Douwen, see [9], the space $\mathbb{T}_\kappa^\omega$ embeds into the (remainder of the) Čech-Stone compactification

$\beta(2^\kappa)$ of the discrete space 2^κ. Consider $X_\kappa^\omega := \mathbb{T}_\kappa^\omega \cup 2^\kappa$ and $X_\kappa^n := \mathbb{T}_\kappa^n \cup 2^\kappa$ as subspaces of $\beta(2^\kappa)$ where we identify both $\mathbb{T}_\kappa^\omega$ and 2^κ with their image in $\beta(2^\kappa)$. Then X_κ^ω and X_κ^n are ED. Moreover, $\beta(2^\kappa)$, and hence $\mathbb{T}_\kappa^\omega$, can be embedded into a closed nowhere dense subspace F of the Gleason cover E of $[0,1]^{2^{2^\kappa}}$, where $[0,1]$ denotes the real unit interval. Identify $\mathbb{T}_\kappa^\omega$ with its image in E. Then the subspace $X_\kappa := \mathbb{T}_\kappa^\omega \cup (E \setminus F)$ is ED.

We recall that $\mathsf{S4.2}$, $\mathsf{S4.1.2}$, $\mathsf{Grz.2}$, and $\mathsf{Grz.2}_n$ are obtained respectively from $\mathsf{S4}$, $\mathsf{S4.1}$, Grz, and Grz_n by postulating the formula $\Diamond\Box p \to \Box\Diamond p$, which expresses that a space is ED. As previously, in the following table the space X satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X, giving the logic of X is L.

L	X
$\mathsf{S4.2}$	X_κ
$\mathsf{S4.1.2}$	X_κ^ω
$\mathsf{Grz.2}$	$\bigoplus_{n\in\omega} X_\kappa^n$
$\mathsf{Grz.2}_{n+2}$	X_κ^n $(n \in \omega)$

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Characterization of metrizable Esakia spaces via some forbidden configurations*

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Priestley duality [3, 4] provides a dual equivalence between the category Dist of bounded distributive lattices and the category Pries of Priestley spaces; and Esakia duality [1] provides a dual equivalence between the category Heyt of Heyting algebras and the category Esa of Esakia spaces. A *Priestley space* is a compact space X with a partial order $\leq$ such that $x \not\leq y$ implies there is a clopen upset U with $x \in U$ and $y \notin U$. An *Esakia space* is a Priestley space in which ${\downarrow}U$ is clopen for each clopen U.

The three spaces Z_1, Z_2, and Z_3 depicted in Figure 1 are probably the simplest examples of Priestley spaces that are not Esakia spaces. Topologically each of the three spaces is homeomorphic to the one-point compactification of the countable discrete space $\{y\} \cup \{z_n \mid n \in \omega\}$, with x being the limit point of $\{z_n \mid n \in \omega\}$. For each of the three spaces, it is straightforward to check that with the partial order whose Hasse diagram is depicted in Figure 1, the space is a Pristley space. On the other hand, neither of the three spaces is an Esakia space because $\{y\}$ is clopen, but ${\downarrow}y = \{x, y\}$ is no longer open.

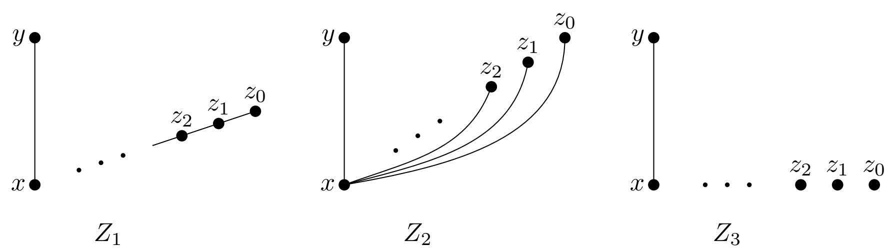

Figure 1: The three Priestley spaces Z_1, Z_2, and Z_3.

We show that a metrizable Priestley space is not an Esakia space exactly when one of these three spaces can be embedded in it. The embeddings we consider are special in that the point y plays a special role. We show that this condition on the embeddings, as well as the metrizability condition, cannot be dropped by presenting some counterexamples. An advantage of our characterization lies in the fact that when a metrizable Priestley space X is presented by a Hasse diagram, it is easy to verify whether or not X contains one of the three "forbidden configurations".

*An expanded version of this abstract, containing the proofs of all reported results, has been submitted for publication.

Definition 1. Let X be a Priestley space. We say that Z_i $(i = 1, 2, 3)$ is a *forbidden configuration* for X if there are a topological and order embedding $e : Z_i \to X$ and an open neighborhood U of $e(y)$ such that $e^{-1}(\downarrow U) = \{x, y\}$.

Theorem 2 (Main Theorem)**.** *A metrizable Priestley space X is not an Esakia space iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X.*

To give the dual statement of Theorem 2, let L_1, L_2, and L_3 be the dual lattices of Z_1, Z_2, and Z_3, respectively. They can be depicted as shown in Figure 2.

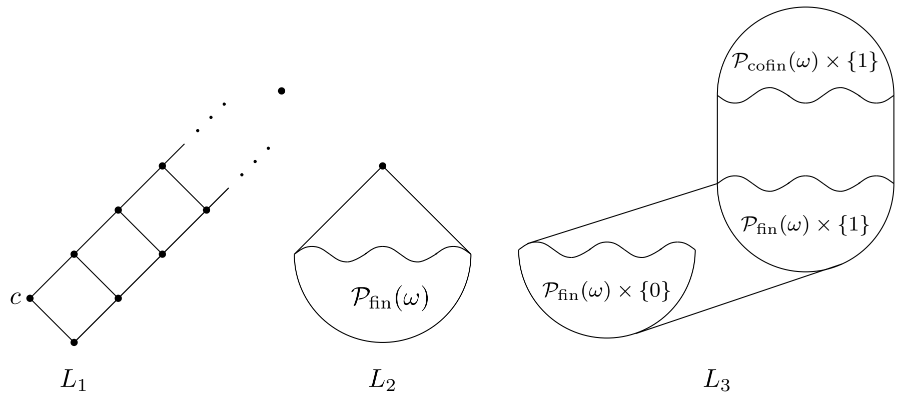

Figure 2: The lattices L_1, L_2 and L_3.

We have that L_2 is isomorphic to the lattice of finite subsets of ω together with a top element, and L_3 is isomorphic to the sublattice of $\mathbf{CF}(\omega) \times \mathbf{2}$ given by the elements of the form (A, n) where A is finite or $n = 1$. Here $\mathbf{CF}(\omega)$ is the Boolean algebra of finite and cofinite subsets of ω and $\mathbf{2}$ is the two-element Boolean algebra.

Neither of L_1, L_2, L_3 is a Heyting algebra: L_1 is not a Heyting algebra because $\neg c$ does not exist; L_2 is not a Heyting algebra because $\neg F$ does not exist for any finite subset F of ω; and L_3 is not a Heyting algebra because $\neg(F, 1)$ does not exist for any finite F.

Definition 3. Let $L \in \mathsf{Dist}$ and let $a, b \in L$. Define

$$I_{a \to b} := \{c \in L \mid c \wedge a \leq b\}$$

It is easy to check that $I_{a\to b}$ is an ideal, and that $I_{a\to b}$ is principal iff $a \to b$ exists in L, in which case $I_{a\to b} = \downarrow(a \to b)$.

Observe that if L is a bounded distributive lattice and X is the Priestley space of L, then X is metrizable iff L is countable. Thus, the following dual statement of Theorem 2 yields a characterization of countable Heyting algebras.

Theorem 4. *Let L be a countable bounded distributive lattice. Then L is not a Heyting algebra iff one of L_i $(i = 1, 2, 3)$ is a homomorphic image of L such that the homomorphism $h_i : L \to L_i$ satisfies the following property: There are $a, b \in L$ such that $h_i[I_{a\to b}] = I_{c_i \to 0}$, where $c_1 = c$, $c_2 = \{0\}$, or $c_3 = (\varnothing, 1)$.*

This characterization easily generalizes to countable p-algebras (=pseudocomplemented distributive lattices). Priestley duality for p-algebras was developed in [5]. We call a Priestley space X a *p-space* provided the downset of each clopen upset is clopen. Then a bounded distributive lattice L is a p-algebra iff its dual Priestley space X is a p-space.

Definition 5. Let X be a Priestley space. We say that Z_i ($i = 1, 2, 3$) is a *p-configuration* for X if Z_i is a forbidden configuration for X and in addition the open neighborhood U of $e(y)$ is an upset.

We point out that neither of the bounded distributive lattices L_1, L_2, L_3 that are dual to Z_1, Z_2, Z_3 is a p-algebra. The next result is a direct generalization of Theorems 2 and 4:

Corollary 6. *Let L be a countable bounded distributive lattice, and let X be its Priestley space, which is then a metrizable space.*

1. *X is not a p-space iff one of Z_1, Z_2, Z_3 is a p-configuration for X.*
2. *L is not a p-algebra iff one of L_i ($i = 1, 2, 3$) is a homomorphic image of L such that the homomorphism $h_i : L \to L_i$ satisfies the following property: There is $a \in L$ such that $h_i[I_{a\to 0}] = I_{c_i\to 0}$, where $c_1 = c$, $c_2 = \{0\}$, or $c_3 = (\varnothing, 1)$.*

We recall that *co-Heyting algebras* are order-duals of Heyting algebras. The Priestley spaces dual to co-Heyting algebras are the ones with the property that the upset of each clopen is clopen [2]. Let Z_1^*, Z_2^*, Z_3^* be the Priestley spaces obtained by reversing the order in Z_1, Z_2, Z_3, respectively. Then dualizing Theorem 2 yields:

Corollary 7. *A metrizable Priestley space X is not the dual of a co-Heyting algebra iff there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.*

We recall that *bi-Heyting algebras* are the lattices which are both Heyting algebras and co-Heyting algebras. Priestley spaces dual to bi-Heyting algebras are the ones in which the upset and downset of each clopen is clopen. Putting together the results for Heyting algebras and co-Heyting algebras yields:

Corollary 8. *A metrizable Priestley space X is not dual to a bi-Heyting algebra iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X or there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.*

We conclude by two examples showing that Theorem 2 is false without the metrizability assumption, and that in Definition 1 the condition on the open neighborhood U of $e(y)$ cannot be dropped.

Example 9. Let ω_1 be the first uncountable ordinal, and let X be the poset obtained by taking the dual order of $\omega_1 + 1$. Endow X with the interval topology. Consider the space Z given by the disjoint union of a singleton space $\{y\}$ and X with the partial order as depicted in Figure 3. Since $\downarrow\{y\}$ is not clopen, Z is not an Esakia space. On the other hand, there is no sequence in $X \setminus \{\omega_1\}$ converging to ω_1. Thus, Z does not contain the three forbidden configurations.

Example 10. Let X be the disjoint union of two copies of the one-point compactification of the discrete space ω, and let the order on X be defined as in Figure 4. It is straightforward to check that X is a metrizable Esakia space, and yet there is a topological and order embedding of Z_1 into X, described by the white dots in the figure.

Analogous examples can be found for all three forbidden configurations.

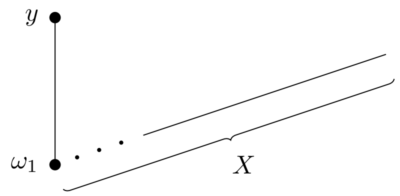

Figure 3: The space Z of Example 9.

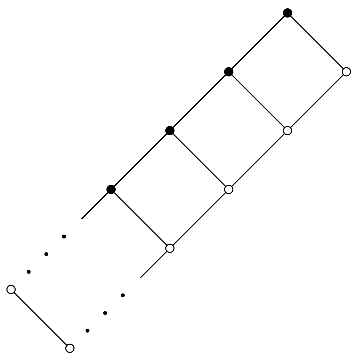

Figure 4: The space X of Example 10. The white dots represent the image of Z_1 under the embedding of Z_1 into X.

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The One-Variable Fragment of Corsi Logic

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It is well-known that the one-variable fragments of first-order classical logic and intuitionistic logic can be understood as notational variants of the modal logic S5 and the intuitionistic modal logic MIPC, respectively. Similarly, the one-variable fragment of first-order Gödel logic may be viewed as a notational variant of the many-valued Gödel modal logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, axiomatized in [4] as an extension of MIPC with the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ and the constant domains axiom $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$. Further results and general methods for establishing correspondences between one-variable fragments of first-order intermediate logics and intermediate modal logics have been obtained in, e.g., [7, 1].

In this work, we establish such a correspondence for a weaker extension of propositional Gödel logic: the first-order logic of totally ordered intuitionistic Kripke models with increasing domains QLC, axiomatized by Corsi in [5] as an extension of first-order intuitionistic logic with the prelinearity axiom, and often referred to as "Corsi logic". We show that its one-variable fragment QLC_1 corresponds both to the Gödel modal logic $\mathsf{S5}(\mathbf{G})$, axiomatized in [4] as an extension of MIPC with the prelinearity axiom, and also to a one-variable fragment of a "Scott logic" studied in, e.g., [6]. Since $\mathsf{S5}(\mathbf{G})$ enjoys an algebraic finite model property (see [1]), validity in both this logic and QLC_1 are decidable, and indeed — as can be shown using methods from [3] — co-NP-complete.

Let us first recall the Kripke semantics for Corsi logic, restricted for convenience to its one-variable fragment. A QLC_1-*model* is a 4-tuple $\mathcal{M} = \langle W, \preceq, D, I\rangle$ such that

- W is a non-empty set;
- $\preceq$ is a total order on W;
- for all $w \in W$, D_w is a non-empty set called the *domain* of w, and $D_w \subseteq D_v$ whenever $w \preceq v$;
- for all $w \in W$, I_w maps each unary predicate P to some $I_w(P) \subseteq D_w$, and $I_w(P) \subseteq I_v(P)$ whenever $w \preceq v$.

We define inductively for $w \in W$ and $a \in D_w$:

$$
\begin{array}{lcl}
\mathcal{M}, w \models^a \bot & \Leftrightarrow & \text{never} \\
\mathcal{M}, w \models^a \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, w \models^a P(x) & \Leftrightarrow & a \in I_w(P) \\
\mathcal{M}, w \models^a \varphi \wedge \psi & \Leftrightarrow & \mathcal{M}, w \models^a \varphi \text{ and } \mathcal{M}, w \models^a \psi \\
\mathcal{M}, w \models^a \varphi \vee \psi & \Leftrightarrow & \mathcal{M}, w \models^a \varphi \text{ or } \mathcal{M}, w \models^a \psi \\
\mathcal{M}, w \models^a \varphi \to \psi & \Leftrightarrow & \mathcal{M}, v \models^a \varphi \text{ implies } \mathcal{M}, v \models^a \psi \text{ for all } v \succeq w \\
\mathcal{M}, w \models^a (\forall x)\varphi & \Leftrightarrow & \mathcal{M}, v \models^b \varphi \text{ for all } v \succeq w \text{ and } b \in D_v \\
\mathcal{M}, w \models^a (\exists x)\varphi & \Leftrightarrow & \mathcal{M}, w \models^b \varphi \text{ for some } b \in D_w.
\end{array}
$$

We write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models^a \varphi$ for all $w \in W$, and $a \in D_w$. We say that a one-variable first-order formula φ is QLC_1-*valid* if $\mathcal{M} \models \varphi$ for all QLC_1-models $\mathcal{M}$. As mentioned above, it follows from results of Corsi [5] that φ is QLC_1-valid if and only if it is derivable in first-order intuitionistic logic extended with the prelinearity axiom.

The semantics for the modal logic $\mathsf{S5}(\mathbf{G})$ is defined for a set of formulas Fm built as usual over the language of intuitionistic logic extended with $\Box$ and $\Diamond$ and a countably infinite set of variables Var, where $\mathbf{G}$ denotes the standard Gödel algebra $\langle [0,1], \wedge, \vee, \to, 0, 1\rangle$. An $\mathsf{S5}(\mathbf{G})$-*model* $\mathfrak{M} = \langle W, R, V\rangle$ consists of a non-empty set of *worlds* W, a $[0,1]$-*accessibility relation* $R\colon W \times W \to [0,1]$ satisfying for all $u, v, w \in W$,

$$Rww = 1, \quad Rwv = Rvw, \quad \text{and} \quad Ruv \wedge Rvw \le Ruw,$$

and a *valuation* map $V\colon \mathrm{Var} \times W \to [0,1]$. The valuation map is extended to $V\colon \mathrm{Fm} \times W \to [0,1]$ by $V(\bot, w) = 0$, $V(\top, w) = 1$, $V(\varphi_1 \star \varphi_2, w) = V(\varphi_1, w) \star V(\varphi_2, w)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$V(\Box\varphi, w) = \bigwedge\{Rwv \to V(\varphi, v) \mid v \in W\}$$
$$V(\Diamond\varphi, w) = \bigvee\{Rwv \wedge V(\varphi, v) \mid v \in W\}.$$

We say that $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-*valid* if $V(\varphi, w) = 1$ for all $\mathsf{S5}(\mathbf{G})$-models $\langle W, R, V\rangle$ and $w \in W$.

Let us make the correspondence between one-variable fragments and modal logics explicit, recalling the following standard translations $(-)^*$ and $(-)^\circ$ between the propositional language of $\mathsf{S5}(\mathbf{G})$ and the one-variable first-order language of QLC_1, assuming $\star \in \{\wedge, \vee, \to\}$:

$$\begin{aligned} \bot^* &= \bot & \bot^\circ &= \bot \\ \top^* &= \top & \top^\circ &= \top \\ (P(x))^* &= p & p^\circ &= P(x) \\ (\varphi \star \psi)^* &= \varphi^* \star \psi^* & (\varphi \star \psi)^\circ &= \varphi^\circ \star \psi^\circ \\ ((\forall x)\varphi)^* &= \Box\varphi^* & (\Box\varphi)^\circ &= (\forall x)\varphi^\circ \\ ((\exists x)\varphi)^* &= \Diamond\varphi^* & (\Diamond\varphi)^\circ &= (\exists x)\varphi^\circ. \end{aligned}$$

Note that the composition of $(-)^\circ$ and $(-)^*$ is the identity map. Therefore to show that $\mathsf{S5}(\mathbf{G})$ corresponds to the one-variable fragment of QLC, it suffices to show that $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if φ° is QLC_1-valid. It is easily shown that the translations under $(-)^\circ$ of the axioms and rules of the axiomatization of $\mathsf{S5}(\mathbf{G})$ given in [4] are QLC_1-valid and preserve QLC_1-validity, respectively. Hence if φ is $\mathsf{S5}(\mathbf{G})$-valid, then φ° is QLC_1-valid. To prove the converse, we proceed contrapositively and show that if $\varphi \in \mathrm{Fm}$ fails in some $\mathsf{S5}(\mathbf{G})$-model, then φ° fails in some QLC_1-model.

Let us say that an $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ is *irrational* if $V(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}$ and $w \in W$. We first prove the following useful lemma.

Lemma 1. *For any countable* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M} = \langle W, R, V\rangle$*, there exists an irrational* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M}' = \langle W, R', V'\rangle$ *such that* $V(\varphi, w) < V(\psi, w)$ *if and only if* $V'(\varphi, w) < V'(\psi, w)$ *for all* $\varphi, \psi \in \mathrm{Fm}$ *and* $w \in W$.

Next we consider any irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ and fix $w_0 \in W$. We let $(0,1)_{\mathbb{Q}}$ denote $(0,1) \cap \mathbb{Q}$ and define a corresponding one-variable Corsi model

$$\mathcal{M}_\circ = \langle (0,1)_{\mathbb{Q}}, \ge, D, I\rangle$$

such that for all $\alpha \in (0,1)_{\mathbb{Q}}$,

- $D_\alpha = \{v \in W \mid Rw_0v \geq \alpha\}$;
- $I_\alpha(P) = \{v \in W \mid V(p,v) \geq \alpha\} \cap D_\alpha$ for each unary predicate P.

We are then able to prove the following lemma by induction on the complexity of $\varphi \in \mathrm{Fm}$. The fact that $\mathcal{M}$ is irrational ensures that $V(\varphi,w) \geq \alpha$ if and only if $V(\varphi,w) > \alpha$ for all $\alpha \in (0,1)_\mathbb{Q}$, which is particularly important when considering the case for $\varphi = \Diamond\psi$.

Lemma 2. *For any $\varphi \in \mathrm{Fm}$, $\alpha \in (0,1)_\mathbb{Q}$, and $w \in D_\alpha$,*

$$\mathcal{M}_\circ, \alpha \models^w \varphi^\circ \iff V(\varphi,w) \geq \alpha.$$

Hence, if $\varphi \in \mathrm{Fm}$ is not $\mathsf{S5}(\mathbf{G})$-valid, there exists, by Lemma 1, an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$ such that $V(\varphi,w) < \alpha < 1$ for some $\alpha \in (0,1)$, and then, by Lemma 2, a QLC_1-model $\mathcal{M}_\circ = \langle (0,1)_\mathbb{Q}, \geq, D, I\rangle$ such that $\mathcal{M}_\circ, \alpha \not\models^w \varphi^\circ$. That is, φ° is not QLC_1-valid, and we obtain the following result.

Theorem 1. *A formula $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if φ° is QLC_1-valid.*

We have also established a correspondence between $\mathsf{S5}(\mathbf{G})$ and the one-variable fragment of a "Scott logic" studied in, e.g., [6], that is closely related to the semantics of a many-valued possibilistic logic defined in [2]. Let us call a SL_1-*model* a triple $\mathcal{M} = \langle D, \pi, I\rangle$ such that

- D is a non-empty set;
- $\pi\colon D \to [0,1]$ is a map satisfying $\pi(a) = 1$ for some $a \in D$;
- for each unary predicate P, $I(P)$ is a map assigning to any $a \in D$ some $I_a(P) \in [0,1]$.

The interpretation I_a is extended to formulas by the clauses $I_a(\bot) = 0$, $I_a(\top) = 1$, $I_a(\varphi \star \psi) = I_a(\varphi) \star I_a(\psi)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$I_a((\forall x)\varphi) = \bigwedge\{\pi(b) \to I_b(\varphi) \mid b \in D\}$$
$$I_a((\exists x)\varphi) = \bigvee\{\pi(b) \wedge I_b(\varphi) \mid b \in D\}.$$

We say that a one-variable first-order formula φ is SL_1-*valid* if $I_a(\varphi) = 1$ for all SL_1-models $\langle D, \pi, I\rangle$ and $a \in D$. Using Theorem 1 and a result from [6] relating Scott logics to first-order logics of totally ordered intuitionistic Kripke models, we obtain the following correspondence

Theorem 2. *A formula $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if $(\Box\varphi)^\circ$ is SL_1-valid.*

Let us mention finally that $\mathsf{S5}(\mathbf{G})$ enjoys an algebraic finite model property (see [1]), and hence validity in this logic and QLC_1 are decidable. Moreover, using a version of the non-standard semantics developed in [3] to obtain a polynomial bound on the size of the algebras to be checked, we are able to obtain the following sharpened result.

Theorem 3. *The validity problem for $\mathsf{S5}(\mathbf{G})$ is co-NP-complete.*

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Universal Objects for Orders on Groups, and their Dual Spaces

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Following on from the success of scheme theory in algebraic geometry, Keimel in 1971 introduced in his doctoral dissertation [10] a notion of spectral space associated to Abelian lattice-ordered groups (cf. [2, Chapter 10]). For an Abelian lattice-ordered group H, the ℓ-spectrum is defined as the set of its prime ℓ-ideals with the spectral topology. The notion of ℓ-spectrum is not limited to the commutative setting, and can also be defined for an arbitrary lattice-ordered group (see, e.g., [6] and [7, Chapter 9]).

A *lattice-ordered group* (briefly, *ℓ-group*) H is a group with a lattice structure compatible with the group operation, i.e., the group operation distributes over the lattice operations. We call an ℓ-group *representable* if it is a subdirect product of chains, and *Abelian* if its underlying group is Abelian. The *ℓ-spectrum* $\operatorname{Spec} H$ of an ℓ-group H is the root system of all its prime convex ℓ-subgroups ordered by inclusion. Here, a *convex ℓ-subgroup* of H is an order-convex sublattice subgroup of H, while an *ℓ-ideal* is a convex ℓ-subgroup which is also normal, i.e., closed under conjugation. For any convex ℓ-subgroup K, the quotient H/K is lattice-ordered by: $\mathrm{K}x \leqslant_{H/\mathrm{K}} \mathrm{K}y$ if, and only if, there exists $t \in \mathrm{K}$ such that $x \leqslant ty$. A convex ℓ-subgroup P is *prime* when the quotient H/P is non-trivial and totally ordered. A prime convex ℓ-subgroup of H is *minimal* if it is inclusion-minimal in $\operatorname{Spec} H$. By an application of Zorn's Lemma, any prime convex ℓ-subgroup of H contains a minimal prime convex ℓ-subgroup. We write $\operatorname{Min} H$ for the set of minimal prime convex ℓ-subgroups of H. Given an ℓ-group H, we consider the following *hull-kernel* (or *spectral*, or *Stone*, or *Zariski*) topology on $\operatorname{Spec} H$. The basic open sets are

$$\mathbb{S}_x = \{\mathrm{P} \in \operatorname{Spec} H \mid x \notin \mathrm{P}\}, \quad \text{for } x \in H$$

and we refer to $\mathbb{S}_x$ as the *support* of $x \in H$. We also endow $\operatorname{Min} H$ with the subspace topology. It can be proved that $\operatorname{Min} H$ is Hausdorff [7, Proposition 49.8].

We adopt the standard notation $x \perp y$—read 'x and y are orthogonal'—to denote $|x| \wedge |y| = e$ for $x, y \in H$, where $|x| = x \vee x^{-1}$ is the *absolute value* of x, and 'e' is the group identity. For $T \subseteq H$, set

$$T^{\perp} = \{x \in H \mid x \perp y \text{ for all } y \in T\},$$

and call those subsets *polars*. We write $\operatorname{Pol} H$ for the Boolean algebra of polars of H, and $\operatorname{Pol_p} H$ for its sublattice of principal polars, namely those of the form $\{x\}^{\perp\perp}$ for some $x \in H$.

The spectral space of an ℓ-group H provides—in the case in which H is representable—a tool for employing sheaf-theoretic methods in the study of ℓ-groups.

Example. A representable ℓ-group H can be embedded into a Hausdorff sheaf of ℓ-groups on the Stone space associated with the complete Boolean algebra $\operatorname{Pol} H$ of polars [7, Proposition 49.21].

Furthermore, topological properties of the ℓ-spectrum $\operatorname{Spec} H$ can have important consequences on the structure of the ℓ-group H.

Example. The space $\operatorname{Min} H$ is compact if, and only if, $\operatorname{Pol_p} H$ is a Boolean algebra [6].[1]

*Based on joint work with Vincenzo Marra (University of Milano, Italy).

[1]Further striking examples—although not relevant for the present abstract—are the following: $\operatorname{Spec} H$ is Hausdorff if, and only if, H is hyperarchimedean [6, 1.2]; $\operatorname{Spec} H$ is compact if, and only if, H has a strong order unit [6, 1.3].

In 2004, Sikora's paper 'Topology on the spaces of orderings of groups' [13] pioneered a different perspective on the study of the interplay between topology and ordered groups, that has led to applications to both orderable groups and algebraic topology (see, e.g., [3]). The basic construction in Sikora's paper is the definition of a topology on the set of right orders on a given right orderable group.

A binary relation R on a group G is *right-invariant* (resp. *left-invariant*) if for all $a, b, t \in G$, whenever aRb then $atRbt$ (resp. $taRtb$). We call a binary relation $\leqslant$ on a set S a (*total*) *order* if it is reflexive, transitive, antisymmetric, and total. A *right order on* G is just a right-invariant order on G, and G is *right orderable* if there exists a right order on G. A submonoid $C \subseteq G$ is a (*total*) *right cone for* G if $G = C \cup C^{-1}$ and $\{e\} = C \cap C^{-1}$. We set $\mathcal{R}(G)$ to be the set of right cones for G. It is elementary that $\mathcal{R}(G)$ is in bijection with the right orders on G via the map that associates to $C \in \mathcal{R}(G)$ the relation: $a \leqslant_C b$ if, and only if, $ba^{-1} \in C$. Hence, we refer to $\mathcal{R}(G)$ as 'the set of right orders on G'.

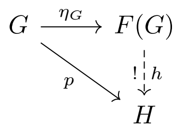
For a right orderable group G, Sikora endowed $\mathcal{R}(G)$ with the subspace topology inherited from the power set 2^G with the Tychonoff topology. A subbasis of clopens for $\mathcal{R}(G)$ is given by the sets

$$\mathbb{R}_a = \{C \in \mathcal{R}(G) \mid a \in C\}, \quad \text{for } a \in G.$$

The subspace $\mathcal{R}(G)$ can be proved to be closed in 2^G, and is therefore a compact totally disconnected Hausdorff space, i.e., it is a Stone space.

The theory of ℓ-groups and the theory of right orderable groups have been proved to be deeply related, and examples of this interdependence can be found almost everywhere in the literature of either field (see, e.g., [9, 11, 8, 4]). For this reason, the question whether a relation can be found between the topological space of right orders on a right orderable group G, and the ℓ-spectrum of some ℓ-group H arises naturally. In this work, we provide a positive answer to this question. In order to give a satisfying result that intrinsically relates the two topological spaces, we employ a fully general and natural construction, involving all the varieties of ℓ-groups. We focus here on the particular result, and only briefly sketch how the latter fits into the general framework.

For a group G, we write $F(G)$ for the free ℓ-group over G (as a group), and $\eta_G \colon G \to F(G)$ for the group homomorphism characterized by the following universal property: for each group homomorphism $p \colon G \to H$, with H an ℓ-group, there is exactly one ℓ-homomorphism $h \colon F(G) \to H$ such that $h \circ \eta_G = p$, i.e., such that the following diagram



commutes. As it turns out, $F(G)$ is the ℓ-group that we were looking for.

Theorem 1. *Given a right orderable group G, the space* $\mathcal{R}(G)$ *of right orders on G is homeomorphic to the minimal layer* $\operatorname{Min} F(G)$ *of the ℓ-spectrum* $\operatorname{Spec} F(G)$*. As a consequence, the lattice* $\operatorname{Pol_p} F(G)$ *of principal polars is a Boolean algebra and* $\mathcal{R}(G)$ *is its dual Stone space.*

We obtain Theorem 1 as a consequence of a result involving the whole of $\operatorname{Spec} F(G)$. For this, it is necessary to consider the broader notion of right pre-order. A binary relation $\preccurlyeq$ on a set S is a (*total*) *pre-order* if it is reflexive, transitive, and total. A *right pre-order on* G is a right-invariant pre-order on G. A submonoid $C \subset G$ is a (*total*) *right pre-cone for* G if $G = C \cup C^{-1}$. We set $\mathcal{P}(G)$ to be the root system of total right pre-cones for G ordered by inclusion. It is again elementary that $\mathcal{P}(G)$ is in bijection with the right pre-orders on G via the map that associates to $C \in \mathcal{P}(G)$ the relation: $a \preccurlyeq_C b$ if, and only if, $ba^{-1} \in C$. Note that if the group G is right orderable, then Sikora's $\mathcal{R}(G)$ is a subset of $\mathcal{P}(G)$. More precisely, if $\mathcal{R}(G) \neq \emptyset$, it can be proved to coincide with the minimal layer of $\mathcal{P}(G)$.

For any group G, we set

$$\mathbb{P}_a = \{C \in \mathcal{P}(G) \mid a \in C \text{ and } a \notin C^{-1}\}, \quad \text{for } a \in G$$

and endow $\mathcal{P}(G)$ with the smallest topology for which all sets $\mathbb{P}_a$ are open. Thus, any open set in this topology is the union of sets of the form $\mathbb{P}_{a_1} \cap \ldots \cap \mathbb{P}_{a_n}$. If G is right orderable, the subspace topology on $\mathcal{R}(G)$ amounts to Sikora's topology, and hence, the minimal layer of $\mathcal{P}(G)$ is compact.

Observe that there is a natural way to obtain an ℓ-group from a right pre-cone C for G. In fact, the set $G_{\gtrless} = C \cap C^{-1}$ is a subgroup of G, and the quotient $G/G_{\gtrless}$ can be totally ordered by: $[a] \leqslant [b]$ if, and only if, $a \preccurlyeq_C b$; if Ω_C is the resulting chain, we can consider the ℓ-subgroup H_C of $Aut(\Omega_C)$ generated by the image of G through the group homomorphism $\pi_C \colon G \to Aut(\Omega_C)$ defined by

$$a \mapsto (\pi_C(a)[b] = [ba]).$$

Note that the ℓ-group H_C allows us to exploit the universal property of $\eta_G \colon G \to F(G)$, inducing the existence of the surjective ℓ-group homomorphism h_C, making the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & F(G) \\ & {}_{\pi_C}\searrow & \big\downarrow{\scriptstyle !\, h_C} \\ & & H_C \end{array}$$

commute. We then make use of h_C to conclude the following result.

Proposition 1. *The topological spaces $\mathcal{P}(G)$ and* $\operatorname{Spec} F(G)$ *are homeomorphic.*

We would like to point out that a correspondence between the two root systems from Proposition 1 for the particular case in which G is a free group can essentially be found in [12].

The proof of Proposition 1 is self-contained, and only uses basic facts about (lattice-)ordered groups. An order-isomorphism between the underlying root systems is built explicitly, and then proved to be a homeomorphism between the corresponding topological spaces. Besides Theorem 1, a further consequence of Proposition 1 is an alternative proof of a fundamental representation theorem for free ℓ-groups, originally proved by Conrad [5]. As already remarked, Proposition 1 is an instance of a much more general result associating a family of right pre-orders to each variety V of ℓ-groups: given a group G, the right pre-orders on G associated with the variety V are exactly those for which $H_C \in \mathsf{V}$.

Without going into details, we conclude by stating a further consequence of the above-mentioned general result. Given a group G, we write $F(G)_{\mathsf{R}}$ for the free representable ℓ-group over the group G and $\eta_G \colon G \to F(G)_{\mathsf{R}}$ for the corresponding universal morphism (i.e., characterized by the universal property with respect to the variety R of representable ℓ-groups). An *order on G* is just a left-invariant right order on G, and G is *orderable* if there exists an order on G. As in the case of right orders, the set $\mathcal{O}(G)$ of orders on a group G can be identified with a set of subsets of G, namely those right cones that are closed under conjugation. Sikora's topological space on $\mathcal{O}(G)$ can then be defined as the smallest topology containing the sets $\{C \in \mathcal{O}(G) \mid a \in C\}$, for $a \in G$.

Theorem 2. *Given an orderable group G, the space $\mathcal{O}(G)$ of orders on G is homeomorphic to the minimal layer* $\operatorname{Min} F(G)_{\mathsf{R}}$ *of the ℓ-spectrum* $\operatorname{Spec} F(G)_{\mathsf{R}}$. *As a consequence, the lattice* $\operatorname{Pol_p} F(G)_{\mathsf{R}}$ *of principal polars is a Boolean algebra and $\mathcal{O}(G)$ is its dual Stone space.*

Theorem 2 is, similarly to Theorem 1, a consequence of the existence of a homeomorphism between the space of representable right pre-orders on G, namely those for which $H_C \in \mathsf{R}$, and the ℓ-spectrum $\operatorname{Spec} F(G)_{\mathsf{R}}$. Its main significance lies in the fact that it provides a new perspective on some open problems in the theory of orderable groups (e.g., it is still unknown whether the equational theory of R is decidable; it is also unknown whether the space $\mathcal{O}(G)$, for G free of finite rank $n \geq 2$, is Cantor [1]).

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Theorems of Alternatives: An Application to Densifiability

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1 Introduction

A variety $\mathcal{V}$ of semilinear residuated lattices is called *densifiable* if it is generated as a quasivariety by its dense chains, or, equivalently, each chain in $\mathcal{V}$ embeds into a dense chain in $\mathcal{V}$ (see [1, 10, 6, 3]). Establishing that some variety is densifiable is a fundamental problem of mathematical fuzzy logic, corresponding to a key intermediate step in proving that a given axiom system is "standard complete": that is, complete with respect to a class of algebras with lattice reduct $[0, 1]$ (see, e.g., [7, 8, 2]).

Densifiability may be established using representation theorems or by providing explicit embeddings of countable chains into dense countable chains of the variety. The latter approach, introduced in [7], has been used to establish densifiability for varieties of integral semilinear residuated lattices, but can be difficult to apply in the non-integral setting. An alternative proof-theoretic method, used in [8, 2] to establish densifiability for a range of integral and non-integral varieties, circumvents the need to give explicit embeddings. Instead, the elimination of a certain density rule for a hypersequent calculus is used to prove that the variety satisfies a property that guarantees densifiability. Remarkably, this method has also been reinterpreted algebraically to obtain explicit embeddings of chains into dense chains [6, 1].

The methods described above are suitable for varieties of semilinear residuated lattices that admit either a useful representation theorem (e.g., via ordered groups) or an analytic hypersequent calculus. In this work, we introduce a method for establishing densifiability for varieties that may not satisfy either of these conditions, but admit instead a "theorem of alternatives" relating validity of equations in the variety to validity of equations in its residuated monoid reduct. Although the scope of this method is fairly narrow — applying so far only to varieties of involutive commutative semilinear residuated lattices — it yields both new and familiar (e.g., abelian ℓ-groups and odd Sugihara monoids) examples of densifiable varieties, and provides perhaps a first step towards addressing the open standard completeness problem for involutive uninorm logic posed in [8].

2 Theorems of Alternatives

Theorems of alternatives can be understood as duality principles stating that either one or another linear system has a solution over the real numbers, but not both (see, e.g., [5]). In particular, the following variant of Gordan's theorem (replacing real numbers with integers) states that

$$\text{for any } M \in \mathbb{Z}^{m\times n},\ \textit{either}\ \ y^T M > 0 \text{ for some } y \in \mathbb{Z}^m\ \ \textit{or}\ \ Mx = 0 \text{ for some } x \in \mathbb{N}^n\backslash\{0\}.$$

This theorem is established in [4] by considering partial orders on free abelian groups and reformulated as the following correspondence between validity in the variety $\mathcal{LA}$ of abelian ℓ-groups of inequations $0 \leq t_1 \vee \ldots \vee t_n$, where $t_1, \ldots, t_n$ are group terms, and equations in the variety $\mathcal{A}$ of abelian groups:

$$\mathcal{LA} \models 0 \leq t_1 \vee \ldots \vee t_n \iff \mathcal{A} \models 0 \approx \lambda_1 t_1 + \cdots + \lambda_n t_n \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ not all } 0.$$

This result may also be understood as generating a one-sided variant of the hypersequent calculus for abelian ℓ-groups introduced in [9].

In this work, we consider how far such theorems of alternatives can be extended to other classes of algebras and corresponding non-classical logics. Let $\mathcal{L}_g$ be the language of abelian groups with connectives $+, -, 0$, defining also $a \to b := -a + b$, $a \cdot b := -(-a + -b)$, $1 := -0$, and, inductively, for $n \in \mathbb{N}$, $0a := 0$, $a^0 := 1$, $(n+1)a = na + a$, and $a^{n+1} = a^n \cdot a$. We take as our starting point the following axiomatization of *Multiplicative Linear Logic* (MLL):

(B)	$(s \to t) \to ((t \to u) \to (s \to u))$	(1 L)	$s \to (1 \to s)$
(C)	$(s \to (t \to u)) \to (t \to (s \to u))$	(1 R)	1
(I)	$s \to s$	($-$L)	$-s \to (s \to 0)$
(INV)	$((s \to 0) \to 0) \to s$	($-$R)	$(s \to 0) \to -s$
($\cdot$ L)	$(s \to (t \to u)) \to ((s \cdot t) \to u)$	(+L)	$(s + t) \to -(-s \cdot -t)$
($\cdot$ R)	$s \to (t \to (s \cdot t))$	(+R)	$-(-s \cdot -t) \to (s + t)$

$$\frac{s \quad s \to t}{t}\ \text{(mp)}$$

We also define $\mathrm{MLL}_{0=1}$ to be the extension of MLL_0 with the axioms $0 \to 1$ and $1 \to 0$.

Algebraic semantics for MLL and its extensions are provided by *involutive commutative residuated pomonoids*: algebras $\langle A, +, -, 0, \leq \rangle$ satisfying (i) $\langle A, +, 0 \rangle$ is a commutative monoid, (ii) $-$ is an involution on $\langle A, \leq \rangle$, (iii) $\leq$ is a partial order on $\langle A, +, 0 \rangle$, and (iv) $a \cdot b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$. For any axiomatic extension L of MLL, let $\mathcal{V}_{\mathrm{L}}$ be the class of involutive commutative residuated pomonoids satisfying $1 \leq s$ whenever $\vdash_{\mathrm{L}} s$. Then for any set of $\mathcal{L}_g$-terms $\Sigma \cup \{s\}$,

$$\Sigma \vdash_{\mathrm{L}} s \iff \{1 \leq t \mid t \in \Sigma\} \models_{\mathcal{V}_{\mathrm{L}}} 1 \leq s.$$

Let L^ℓ be the logic defined over the language $\mathcal{L}$ with connectives $+, -, 0, \wedge, \vee$ obtained by extending the axiomatization of L with the following axiom schema and rule:

($\wedge$1)	$(s \wedge t) \to s$	($\vee$1)	$s \to (s \vee t)$
($\wedge$2)	$(s \wedge t) \to t$	($\vee$2)	$t \to (s \vee t)$
($\wedge$3)	$((s \to t) \wedge (s \to u)) \to (s \to (t \wedge u))$	($\vee$3)	$((s \to u) \wedge (t \to u)) \to ((s \vee t) \to u)$
(PRL)	$(s \to t) \vee (t \to s)$	(DIS)	$((s \wedge (t \vee u)) \to ((s \wedge t) \vee (s \wedge u))$

$$\frac{s \quad t}{s \wedge t}\ \text{(adj)}$$

In particular, MLL^ℓ is involutive uninorm logic IUL formulated without the constants $\bot$ and $\top$ (see [8]).

Let $\mathcal{V}_{\mathrm{L}}^\ell$ be the variety generated by the totally ordered members of $\mathcal{V}_{\mathrm{L}}$ equipped with the binary meet and join operations $\wedge$ and $\vee$. Then for any set of $\mathcal{L}$-terms $\Sigma \cup \{s\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} s \iff \{1 \leq t \mid t \in \Sigma\} \models_{\mathcal{V}_{\mathrm{L}}^\ell} 1 \leq s.$$

Note that if $\mathcal{V}_{\mathrm{L}}$ is axiomatized over the class of involutive commutative residuated pomonoids by a set of equations E, then $\mathcal{V}_{\mathrm{L}}^\ell$ is axiomatized by E over the variety of involutive commutative semilinear residuated lattices.

We say that an axiomatic extension L of MLL admits a *theorem of alternatives* if for any set of $\mathcal{L}_g$-terms $\Sigma \cup \{t_1, \ldots, t_n\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} t_1 \vee \ldots \vee t_n \iff \Sigma \vdash_{\mathrm{L}} \lambda_1 t_1 + \cdots + \lambda_n t_n \ \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ not all } 0.$$

This property can also be reformulated as a conservative extension property for L^ℓ over L.

Proposition 2.1. *An axiomatic extension* L *of* MLL *admits a theorem of alternatives if and only if* $\vdash_{\mathrm{L}^\ell} x \vee -x$ *and for any set of* $\mathcal{L}_g$*-terms* $\Sigma \cup \{t\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} t \iff \Sigma \vdash_{\mathrm{L}} \lambda t \text{ for some } \lambda \in \mathbb{N}\backslash\{0\}.$$

Note that the condition $\vdash_{\mathrm{L}^\ell} x \vee -x$ is immediate when L is an axiomatic extension of $\mathrm{MLL}_{0=1}$, and we will therefore assume this in what follows (even when more general results can be formulated).

The next result provides characterizations of logics admitting theorems of alternatives in terms of both consequences and valid formulas.

Theorem 2.2. *An axiomatic extension* L *of* $\mathrm{MLL}_{0=1}$ *admits a theorem of alternatives if and only if for all* $n \in \mathbb{N}\backslash\{0\}$,

$$\{nx,\, n(-x)\} \vdash_{\mathrm{L}} x^n + (-x)^n,$$

or, equivalently, if for all $n \in \mathbb{N}$*, there exist* $m \in \mathbb{N}\backslash\{0\}$*,* $k \in \mathbb{N}$ *such that* $\vdash_{\mathrm{L}} (nx)^k \to mx^n$.

In particular, any axiomatic extension L of the logic obtained by extending $\mathrm{MLL}_{0=1}$ with the axiom schema $nx \to x^n$ ($n \in \mathbb{N}\backslash\{0\}$) admits a theorem of alternatives. Moreover, the corresponding varieties $\mathcal{V}^\ell_{\mathrm{L}}$ of semilinear residuated lattices are exactly those axiomatized by group equations over the variety of involutive commutative semilinear residuated lattices satisfying $0 \approx 1$ and $nx \approx x^n$ ($n \in \mathbb{N}\backslash\{0\}$). These include the varieties of abelian ℓ-groups and odd Sugihara monoids.

3 Densifiability

We make use of the following lemma, originating in [8] (see also [2, 1, 10, 6, 3]).

Lemma 3.1. *A variety* $\mathcal{V}$ *of commutative semilinear residuated lattices is densifiable if and only if for any* $\mathcal{L}_g$*-terms* $s, t, u_1, \ldots, u_n$ *not containing the variable* x,

$$\mathcal{V} \models 1 \leq (s \to x) \vee (x \to t) \vee u_1 \vee \ldots \vee u_n \implies \mathcal{V} \models 1 \leq (s \to t) \vee u_1 \vee \ldots \vee u_n.$$

Consider any axiomatic extension L of $\mathrm{MLL}_{0=1}$ that admits a theorem of alternatives. Suppose that $\mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq (s \to x) \vee (x \to t) \vee u_1 \vee \ldots \vee u_n$ where $s, t, u_1, \ldots, u_n$ are $\mathcal{L}_g$-terms not containing the variable x. Since L admits a theorem of alternatives, there exist $\lambda, \mu, \gamma_1, \ldots, \gamma_n \in \mathbb{N}$ not all 0 such that

$$\mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq \lambda(s \to x) + \mu(x \to t) + \gamma_1 u_1 + \ldots + \gamma_n u_n.$$

Substituting, on the one hand x with 0, and on the other, all other variables with 0, yields

$$\mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq \lambda(-s) + \mu t + \gamma_1 u_1 + \ldots + \gamma_n u_n \quad \text{and} \quad \mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq \lambda x + \mu(-x).$$

Substituting x with s^λ in the second inequation and rewriting both inequations then yields

$$\mathcal{V}^\ell_{\mathrm{L}} \models \lambda(s^\lambda) \leq \lambda\mu t + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n \quad \text{and} \quad \mathcal{V}^\ell_{\mathrm{L}} \models s^{\lambda\mu} \leq \lambda(s^\lambda).$$

By transitivity, we obtan $\mathcal{V}^\ell_{\mathrm{L}} \models s^{\lambda\mu} \leq \lambda\mu t + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n$, which can be rewritten as

$$\mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq \lambda\mu(s \to t) + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n.$$

But then, by the theorem of alternatives,

$$\mathcal{V}^\ell_{\mathrm{L}} \models 1 \leq (s \to t) \vee u_1 \vee \ldots \vee u_n.$$

Hence, by the lemma, we obtain our main result.

Theorem 3.2. *Let* L *be any axiomatic extension of* $\mathrm{MLL}_{0=1}$ *that admits a theorem of alternatives. Then the variety* $\mathcal{V}^{\ell}_{\mathrm{L}}$ *is densifiable.*

In particular, any variety $\mathcal{V}^{\ell}_{\mathrm{L}}$ axiomatized by group equations over the variety of involutive commutative semilinear residuated lattices satisfying $0 \approx 1$ and $nx \approx x^n$ ($n \in \mathbb{N}\backslash\{0\}$) is densifiable, including (as is already well-known) the varieties of abelian ℓ-groups and odd Sugihara monoids.

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Multialgebraic First-Order Structures for **QmbC**

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Abstract

In this paper we present an adequate semantics for the first-order paraconsistent logic **QmbC**. That semantics is based on multialgebras known as *swap structures*.

1 The logic QmbC

The class of paraconsistent logics known as *Logics of Formal Inconsistency* (**LFI**s, for short) was introduced by W. Carnielli and J. Marcos in [3] and studied in [2] and [1]. **LFI**s are characterized for having a (primitive or derived) *consistency connective* $\circ$ which allows to recover the explosion law in a controlled way. The logic **mbC** is the weakest system in the hierarchy of **LFI**s and the system **QmbC** is the extension of **mbC** to first-order languages. The goal of this paper is to introduce an algebraic-like semantics for **QmbC** based on multialgebraic structures called *swap structures* (see [1]), which naturally induce non-deterministic matrices. The semantical framework for **QmbC** we present here can be seen as a generalization of the standard semantics approach for classical first-order logic, in which a logical matrix induced by a Boolean algebra is replaced by a non-deterministic matrix induced by a Boolean algebra.

The logic **mbC** (see [2, 1]) is defined over the propositional signature $\Sigma = \{\wedge, \vee, \to, \neg, \circ\}$ by adding to $\mathbf{CPL}^+$ (positive classical propositional logic) the following axiom schemas:

(Ax10) $\alpha \vee \neg\alpha$ and **(Ax11)** $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$.

Recall that a *multialgebra* (or *hyperalgebra*) over a signature Σ' is a pair $\mathcal{A} = \langle A, \sigma_{\mathcal{A}}\rangle$ such that A is a nonempty set (the *support* of $\mathcal{A}$) and $\sigma_{\mathcal{A}}$ is a mapping assigning to each n-ary $\#$ in Σ', a function (called *multioperation* or *hyperoperation*) $\#^{\mathcal{A}} : A^n \to (\mathcal{P}(A) - \{\emptyset\})$. In particular, $\emptyset \neq \#^{\mathcal{A}} \subseteq A$ if $\#$ is a constant in Σ'. A *non-deterministic matrix (or Nmatrix)* over Σ' is a pair $\mathcal{M} = \langle \mathcal{A}, D\rangle$ such that $\mathcal{A}$ is a multialgebra over Σ' with support A, and D is a subset of A. The elements in D are called *designated* elements.

Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a complete Boolean algebra, and $B_{\mathcal{A}} = \{z \in A^3 \ : \ z_1 \vee z_2 = 1 \text{ and } z_1 \wedge z_2 \wedge z_3 = 0\}$, where z_i denote the *ith*-projection of z. A swap structure for **mbC** over $\mathcal{A}$ is any multialgebra $\mathcal{B} = (B, \tilde{\wedge}, \tilde{\vee}, \tilde{\to}, \tilde{\neg}, \tilde{\circ})$ over Σ, such that $B \subseteq B_{\mathcal{A}}$ and, for every z and w in B:

(i) $\emptyset \neq z\tilde{\#}w \subseteq \{u \in B \ : \ u_1 = z_1 \# w_1\}$, for each $\# \in \{\wedge, \vee, \to\}$;
(ii) $\emptyset \neq \tilde{\neg}z \subseteq \{u \in B \ : \ u_1 = z_2\}$;
(iii) $\emptyset \neq \tilde{\circ}z \subseteq \{u \in B \ : \ u_1 = z_3\}$.

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The support B will be also denoted by $|\mathcal{B}|$. The *full swap structure for **mbC** over $\mathcal{A}$*, denoted by $\mathcal{B}_{\mathcal{A}}$, is the unique swap structure for **mbC** over $\mathcal{A}$ with domain $B_{\mathcal{A}}$, in which '$\subseteq$' is replaced by '$=$' in items (i)-(iii) above.

Definition 1.1 ([1], Definition 7.1.5)**.** Let Θ be a first-order signature. The logic **QmbC** over Θ is obtained from the Hilbert calculus **mbC** by adding the following axioms and rules:

(Ax12) $\varphi[x/t] \to \exists x\varphi$, if t is a term free for x in φ

(Ax13) $\forall x\varphi \to \varphi[x/t]$, if t is a term free for x in φ

(Ax14) $\alpha \to \beta$, whenever α is a variant[1] of β

($\exists$-In) $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$, where x does not occur free in ψ

($\forall$-In) $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$, where x does not occur free in φ

If Θ is a first-order signature then $For(\Theta)$, $Sen(\Theta)$ and $CTer(\Theta)$ will denote the set of formulas, closed formulas and closed terms over Θ. Var is the set of individual variables. The consequence relation of **QmbC** will be denoted by $\vdash_{\mathbf{QmbC}}$. If $\Gamma \cup \{\varphi\} \subseteq For(\Theta)$, then $\Gamma \vdash_{\mathbf{QmbC}} \varphi$ will denote that there exists a derivation in **QmbC** of φ from Γ.

2 First-Order Swap Structures: Soundness

Given a swap structure $\mathcal{B}$ for **mbC**, the non-deterministic matrix induced by $\mathcal{B}$ is $\mathcal{M}(\mathcal{B}) = (\mathcal{B}, D)$ such that $D = \{z \in |\mathcal{B}| \ : \ z_1 = 1\}$. The logic **mbC** is sound and complete w.r.t. swap structures semantics, see [1], Chapter 6.

A *(first-order) structure* over $\mathcal{M}(\mathcal{B})$ and Θ is a pair $\mathfrak{A} = \langle U, I_{\mathfrak{A}} \rangle$ such that U is a nonempty set (the domain of the structure) and $I_{\mathfrak{A}}$ is an interpretation mapping which assigns to each individual constant $c \in \mathcal{C}$, an element $c^{\mathfrak{A}}$ of U; to each function symbol f of arity n, a function $f^{\mathfrak{A}} : U^n \to U$; and to each predicate symbol P of arity n, a function $P^{\mathfrak{A}} : U^n \to |\mathcal{B}|$.

Let $\mathfrak{A}$ be a structure over $\mathcal{M}(\mathcal{B})$ and Θ. A function $\mu : Var \to U$ is called an *assignment* over $\mathfrak{A}$. Let $\mathfrak{A}$ be a structure and let $\mu : Var \to U$ be an assignment. For each term t, we define $||t||^{\mathfrak{A}}_{\mu} \in U$ such that: $||c||^{\mathfrak{A}}_{\mu} = c^{\mathfrak{A}}$ if c is an individual constant; $||x||^{\mathfrak{A}}_{\mu} = \mu(x)$ if x is a variable; $||f(t_1, \ldots, t_n)||^{\mathfrak{A}}_{\mu} = f^{\mathfrak{A}}(||t_1||^{\mathfrak{A}}_{\mu}, \ldots, ||t_n||^{\mathfrak{A}}_{\mu})$ if f is a function symbol of arity n and $t_1, \ldots, t_n$ are terms. If t is closed (i.e., has no variables) we will simply write $||t||^{\mathfrak{A}}$, since μ plays no role.

Given a structure $\mathfrak{A}$ over Θ, the signature Θ_U is obtained from $\mathfrak{A}$ by adding to Θ a new individual constant $\bar{a}$ for any $a \in U$. The expansion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ to Θ_U is defined by interpreting $\bar{a}$ as a. Any assignment $\mu : Var \to U$ induces a function $\widehat{\mu} : For(\Theta_U) \to Sen(\Theta_U)$ such that $\widehat{\mu}(\varphi)$ is the sentence obtained from φ by replacing any free variable x by the constant $\overline{\mu(x)}$.

Definition 2.1 (**QmbC**-valuations)**.** Let $\mathcal{M}(\mathcal{B}) = (\mathcal{B}, D)$ be the non-deterministic matrix induced by a swap structure $\mathcal{B}$ for **mbC**, and let $\mathfrak{A}$ be a structure over Θ and $\mathcal{M}(\mathcal{B})$. A mapping $v : Sen(\Theta_U) \to |\mathcal{B}|$ is a **QmbC**-*valuation* over $\mathfrak{A}$ and $\mathcal{M}(\mathcal{B})$, if it satisfies the following clauses:
(i) $v(P(t_1, \ldots, t_n)) = P^{\widehat{\mathfrak{A}}}(||t_1||^{\widehat{\mathfrak{A}}}, \ldots, ||t_n||^{\widehat{\mathfrak{A}}})$, if $P(t_1, \ldots, t_n)$ is atomic;
(ii) $v(\#\varphi) \in \tilde{\#} v(\varphi)$, for every $\# \in \{\neg, \circ\}$;

[1] That is, φ can be obtained from ψ by means of addition or deletion of void quantifiers, or by renaming bound variables (keeping the same free variables in the same places).

(iii) $v(\varphi\#\psi) \in v(\varphi)\tilde{\#}v(\psi)$, for every $\# \in \{\wedge, \vee, \to\}$;
(iv) $v(\forall x\varphi) \in \{z \in |\mathcal{B}| \ : \ z_1 = \bigwedge\{v(\varphi[x/\bar{a}]) \ : \ a \in U\}\}$;
(v) $v(\exists x\varphi) \in \{z \in |\mathcal{B}| \ : \ z_1 = \bigvee\{v(\varphi[x/\bar{a}]) \ : \ a \in U\}\}$;
(vi) Let t be free for z in φ and ψ, μ an assignment and $b = ||t||^{\widehat{\mathfrak{A}}}_{\mu}$. Then:
(vi.1) if $v(\widehat{\mu}(\varphi[z/t])) = v(\widehat{\mu}(\varphi[z/\bar{b}]))$, then $v(\widehat{\mu}(\#\varphi[z/t])) = v(\widehat{\mu}(\#\varphi[z/\bar{b}]))$, for $\# \in \{\neg, \circ\}$;
(vi.2) if $v(\widehat{\mu}(\varphi[z/t])) = v(\widehat{\mu}(\varphi[z/\bar{b}]))$ and $v(\widehat{\mu}(\psi[z/t])) = v(\widehat{\mu}(\psi[z/\bar{b}]))$, then $v(\widehat{\mu}(\varphi\#\psi[z/t])) = v(\widehat{\mu}(\varphi\#\psi[z/\bar{b}]))$, for $\# \in \{\wedge, \vee, \to\}$;
(vi.3) let x be such that $x \neq z$ and x does not occur in t, and let μ^x_a such that $\mu^x_a(y) = \mu(y)$, if $y \neq x$ and $\mu^x_a(y) = a$, if $y = x$. If $v(\widehat{\mu^x_a}(\varphi[z/t])) = v(\widehat{\mu^x_a}(\varphi[z/\bar{b}]))$, for every $a \in U$, then $v(\widehat{\mu}((Qx\varphi)[z/t])) = v(\widehat{\mu}((Qx\varphi)[z/\bar{b}]))$, for $Q \in \{\forall, \exists\}$;
(vii) If α and α' are variant, then $v(\alpha) = v(\alpha')$.

Given μ and v let $v_\mu : For(\Theta_U) \to |\mathcal{B}|$ such that $v_\mu(\varphi) = v(\widehat{\mu}(\varphi))$ for every φ.

Theorem 2.2 (Substitution Lemma)**.** *Given $\mathcal{M}(\mathcal{B})$, $\mathfrak{A}$, a **QmbC**-valuation v and an assignment μ, if t is free for z in φ and $b = ||t||^{\widehat{\mathfrak{A}}}_{\mu}$, then: $v_\mu(\varphi[z/t]) = v_\mu(\varphi[z/\bar{b}])$.*

Definition 2.3. Given $\mathcal{M}(\mathcal{B})$ and $\mathfrak{A}$, let $\Gamma \cup \{\varphi\} \subseteq For(\Theta_U)$. Then φ is a *semantical consequence of* Γ *over* $(\mathfrak{A}, \mathcal{M}(\mathcal{B}))$, denoted by $\Gamma \models_{(\mathfrak{A},\mathcal{M}(\mathcal{B}))} \varphi$, if the following holds: for every **QmbC**-valuation v over $(\mathfrak{A}, \mathcal{M}(\mathcal{B}))$, if $v_\mu(\gamma) \in D$, for every $\gamma \in \Gamma$ and every μ, then $v_\mu(\varphi) \in D$, for every μ. And φ is said to be a *semantical consequence of* Γ *in **QmbC** w.r.t. first-order swap structures*, denoted by $\Gamma \models_{\mathbf{QmbC}} \varphi$, if $\Gamma \models_{(\mathfrak{A},\mathcal{M}(\mathcal{B}))} \varphi$ for every $(\mathfrak{A}, \mathcal{M}(\mathcal{B}))$.

Theorem 2.4 (Soundness of **QmbC** w.r.t. first-order swap structures)**.** *For every set $\Gamma \cup \{\varphi\} \subseteq For(\Theta)$: if $\Gamma \vdash_{\mathbf{QmbC}} \varphi$, then $\Gamma \models_{\mathbf{QmbC}} \varphi$.*

3 First-Order Swap Structures: Completeness

Let $\Delta \subseteq Sen(\Theta)$ and let C be a nonempty set of constants of the signature Θ. Then, Δ is called a *C-Henkin theory* in **QmbC** if it satisfies the following: for every sentence of the form $\exists x\varphi$ in $Sen(\Theta)$, there exists a constant c in C such that if $\Delta \vdash_{\mathbf{QmbC}} \exists x\varphi$ then $\Delta \vdash_{\mathbf{QmbC}} \varphi[x/c]$.

Let Θ_C be the signature obtained from Θ by adding a set C of new individual constants. The consequence relation $\vdash^C_{\mathbf{QmbC}}$ is the consequence relation of **QmbC** over the signature Θ_C.

Recall that, given a Tarskian and finitary logic $\mathbf{L} = \langle For, \vdash\rangle$ (where For is the set of formulas of $\mathbf{L}$), and given a set $\Gamma \cup \{\varphi\} \subseteq For$, the set Γ is said to be *maximally non-trivial with respect to φ in* $\mathbf{L}$ if the following holds: (i) $\Gamma \nvdash \varphi$, and (ii) $\Gamma, \psi \vdash \varphi$ for every $\psi \notin \Gamma$.

Proposition 3.1 ([1], Corollary 7.5.4)**.** *Let $\Gamma \cup \{\varphi\} \subseteq Sen(\Theta)$ such that $\Gamma \nvdash_{\mathbf{QmbC}} \varphi$. Then, there exists a set of sentences $\Delta \subseteq Sen(\Theta)$ which is maximally non-trivial with respect to φ in **QmbC** (by restricting $\vdash_{\mathbf{QmbC}}$ to sentences) and such that $\Gamma \subseteq \Delta$.*

Definition 3.2. Let $\Delta \subseteq Sen(\Theta)$ be a C-Henkin and non-trivial theory in **QmbC**. Let $\equiv_\Delta \subseteq Sen(\Theta)^2$ be the relation in $Sen(\Theta)$ defined as follows: $\alpha \equiv_\Delta \beta$ iff $\Delta \vdash_{\mathbf{QmbC}} \alpha \to \beta$ and $\Delta \vdash_{\mathbf{QmbC}} \beta \to \alpha$.

Clearly $\equiv_\Delta$ is an equivalence relation. In the quotient set $A_\Delta \stackrel{\text{def}}{=} Sen(\Theta)/_{\equiv_\Delta}$ define $\wedge, \vee, \to$ as follows: $[\alpha]_\Delta\#[\beta]_\Delta \stackrel{\text{def}}{=} [\alpha\#\beta]_\Delta$ for any $\# \in \{\wedge, \vee, \to\}$, where $[\alpha]_\Delta$ is the equivalence class of α w.r.t. Δ. Then, $\mathcal{A}_\Delta \stackrel{\text{def}}{=} \langle A_\Delta, \wedge, \vee, \to, 0_\Delta, 1_\Delta\rangle$ is a Boolean algebra with $0_\Delta \stackrel{\text{def}}{=} [\varphi \wedge \neg\varphi \wedge \circ\varphi]_\Delta$ and $1_\Delta \stackrel{\text{def}}{=} [\varphi \vee \neg\varphi]_\Delta$, for any φ. Moreover, for every formula $\psi(x)$ with at most x occurring free, $[\forall x\psi]_\Delta = \bigwedge_{\mathcal{A}_\Delta}\{[\psi[x/t]_\Delta \ : \ t \in CTer(\Theta)\}$, and $[\exists x\psi]_\Delta = \bigvee_{\mathcal{A}_\Delta}\{[\psi[x/t]_\Delta \ : \ t \in CTer(\Theta)\}$.

Let $\mathcal{CA}_\Delta$ be the MacNeille-Tarski completion of $\mathcal{A}_\Delta$ and let $* : \mathcal{A}_\Delta \to \mathcal{CA}_\Delta$ be the associated monomorphism. Let $\mathcal{B}_\Delta$ be the full swap structure over $\mathcal{CA}_\Delta$ with associated Nmatrix $\mathcal{M}(\mathcal{B}_\Delta) \stackrel{\text{def}}{=} (\mathcal{B}_\Delta, D_\Delta)$. Note that $(([\alpha]_\Delta)^*, ([\beta]_\Delta)^*, ([\gamma]_\Delta)^*) \in D_\Delta$ iff $\Delta \vdash_{\mathbf{QmbC}} \alpha$.

Definition 3.3. Let $\Delta \subseteq Sen(\Theta)$ be C-Henkin and non-trivial in **QmbC**, let $\mathcal{M}(\mathcal{B}_\Delta)$ be as above, and let $U = CTer(\Theta)$. The *canonical structure induced by* Δ is the structure $\mathfrak{A}_\Delta = \langle U, I_{\mathfrak{A}_\Delta}\rangle$ over $\mathcal{M}(\mathcal{B}_\Delta)$ and Θ such that: $c^{\mathfrak{A}_\Delta} = c$ for each constant c; $f^{\mathfrak{A}_\Delta} : U^n \to U$ is such that $f^{\mathfrak{A}_\Delta}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for each n-ary function symbol f; and $P^{\mathfrak{A}_\Delta}(t_1, \ldots, t_n) = (([\varphi]_\Delta)^*, ([\neg\varphi]_\Delta)^*, ([\circ\varphi]_\Delta)^*)$ where $\varphi = P(t_1, \ldots, t_n)$, for each n-ary predicate symbol P.

Let $(\cdot)^\triangleright : Ter(\Theta_U) \to Ter(\Theta)$ be the mapping such that $(t)^\triangleright$ is the term obtained from t by substituting every occurrence of a constant $\bar{s}$ by the term s itself. Observe that $(t)^\triangleright = \widehat{||t||^{\mathfrak{A}_\Delta}}$ for every $t \in CTer(\Theta_U)$. This mapping can be naturally extended to a mapping $(\cdot)^\triangleright : For(\Theta_U) \to For(\Theta)$ such that $(\varphi)^\triangleright$ is the formula in $For(\Theta)$ obtained from $\varphi \in For(\Theta_U)$ by substituting every occurrence of a constant $\bar{t}$ by the term t itself.

Definition 3.4. Let $\Delta \subseteq Sen(\Theta)$ be a C-Henkin theory in **QmbC** for a nonempty set C of individual constants of Θ, such that Δ is maximally non-trivial with respect to φ in **QmbC**, for some sentence φ. The *canonical **QmbC**-valuation induced by* Δ *over* $\mathfrak{A}_\Delta$ *and* $\mathcal{M}(\mathcal{B}_\Delta)$ is the mapping $v_\Delta : Sen(\Theta_U) \to |\mathcal{B}_\Delta|$ such that $v_\Delta(\psi) = (([(\psi)^\triangleright]_\Delta)^*, ([\neg(\psi)^\triangleright]_\Delta)^*, ([\circ(\psi)^\triangleright]_\Delta)^*)$.

Remark 3.5. Note that $v_\Delta(\psi) \in D_\Delta$ iff $\Delta \vdash_{\mathbf{QmbC}} \psi$, for every sentence $\psi \in Sen(\Theta)$.

Theorem 3.6. *The canonical **QmbC**-valuation* v_Δ *is a **QmbC**-valuation over* $\mathfrak{A}_\Delta$ *and* $\mathcal{M}(\mathcal{B}_\Delta)$.

Theorem 3.7 (Completeness for sentences of **QmbC** w.r.t. first-order swap structures)**.** *Let* $\Gamma \cup \{\varphi\} \subseteq Sen(\Theta)$. *If* $\Gamma \vDash_{\mathbf{QmbC}} \varphi$ *then* $\Gamma \vdash_{\mathbf{QmbC}} \varphi$.

Proof. Suppose that $\Gamma \nvdash_{\mathbf{QmbC}} \varphi$. Then, there exists a C-Henkin theory Δ^H over Θ_C in **QmbC** for a nonempty set C of new individual constants such that $\Gamma \subseteq \Delta^H$ and, for every $\alpha \in Sen(\Theta)$: $\Gamma \vdash_{\mathbf{QmbC}} \alpha$ iff $\Delta^H \vdash^C_{\mathbf{QmbC}} \alpha$. Hence, $\Delta^H \nvdash^C_{\mathbf{QmbC}} \varphi$ and so, by Proposition 3.1, there exists a set of sentences $\overline{\Delta^H}$ in Θ_C extending Δ^H which is maximally non-trivial with respect to φ in **QmbC**, such that $\overline{\Delta^H}$ is a C-Henkin theory over Θ_C in **QmbC**. Now, let $\mathcal{M}(\mathcal{B}_{\overline{\Delta^H}})$, $\mathfrak{A}_{\overline{\Delta^H}}$ and $v_{\overline{\Delta^H}}$ as above. Then, $v_{\overline{\Delta^H}}(\alpha) \in D_{\overline{\Delta^H}}$ iff $\overline{\Delta^H} \vdash^C_{\mathbf{QmbC}} \alpha$, for every α in $Sent(\Theta_C)$. From this, $v_{\overline{\Delta^H}}[\Gamma] \subseteq D_{\overline{\Delta^H}}$ and $v_{\overline{\Delta^H}}(\varphi) \notin D_{\overline{\Delta^H}}$. Finally, let $\mathfrak{A}$ and v be the restriction to Θ of $\mathfrak{A}_{\overline{\Delta^H}}$ and $v_{\overline{\Delta^H}}$, respectively. Then, $\mathfrak{A}$ is a structure over $\mathcal{M}(\mathcal{B}_{\overline{\Delta^H}})$, and v is a valuation for **QmbC** over $\mathfrak{A}$ and $\mathcal{M}(\mathcal{B}_{\overline{\Delta^H}})$ such that $v[\Gamma] \subseteq D_{\overline{\Delta^H}}$ but $v(\varphi) \notin D_{\overline{\Delta^H}}$. This shows that $\Gamma \nvDash_{\mathbf{QmbC}} \varphi$. □

It is easy to extend **QmbC** by adding a standard equality predicate $\approx$ such that $(a \approx^{\mathfrak{A}} b) \in D$ iff $a = b$. On the other hand, the extension of **QmbC** to other first-order **LFI**s based on well-known axiomatic extensions of **mbC** is straightforward, both syntactically and semantically.

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Prime numbers and implication free reducts of MV_n-chains

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Abstract

Let $\mathbf{Ł}_{n+1}$ be the MV-chain on the $n+1$ elements set $Ł_{n+1} = \{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ in the algebraic language $\{\to, \neg\}$ [3]. As usual, further operations on $Ł_{n+1}$ are definable by the following stipulations: $1 = x \to x$, $0 = \neg 1$, $x \oplus y = \neg x \to y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \wedge y = x \odot (x \to y)$, $x \vee y = \neg(\neg x \wedge \neg y)$. Moreover, we will pay special attention to the also definable unary operator $^*x = x \odot x$.

In fact, the aim of this paper is to continue the study initiated in [4] of the $\{^*, \neg, \vee\}$-reducts of the MV-chains $\mathbf{Ł}_{n+1}$, denoted $\mathbf{Ł}^*_{n+1}$. In fact $\mathbf{Ł}^*_{n+1}$ is the algebra on $Ł_{n+1}$ obtained by replacing the implication operator $\to$ by the unary operation * which represents the square operator $^*x = x \odot x$ and which has been recently used in [5] to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \langle \mathbf{Ł}_4, \{1/3, 2/3, 1\}\rangle$. In this contribution we make a step further in studying the expressive power of the * operation, in particular our main result provides a full characterization of those prime numbers n for which the structures $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. In other words, we characterize for which n the Łukasiewicz implication $\to$ is definable in $\mathbf{Ł}^*_{n+1}$, or equivalenty, for which n $\mathbf{Ł}^*_{n+1}$ is in fact an MV-algebra. We also recall that, in any case, the matrix logics $\langle \mathbf{Ł}^*_{n+1}, F\rangle$, where F is an order filter, are algebraizable.

Term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$

Let X be a subset of $Ł_{n+1}$. We denote by $\langle X\rangle^*$ the subalgebra of $\mathbf{Ł}^*_{n+1}$ generated by X (in the reduced language $\{^*, \neg, \vee\}$). For $n \geq 1$ define recursively $(^*)^n x$ as follows: $(^*)^1 x = {^*x}$, and $(^*)^{i+1} x = {^*((^*)^i x)}$, for $i \geq 1$.

A nice feature of the $\mathbf{Ł}^*_{n+1}$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in Ł_{n+1} \mid a \leq b\}$, for every $a \in Ł_{n+1}$. A proof of the following result can be found in [4].

Proposition 1. *For each $a \in L_{n+1}$, the unary operation Δ_a defined as*

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in $\mathbf{Ł}^*_{n+1}$. Therefore, for every $a \in L_{n+1}$, the operation χ_a, i.e., the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

It is now almost immediate to check that the following implication-like operation is definable in every $\mathbf{Ł}^*_{n+1}$: $x \Rightarrow y = 1$ if $x \leq y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0 \leq i \leq j \leq n} (\chi_{i/n}(x) \wedge \chi_{j/n}(y)).$$

Actually, one can also define Gödel implication on $\mathbf{Ł}^*_{n+1}$ by putting $x \Rightarrow_G y = (x \Rightarrow y) \vee y$.

It readily follows from Proposition 1 that all the $\mathbf{Ł}^*_{n+1}$ algebras are simple as, if $a > b \in Ł_{n+1}$ would be congruent, then $\Delta_a(a) = 1$ and $\Delta_a(b) = 0$ should be so. Recall that an algebra is called *strictly simple* if it is simple and does not contain proper subalgebras. It is clear that if $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are strictly simple, then $\{0, 1\}$ is their only proper subalgebra.

Remark 2. It is well-known that $\mathbf{Ł}_{n+1}$ is strictly simple iff n is prime. Note that, for every n, if $\mathbf{B} = (B, \neg, \to)$ is an MV-subalgebra of $\mathbf{Ł}_{n+1}$, then $\mathbf{B}^* = (B, \vee, \neg, *)$ is a subalgebra of $\mathbf{Ł}^*_{n+1}$ as well. Thus, if $\mathbf{Ł}_{n+1}$ is not strictly simple, then $\mathbf{Ł}^*_{n+1}$ is not strictly simple as well. Therefore, if n is not prime, $\mathbf{Ł}^*_{n+1}$ is not strictly simple. However, in contrast with the case of $\mathbf{Ł}_{n+1}$, n being prime is not a sufficient condition for $\mathbf{Ł}^*_{n+1}$ being strictly simple.

We now introduce the following procedure $\mathtt{P}$: given n and an element $a \in Ł^*_{n+1} \setminus \{0, 1\}$, it iteratively computes a sequence $[a_1, \dots, a_k, \dots]$ where $a_1 = a$ and for every $k \geq 1$,

$$a_{k+1} = \begin{cases} {}^*(a_k), & \text{if } a_k > 1/2 \\ \neg(a_k), & \text{otherwise (i.e, if } a_k < 1/2) \end{cases}$$

until it finds an element a_i such that $a_i = a_j$ for some $j < i$, and then it stops. Since $Ł^*_{n+1}$ is finite, this procedure always stops and produces a finite sequence $[a_1, a_2, \dots, a_m]$, where $a_1 = a$ and a_m is such that $\mathtt{P}$ stops at a_{m+1}. In the following, we will denote this sequence by $\mathtt{P}(n, a)$.

Lemma 3. *For each odd number n, let $a_1 = (n-1)/n$. Then the procedure $\mathtt{P}$ stops after reaching $1/n$, that is, if $\mathtt{P}(n, a_1) = [a_1, a_2, \dots, a_m]$ then $a_m = 1/n$.*

Furthermore, for any $a \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$, the set A_1 of elements reached by $\mathtt{P}(n, a)$, i.e. $A_1 = \{b \in Ł^*_{n+1} \mid b \text{ appears in } \mathtt{P}(n, a)\}$, together with the set A_2 of their negations, 0 and 1, define the domain of a subalgebra of $\mathbf{Ł}^*_{n+1}$.

Lemma 4. *$\mathbf{Ł}^*_{n+1}$ is strictly simple iff $\langle (n-1)/n \rangle^* = \mathbf{Ł}^*_{n+1}$.*

Proof. (Sketch) The 'if' direction is trivial. As for the other direction, call $a_1 = (n-1)/n$ and assume that $\langle a_1 \rangle^* = \mathbf{Ł}^*_{n+1}$. Launch the procedure $\mathtt{P}(n, a_1)$ and let $\mathbf{A}$ be the subalgebra of $\mathbf{Ł}^*_{n+1}$ whose universe is $A_1 \cup A_2 \cup \{0, 1\}$ defined as above. Clearly $a_1 \in A$, hence $\langle a_1 \rangle^* \subseteq \mathbf{A}$. But $\mathbf{A} \subseteq \langle a_1 \rangle^*$, by construction. Therefore $\mathbf{A} = \langle a_1 \rangle^* = \mathbf{Ł}^*_{n+1}$.

Fact: Under the current hypothesis (namely, $\langle a_1 \rangle^* = \mathbf{Ł}^*_{n+1}$) if n is even, then $n = 2$ or $n = 4$.

Thus, assume n is odd, and hence Lemma 3 shows that $1/n \in A_1$. Now, let $c \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$ such that $c \neq a_1$. If $c \in A_1$ then the process of generation of A from c will produce the same set A_1 and so $\mathbf{A} = \mathbf{Ł}^*_{n+1}$, showing that $\langle c \rangle^* = \mathbf{Ł}^*_{n+1}$. Otherwise, if $c \in A_2$ then $\neg c \in A_1$ and, by the same argument as above, it follows that $\langle c \rangle^* = \mathbf{Ł}^*_{n+1}$. This shows that $\mathbf{Ł}^*_{n+1}$ is strictly simple. □

Lemma 5 ([4])**.** *If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then:*
(i) $\mathbf{Ł}^*_{n+1}$ is strictly simple.
(ii) n is prime

Theorem 6. *$\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ iff $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. The 'only if' part is (i) of Lemma 5. For the 'if' part, since $\mathbf{Ł}^*_{n+1}$ is strictly simple then, for each $a, b \in Ł_{n+1}$ where $a \notin \{0, 1\}$ there is a definable term $\mathbf{t}_{a,b}(x)$ such that $\mathbf{t}_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0, 1\}$ and $b \in Ł_{n+1}$ there is no such term then $\mathbf{A} = \langle a \rangle^*$ would be a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (since $b \notin \mathbf{A}$) different from $\{0,1\}$, a contradiction. By Proposition 1 the operations $\chi_a(x)$ are definable for each $a \in Ł_{n+1}$, then in $\mathbf{Ł}^*_{n+1}$ we can define Łukasiewicz implication $\to$ as follows:

$$x \to y = (x \Rightarrow y) \vee \left(\bigvee_{n>i>j\geq 0} \chi_{i/n}(x) \wedge \chi_{j/n}(y) \wedge \mathbf{t}_{i/n,a_{ij}}(x) \right) \vee \left(\bigvee_{n>j\geq 0} \chi_1(x) \wedge \chi_{j/n}(y) \wedge y \right)$$

where $a_{ij} = 1 - i/n + j/n$. □

We have seen that n being prime is a necessary condition for $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent and this is the case, for instance, of $n = 17$.

Definition 7. Let Π be the set of odd primes n such that 2^m is not congruent with ± 1 mod n for all m such that $0 < m < (n-1)/2$.

Since, for every odd prime n, 2^m is congruent with ± 1 mod n for $m = (n-1)/2$ then n is in Π iff n is an odd prime such that $(n-1)/2$ is the least $0 < m$ such that 2^m is congruent with ± 1 mod n.

The following is our main result and it characterizes the class of prime numbers for which the Łukasiewicz implication is definable in $\mathbf{Ł}^*_{n+1}$.

Theorem 8. *For every prime number $n > 5$, $n \in \Pi$ iff $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent.*

The proof of theorem above makes use of the procedure $\mathtt{P}$ defined above. Let $a_1 = (n-1)/n$ and let $\mathtt{P}(n, a_1) = [a_1, \ldots, a_l]$. By the definition of the procedure $\mathtt{P}$, the sequence $[a_1, \ldots, a_l]$ is the concatenation of a number r of subsequences $[a^1_1, \ldots, a^1_{l_1}]$, $[a^2_1, \ldots, a^2_{l_2}]$, $\ldots$, $[a^r_1, \ldots, a^r_{l_r}]$, with $a^1_1 = a_1$ and $a^r_{l_r} = a_l$, where for each subsequence $1 \leq j \leq r$, only the last element $a^i_{l_i}$ is below $1/2$, while the rest of elements are above $1/2$.

Now, by the very definition of *, it follows that the last elements $a^j_{l_j}$ of every subsequence are of the form

$$a^j_{l_j} = \begin{cases} \frac{kn-2^m}{n}, \text{ if } j \text{ is odd} \\ \frac{2^m-kn}{n}, \text{ otherwise, i.e. if } j \text{ is even} \end{cases}$$

for some $m, k > 0$, where in particular m is the number of strictly positive elements of $\mathbf{Ł}_{n+1}$ which are obtained by the procedure before getting $a^j_{l_j}$.

Now, Lemma 3 shows that if n is odd then $1/n$ is reached by $\mathtt{P}$, i.e. $a_l = a^r_{l_r} = 1/n$. Thus,

$$\begin{cases} kn - 2^m = 1, \text{ if } r \text{ is odd (i.e., } 2^m \equiv -1 \pmod{n} \text{ if } r \text{ is odd)} \\ 2^m - kn = 1, \text{ otherwise (i.e., } 2^m \equiv 1 \pmod{n} \text{ if } r \text{ is even)} \end{cases}.$$

where m is now the number of strictly positive elements in the list $\mathtt{P}(n, a_1)$, i.e. that are reached by the procedure.

Therefore 2^m is congruent with ± 1 mod n. If n is a prime such that $\mathbf{Ł}^*_{n+1}$ is strictly simple, the integer m must be exactly $(n-1)/2$, for otherwise $\langle a_1 \rangle^*$ would be a proper subalgebra of $Ł^*_{n+1}$ which is absurd. Moreover, for no $m' < m$ one has that $2^{m'}$ is congruent with ± 1 mod n because, in this case, the algorithm would stop producing a proper subalgebra of $\mathbf{Ł}^*_{n+1}$. This result, together with Theorem 6, shows the right-to-left direction of Theorem 8.

In order to show the other direction assume, by Theorem 6, that $\mathbf{Ł}^*_{n+1}$ is not strictly simple. Thus, by Lemma 4, $\langle a_1\rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ and hence the algorithm above stops, in $1/n$, after reaching $m < (n-1)/2$ strictly positive elements of $\mathbf{Ł}^*_{n+1}$. Thus, 2^m is congruent with ± 1 (depending on whether r is even or odd, where r is the number of subsequences in the list $\mathtt{P}(n, a_1)$ as described above) mod n, showing that $n \notin \Pi$.

Algebraizability of $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$

Given the algebra $\mathbf{Ł}^*_{n+1}$, it is possible to consider, for every $1 \leq i \leq n$, the matrix logic $Ł^*_{i,n+1} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$. In this section we recall from [4] that all the $Ł^*_{i,n+1}$ logics are algebraizable in the sense of Blok-Pigozzi [1], and that, for every i, j, the quasivarieties associated to $Ł^*_{i,n+1}$ and $Ł^*_{j,n+1}$ are the same.

Observe that the operation $x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise is definable in $\mathbf{Ł}^*_{n+1}$. Indeed, it can be defined as $x \approx y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Also observe that $x \approx y = \Delta_1((x \Rightarrow_G y) \wedge (y \Rightarrow_G x))$ as well.

Lemma 9. *For every n, the logic $Ł^*_{n+1} := Ł^*_{n,n+1} = \langle \mathbf{Ł}^*_{n+1}, \{1\}\rangle$ is algebraizable.*

Proof. It is immediate to see that the set of formulas $\Delta(p, q) = \{p \approx q\}$ and the set of pairs of formulas $E(p, q) = \{\langle p, \Delta_0(p)\rangle\}$ satisfy the requirements of algebraizability. □

Blok and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems S_1 and S_2 in the same language are *equivalent* if there are translations $\tau_i : S_i \to S_j$ for $i \neq j$ such that: $\Gamma \vdash_{S_i} \varphi$ iff $\tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv\vdash_{S_i} \tau_j(\tau_i(\varphi))$. From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the algebraic point of view, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This can be applied to $Ł^*_{i,n+1}$.

Lemma 10. *For every n and every $1 \leq i \leq n-1$, the logics $Ł^*_{n+1}$ and $Ł^*_{i,n+1}$ are equivalent.*

Indeed, it is enough to consider the translation mappings $\tau_1 : Ł^*_{n+1} \to Ł^*_{i,n+1}$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : Ł^*_{i,n+1} \to Ł^*_{n+1}$, $\tau_{i,2}(\varphi) = \Delta_{i/n}(\varphi)$. Therefore, as a direct consequence of Lemma 9, Lemma 10 and the observations above, it follows the algebraizability of $Ł^*_{i,n+1}$.

Theorem 11. *For every n and for every $1 \leq i \leq n$, the logic $Ł^*_{i,n+1}$ is algebraizable.*

Therefore, for each logic $Ł^*_{i,n+1}$ there is a quasivariety $\mathcal{Q}(i, n)$ which is its equivalent algebraic semantics. Moreover, by Lemma 10 and by Blok and Pigozzi's results, $\mathcal{Q}(i, n)$ and $\mathcal{Q}(j, n)$ coincide, for every i, j. The question of axiomatizing $\mathcal{Q}(i, n)$ is left for future work.

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Infinitary connectives and Borel functions in Łukasiewicz logic

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Riesz Spaces are lattice-ordered linear spaces over the field of real numbers $\mathbb{R}$ [3]. They have had a predominant rôle in the development of functional analysis over ordered structures, due to the simple remark that most of the spaces of functions one can think of are indeed Riesz Spaces. Such spaces are also related to expansions of Łukasiewicz infinite-valued logic.

In particular, one can consider MV-algebras – the variety of algebras that model Łukasiewicz logic – and endow them with a scalar multiplication, where scalars are elements of the standard MV-algebra $[0,1]$. Such MV-algebras with scalar multiplication form a variety and they are known in literature with the name of Riesz MV-algebras [1]. Moreover, Riesz MV-algebras are categorical equivalent with Riesz Spaces with a strong unit.

In this talk will exploit the connection between Riesz Spaces and MV-algebras as a bridge between algebras of Borel-measurable functions and Łukasiewicz logic. To do so we will define the infinitary logical systems $\mathcal{IRL}$, whose models are algebras of $[0,1]$-valued continuous functions defined over some basically-disconnected compact Hausdorff space X. We will further discuss completeness of $\mathcal{IRL}$ with respect to σ-complete Riesz MV-algebras and characterize the Lindenbaum-Tarski algebra of it by means of Borel-measurable functions.

The logical system $\mathcal{IRL}$ is obtained in [2] starting from an expansion of Łukasiewicz logic introduced in [1], namely $\mathcal{RL}$, and by adding an infinitary operator that models a countable disjunction. In more details, we can consider a countable set of propositional variables and the connectives $\neg$, $\rightarrow$, $\{\nabla_\alpha\}_{\alpha\in[0,1]}$, $\bigvee$. The connectives $\neg$, $\rightarrow$, $\{\nabla_\alpha\}_{\alpha\in[0,1]}$ are inherited from the logic $\mathcal{RL}$, while the latter is a connective of arity less or equal to ω, i.e. it is defined for any set of formulas which is at most countable. Consider now the following set of axioms:

(L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(L3) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$
(L4) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
(R1) $\nabla_r(\varphi \rightarrow \psi) \leftrightarrow (\nabla_r\varphi \rightarrow \nabla_r\psi)$
(R2) $\nabla_{(r\odot q^*)}\varphi \leftrightarrow (\nabla_q\varphi \rightarrow \nabla_r\varphi)$
(R3) $\nabla_r(\nabla_q\varphi) \leftrightarrow \nabla_{r\cdot q}\varphi$
(R4) $\nabla_1\varphi \leftrightarrow \varphi$
(S1) $\varphi_k \rightarrow \bigvee_{n\in\mathbb{N}}\varphi_n$, for any $k \in \mathbb{N}$.

The logic $\mathcal{IRL}$ is obtained from these axioms, Modus Ponens and the following deduction rule.

$$\text{(SUP)} \qquad \frac{(\varphi_1 \rightarrow \psi), \ldots, (\varphi_k \rightarrow \psi) \ldots}{\bigvee_{n\in\mathbb{N}}\varphi_n \rightarrow \psi}$$

Axioms (L1)-(L4) are the axioms of Łukasiewicz logic, axioms (R1)-(R4) make the connectives $\{\nabla_r\}_{r\in[0,1]}$ into the "de Morgan dual" of a scalar multiplication (that is, $\neg\nabla_r\neg$ behaves like a scalar multiplication in the sense of Riesz MV-algebras), axioms (S1) and (SUP) ensure that the new connective behaves as a least upper bound for a given sequence.

The new system has σ-complete Riesz MV-algebras as models and the following results hold.

Theorem 1. *[2] (1) IRL, the Lindenbaum-Tarski algebra of $\mathcal{IRL}$, is a σ-complete Riesz MV-algebra and it is the smallest σ-complete algebra that contains the Lindenbaum-Tarski algebra of $\mathcal{RL}$.*

(2) $\mathcal{IRL}$ is complete with respect to all algebras in $\mathbf{RMV}_\sigma$, the class of σ-complete Riesz MV-algebras;

A functional description of the Linbenbaum-Tarski algebra IRL is possible by recalling that any σ-complete Riesz MV-algebra is semisimple. Kakutani's duality, a result by Nakano and the duality between Riesz MV-algebras and vector lattices entail the following theorem.

Theorem 2. *Let A be a Dedekind σ-complete Riesz MV-algebra. There exists a basically disconnected compact Hausdorff space (i.e. it has a base of open F_σ sets) X such that $A \simeq C(X)$, where $C(X)$ is the algebra of $[0,1]$-valued and continuous functions defined over X.*

In particular, $IRL \simeq C(X)$ for some basically disconnected compact Hausdorff space X.

The above theorem, as strong as it is, does not allow for a more concrete description of the algebra IRL in the spirit of functional representation that holds for Łukasiewicz logic. Indeed, the Linbenbaum-Tarski algebras of Łukasiewicz logic and of the logic $\mathcal{RL}$ have both a clear-cut description: they are the algebras of all piecewise linear functions (in the first case, with integer coefficient) over some unit cube $[0,1]^\mu$. Having this in mind, to obtain a description of IRL as an appropriate subalgebra of $[0,1]^{[0,1]^\mu}$, we need to develop the algebraic theory of σ-complete Riesz MV-algebras.

To this end, one can consider the work of Słomiński on infinitary algebras. It turns out that σ-complete Riesz MV-algebras are a proper class of infinitary algebras in the sense of [4] and we can prove the following results.

Theorem 3. *[2] The following hold.*

(1) The ω-generated free algebra in the class of σ-complete Riesz MV-algebras exists and it is isomorphic with IRL.

(2) Consider the algebra of term functions of $\mathbf{RVM}_\sigma$ in n variables, denoted by $\mathcal{RT}_n$. If we consider the elements of $\mathcal{RT}_n$ as functions from $[0,1]^n$ to $[0,1]$, $\mathcal{RT}_n$ becomes a Riesz MV-algebras closed to pointwise defined countable suprema.

(3) The algebra $\mathcal{RT}_n$ is isomorphic with the Lindenbaum-Tarski algebra of $\mathcal{IRL}$ build upon n-propositional variables.

Finally, using the previous results, we can prove the following.

Theorem 4. *$\mathcal{RT}_n$ is isomorphic with the algebra of $[0,1]$-valued Borel-measurable functions defined over $[0,1]^n$. Whence, IRL is isomorphic to the algebra of all Borel measurable functions from $[0,1]^n$ to $[0,1]$.*

Moreover, the Loomis-Sikorski theorem holds for Riesz MV-algebras.

Theorem 5. *[2] Let $A \subseteq C(X)$ be a σ-complete Riesz MV-algebra, where $X = Max(A)$, and let $\mathcal{T} \subseteq [0,1]^X$ be the set of functions f that are essentially equal to some function of A. Then $\mathcal{T}$ is a Riesz tribe, each $f \in \mathcal{T}$ is essentially equal to a unique $f^* \in A$ and the map $f \mapsto f^*$ is a σ-homomorphism of $\mathcal{T}$ onto A.*

As a consequence, $\mathbf{RMV}_\sigma$ is an infinitary variety and the logic $\mathcal{IRL}$ is standard complete.

Theorem 6. *[2]* $\mathbf{RMV}_\sigma$ *is the infinitary variety generated by* $[0,1]$.

For the reader convenience, we summarize the main results of this abstract.

1. we introduce an infinitary logic starting from the logic of Riesz MV-algebras,
2. we discuss its models looking at them as infinitary algebras and via Kakutani's duality with compact Hausdorff spaces,
3. we use notions from infinitary universal algebra to obtain a characterization of the Linbenbaum-Tarski algebra of the infinitary logic as algebra of Borel-measurable functions,
4. via the Loomis-Sikorski theorem for Riesz MV-algebras, we prove that σ-complete Riesz MV-algebras are the infinitary variety generated by $[0,1]$ and we infer the standard completeness of $\mathcal{IRL}$.

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(Co)Algebraic Techniques for Markov Decision Processes

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1 Introduction

Markov Decision Processes (MDPs) [11] are a family of probabilistic, state-based models used in planning under uncertainty and reinforcement learning. Informally, an MDP models a situation in which an agent (the decision maker) makes choices at each state of a process, and each choice leads to some reward and a probabilistic transition to a next state. The aim of the agent is to find an optimal policy, i.e., a way of choosing actions that maximizes future expected rewards.

The classic theory of MDPs with discounting is well-developed (see [11, Chapter 6]), and indeed we do not prove any new results about MDPs as such. Our work is inspired by Bellman's principle of optimality, which states the following: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" [2, Chapter III.3]. This principle has clear coinductive overtones, and our aim is to situate it in a body of mathematics that is also concerned with infinite behavior and coinductive proof principles, i.e., in coalgebra.

Probabilistic systems of similar type have been studied extensively, also coalgebraically, in the area of program semantics (see for instance [5, 6, 14, 15]). Our focus is not so much on the observable behavior of MDPs viewed as computations, but on their role in solving optimal planning problems.

This abstract is based on [7] to which we refer for a more detailed account.

2 Markov Decision Processes

We briefly introduce the relevant basic concepts from the classic theory MDPs[11]. Letting ΔS denote the set of probability distributions with finite support on the set S, we define MDPs[1] and policies as follows.

Definition 2.1 (MDP, Policy) *Let Act be a finite set of* actions. *A* Markov decision process (MDP) $m = \langle S, u, t\rangle$ *consists of a finite set S of* states, *a reward function $u\colon S \to \mathbb{R}$, and a probabilistic transition structure $t\colon S \to (\Delta S)^{Act}$. A* policy *is a function $\sigma\colon S \to Act$.*

That is, in state s when the agent chooses action a, there is a probability distribution $t(s)(a)$ over states. Furthermore, in each state s, the agent collects a reward (or utility) specified by a real number $u(s)$.

We shall often leave S implicit and simply write $m = \langle u, t\rangle$. Given a probabilistic transition structure $t\colon S \to (\Delta S)^{Act}$ and a policy $\sigma \in Act^S$, we write $t_\sigma\colon S \to \Delta S$ for the map defined by $t_\sigma(s) = t(s)(\sigma(s))$, and t_a when σ is constant equal to a.

[1]Our simple type of MDPs is known as time-homogeneous, infinite-horizon MDPs with finite state and action spaces, and our policies as stationary, memoryless deterministic policies.

There are several criteria for evaluating the long-term rewards expected by following a given policy. A classic criterion uses discounting. The idea is that rewards collected tomorrow are worth less than rewards collected today.

Definition 2.2 *Let γ be a fixed real number with $0 \leq \gamma < 1$. Such a γ is called a* discount factor*. Let an MDP $m = \langle u, t \rangle$ be given. The* long-term value *of a policy σ (for m) according to the* discounted sum criterion *is the function $\mathrm{LTV}_\sigma \colon S \to \mathbb{R}$ defined as follows:*

$$\mathrm{LTV}_\sigma(s) = r_0^\sigma(s) + \gamma \cdot r_1^\sigma(s) + \cdots + \gamma^n \cdot r_n^\sigma(s) + \cdots \tag{1}$$

where $r_n^\sigma(s)$ is the expected reward at time step n.

Note that $r_0^\sigma(s) = u(s)$ for all $s \in S$, and since S is finite, $\max_s r_0^\sigma(s) < \infty$. This boundedness property entails that the infinite sum in (1) is convergent.

It will be convenient to work with the map ℓ_σ that takes the expected value of LTV_σ relative to some distribution. Formally, $\ell_\sigma \colon \Delta S \to \mathbb{R}$ is defined for all $\varphi \in \Delta S$ by

$$\ell_\sigma(\varphi) \;=\; \sum_{s \in S} \varphi(s) \cdot \mathrm{LTV}_\sigma(s). \tag{2}$$

Observe that for each state s, $\mathrm{LTV}_\sigma(s)$ is equal to the immediate rewards plus the discounted future expected rewards. Seen this way, (1) may be re-written to the corecursive equation

$$\mathrm{LTV}_\sigma(s) = u(s) + \gamma \cdot \left(\sum_{s' \in S} t_\sigma(s)(s') \cdot \mathrm{LTV}_\sigma(s') \right) = u(s) + \gamma \cdot \ell_\sigma(t_\sigma(s)). \tag{3}$$

Viewing LTV_σ as a column vector in $\mathbb{R}^S$ and t_σ as a column-stochastic matrix P_σ, the equation in (3) shows that LTV_σ is a fixpoint of the (linear) operator

$$\Psi_\sigma \colon \mathbb{R}^S \to \mathbb{R}^S \qquad \Psi_\sigma(v) = u + \gamma P_\sigma v. \tag{4}$$

By the Banach Fixpoint Theorem, this fixpoint unique, since Ψ_σ is contractive (due to $0 \leq \gamma < 1$), and $\mathbb{R}^S$ is a complete metric space. The long-term value induces a preorder on policies: $\sigma \leq \tau$ if $\mathrm{LTV}_\sigma \leq \mathrm{LTV}_\tau$ in the pointwise order on $\mathbb{R}^S$. A policy σ is *optimal* if for all policies τ, we have $\tau \leq \sigma$.

Given an MDP m, the *optimal value of m* is the map $V^* \colon S \to \mathbb{R}$ that for each state gives the best long-term value that can be obtained for any policy [11]:

$$V^*(s) = \max_{\sigma \in Act^S} \{\mathrm{LTV}_\sigma(s)\}.$$

It is an important classic result that V^* is the unique (bounded) map that satisfies *Bellman's optimality equation* [2, 11]:

$$V^*(s) = u(s) + \gamma \cdot \max_{a \in Act} \left\{ \sum_{s' \in S} t_a(s)(s') \cdot V^*(s') \right\}.$$

3 Main Contributions

3.1 Policy Improvement via Contraction Coinduction

For our simple model of MDPs with discounting, it is known that the simple type of policies that we consider here, are sufficient. In other words, an optimal policy can always be found among stationary, memoryless, deterministic policies [11, Theorem 6.2.7]. This result together with the optimality equation forms the basis for an effective algorithm for finding optimal policies, known as *policy iteration* [8]. The algorithm starts from any policy $\sigma \in Act^S$, and iteratively improves σ to some τ such that $\sigma \leq \tau$. This leads to an increasing sequence of policies in the preorder of all policies $(S^{Act}, \leq)$. Since this preorder is finite, this process will at some point stabilize. The correctness of policy iteration follows from the following theorem.

Theorem 3.1 (Policy Improvement) *Let an MDP be given by* $t \colon S \to (\Delta S)^{Act}$ *and* $u \colon S \to \mathbb{R}$. *Let* σ *and* τ *be policies. If* $\ell_\sigma \circ t_\tau \geq \ell_\sigma \circ t_\sigma$, *then* $\mathrm{LTV}_\tau \geq \mathrm{LTV}_\sigma$. *Similarly, if* $\ell_\sigma \circ t_\tau \leq \ell_\sigma \circ t_\sigma$, *then* $\mathrm{LTV}_\tau \leq \mathrm{LTV}_\sigma$.

We present a coinductive proof of the Policy Improvement theorem. This leads us to formulate a coinductive proof principle that we have named *contraction (co)induction*. The contraction coinduction principle is a variation of the classic Banach Fixpoint Theorem, asserting that any contractive mapping on a complete metric space has a unique fixpoint. We need a version of this theorem which, in addition to a complete metric, also has an order.

Definition 3.2 *An* ordered metric space *is a structure* $(M, d, \leq)$ *such that* d *is a metric on* M *and* $\leq$ *is a partial order on* M, *satisfying the extra property that for all* $y \in M$, $\{z \mid z \leq y\}$ *and* $\{z \mid y \leq z\}$ *are closed sets in the metric topology. This space is said to be* complete *if it is complete as a metric space.*

Theorem 3.3 (Contraction (Co)Induction) *Let* M *be a non-empty, complete ordered metric space. If* $f \colon M \to M$ *is both contractive and order-preserving, then the fixpoint* x^* *of* f *is a least pre-fixpoint (if* $f(x) \leq x$, *then* $x^* \leq x$*), and also a greatest post-fixpoint (if* $x \leq f(x)$, *then* $x \leq x^*$*).*

Theorem 3.3 follows from the Metric Coinduction Principle [10, 12]. Our aim is not the highest level of generality. Rather, we see contraction (co)induction as a particular instance of Metric Coinduction that suffices to prove interesting results about MDPs. We also believe contraction (co)induction should have applications far beyond the topic of MDPs.

3.2 Long-Term Values via b-Corecursive Algebras

We now take a coalgebraic perspective on MDPs and long-term value functions. Let Δ be the Set-monad of finitely supported probability distributions, and let H be the Set-functor $H = \mathbb{R} \times \mathrm{Id}$. A leading observation of this paper is that we can re-express (3) by saying that $\mathrm{LTV}_\sigma \colon S \to \mathbb{R}$ makes the following diagram commute:

$$\begin{array}{ccccc} S & & \xrightarrow{\quad m_\sigma \quad} & & \mathbb{R} \times \Delta S \\ \Big\downarrow{\scriptstyle \mathrm{LTV}_\sigma} & & & & \Big\downarrow{\scriptstyle \mathbb{R} \times \Delta(\mathrm{LTV}_\sigma)} \\ \mathbb{R} & \xleftarrow[\alpha_\gamma]{} & \mathbb{R} \times \mathbb{R} & \xleftarrow[\mathbb{R} \times \mathrm{E}]{} & \mathbb{R} \times \Delta\mathbb{R} \end{array} \tag{5}$$

Here, $\mathrm{E} \colon \Delta\mathbb{R} \to \mathbb{R}$ is the expected value function and $\alpha_\gamma \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the H-algebra

$$\alpha_\gamma \colon H\mathbb{R} \to \mathbb{R} \qquad \alpha_\gamma(x, y) = x + \gamma \cdot y. \tag{6}$$

This means that LTV_σ is an $H\Delta$-coalgebra-to-algebra map. We naturally wonder whether the $H\Delta$-algebra at the bottom of the diagram is a *corecursive algebra* [4]: for every coalgebra $f \colon X \to H\Delta X$ (where X is possibly infinite), is there a unique map $f^\dagger \colon S \to \mathbb{R}$ making the diagram commute? This turns out not to be the case for arbitrary state spaces, as problems can arise when reward values are unbounded. To remedy this, we introduce the notions of b-categories and b-corecursive algebras with which we aim to give a sparse categorification of boundedness. Combining these with techniques from coinductive specification ([3]) and trace sematics [1, 9], we can show that $\alpha_\gamma \circ (\mathbb{R} \times \mathrm{E})$ is a b-corecursive algebra, and thereby obtain LTV_σ from its universal property. The optimal value function V^* can be characterised in a similar way via a b-corecursive algebra.

This categorical approach emphasizes compositional reasoning about functions and functors. The classical theory of MDPs does not do this; it directly proves properties (such as boundedness) of composites viewed as monolithic entities, instead of deriving them from preservation properties of their constituents. So it neither needs nor uses the extra information that we obtained by working in a categorical setting. Indeed, most of our work is devoted to this extra information, and we hope that it will be useful in settings beyond MDPs. We have some pilot results in this direction for stochastic games [13].

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An algebraic study of the First Order intuitionistic fragment of 3-valued Łukasiewicz logic

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Abstract

MV-algebras are semantic for Łukasiewicz logic and MV-algebras generated for finite chain are Heyting algebras where the Gödel implication can be written in terms of De Morgan and Moisil's modal operators. In our work, a fragment of Łukasiewicz logic is studied in the trivalent case. The propositional and first order logic is presented. The maximal consistent theories is studied as Monteiro's maximal deductive system of the Lindenbaum-Tarski algebra, in both cases. Consequently, the strong adequacy theorem with respect to the suitable algebraic structures is proven. Our algebraic strong completeness theorem does not need a negation in the language, in this sense Rasiowa's work is improved. The techniques presented in this work are adaptable to the other algebrizable logics where the variety of algebras from these logics is semisimple.

1 Trivalent modal Hilbert algebras with supremum

In this section, we shall introduce and study $\{\to, \vee, \triangle, 1\}$-reduct of 3-valued MV-algebra.

Definition 1 *An algebra $\langle A, \to, \vee, \triangle, 1\rangle$ is trivalent modal Hilbert algebra with supremum (for short, $H_3^{\vee,\triangle}$-algebra) if the following properties hold:*

(1) *the reduct $\langle A, \vee, 1\rangle$ is a join-semilattice with greatest element* 1*, and the conditions* (a) $x \to (x \vee y) = 1$ *and* (b) $(x \to y) \to ((x \vee y) \to y) = 1$ *hold.*

(2) *The reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra that verifies: $((x \to y) \to z) \to ((z \to x) \to z) \to z = 1$, and the operator $\triangle$ verify the following identities:* **(M1)** $\triangle x \to x = 1$, **(M2)** $((y \to \triangle y) \to (x \to \triangle\triangle x)) \to \triangle(x \to y) = \triangle x \to \triangle\triangle y$, *and* **(M3)** $(\triangle x \to \triangle y) \to \triangle x = \triangle x$.

Theorem 2 *The variety of $H_3^{\vee,\triangle}$-algebras is semisimple. The simple algebras are $\mathbb{C}_3^{\to,\vee}$ and $\mathbb{C}_2^{\to,\vee}$.*

Let $\mathfrak{Fm}_s$ be the absolutely free algebra over the language $\Sigma = \{\to, \vee, \triangle\}$ generated by a set Var of variables. Consider now the following logic:

Definition 3 *We denote by $\mathcal{H}^3_{\vee,\triangle}$ the Hilbert calculus determined by the followings axioms and inference rules, where $\alpha, \beta, \gamma, ... \in Fm$:*

Axiom schemas

(Ax1) $\alpha \to (\beta \to \alpha)$, **(Ax2)** $(\alpha \to (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$, **(Ax3)** $((\alpha \to (\beta \to \gamma)) \to (((\gamma \to \alpha) \to \gamma) \to \gamma)$, **(Ax4)** $\alpha \to (\alpha \vee \beta)$, **(Ax5)** $\beta \to (\alpha \vee \beta)$,

(Ax6) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$, **(Ax7)** $\triangle\alpha \to \alpha$, **(Ax8)** $\triangle(\triangle\alpha \to \beta) \to (\triangle\alpha \to \triangle\beta)$, **(Ax9)** $((\beta \to \triangle\beta) \to (\alpha \to \triangle(\alpha \to \beta))) \to \triangle(\alpha \to \beta)$, **(Ax10)** $((\triangle\alpha \to \beta) \to \gamma) \to ((\triangle\alpha \to \gamma) \to \gamma)$.

Inference rules

(MP), **(NEC-S)** $\dfrac{\Gamma \vdash_{\vee} \alpha}{\Gamma \vdash_{\vee} \triangle\alpha}$.

Let $\Gamma \cup \{\alpha\}$ be a set formulas of $\mathcal{H}^3_{\vee,\triangle}$, we define the derivation of α from Γ in usual way and denote by $\Gamma \vdash_{\vee} \alpha$.

Theorem 4 (Lindenbaum-Łos) *Let $\mathcal{L}$ be a Tarskian and finitary logic (see [2, pag. 48]) over the language $\mathbb{L}$. Let $\Gamma \cup \{\varphi\} \subseteq \mathbb{L}$ be such that $\Gamma \nvdash \varphi$. Then exists a set Ω such that $\Gamma \subseteq \Omega \subseteq \mathbb{L}$ with Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

Theorem 5 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. Let $\Gamma/\equiv_{\vee} = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum $\mathfrak{Fm}/\equiv_{\vee}$, then:* **1.** *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$ then $\beta \in \Gamma$,* **2.** *$\Gamma/\equiv_{\vee}$ is a modal deductive system of $\mathfrak{Fm}/\equiv_{\vee}$. Also, if $\varphi \notin \Gamma/\equiv_{\vee}$ and for any modal deductive system $\overline{D}$ which contains properly to $\Gamma/\equiv_{\vee}$, then $\overline{\varphi} \in \overline{D}$.*

The notion deductive systems considered in the last Theorem, part 2, was named *Systèmes deductifs liés à "a"* by A. Monteiro, where a is an element of some given algebra such that the congruences are determined by deductive systems [3, pag. 19]. This was studied by Monteiro himself and other authors for diferent algebraic system where it is possible to define an implication in terms of the operations of lenguage form this systems.

Lemma 6 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. If $\alpha \notin \Gamma$ then $\triangle\alpha \to \beta \in \Gamma$ for any $\beta \in \mathfrak{Fm}_s$.*

Theorem 7 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. The map $v : \mathfrak{Fm}_s \to \mathbb{C}_3$, defined by:*

$$v(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Gamma_0 \\ 1/2 & \text{if } \alpha \in \Gamma_{1/2} \\ 1 & \text{if } \alpha \in \Gamma \end{cases}$$

for all $\alpha \in \mathfrak{Fm}_s$ it is a valuation for $\mathcal{H}^3_{\vee,\triangle}$, where $\Gamma_{1/2} = \{\alpha \in \mathfrak{Fm}_s : \alpha \notin \Gamma, \nabla\alpha \in \Gamma\}$ and $\Gamma_0 = \{\alpha \in \mathfrak{Fm}_s : \alpha, \nabla\alpha \notin \Gamma\}$.

Theorem 8 *(Soundness and completeness of $\mathcal{H}^3_{\vee,\triangle}$ w.r.t. $H_3^{\vee,\triangle}$-algebras) Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, $\Gamma \vdash_{\vee} \varphi$ if and only if $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$.*

2 Model Theory and first order logics of $\mathcal{H}_3^{\vee,\triangle}$ without identities

Let Λ be the propositional signature of $\mathcal{H}_3^{\vee,\triangle}$, the simbols $\forall$ (universal quantifier) and $\exists$ (existential quantifier), with the punctuation marks (commas and parenthesis). Let $Var = \{v_1, v_2, ...\}$ a numerable set of individual variables. A first order signature $\Sigma = \langle \mathcal{P}, \mathcal{F}, \mathcal{C}\rangle$ consists of: a set $\mathcal{C}$ of individual constants; for each $n \geq 1$, $\mathcal{F}$ a set of functions with n-ary, for each $n \geq 1$, $\mathcal{P}$ a set of predicates with n-ary. It will be denoted by T_Σ and $\mathfrak{Fm}_\Sigma$ the sets of all terms and formulas, respectively.

Let Σ be a first order signature. The logic $\mathcal{QH}_3^{\vee,\triangle}$ over Σ is obtained from the axioms and rules of $\mathcal{H}_3^{\vee,\triangle}$ by substituting variables by formulas of $\mathfrak{Fm}_\Sigma$, by extending the following axioms and rules:

Axioms Schemas (Ax11) $\varphi_x^t \to \exists x\varphi$, if t is a term free for x in φ, **(Ax12)** $\forall x\varphi \to \varphi_x^t$, if t is a term free for x in φ, **(Ax13)** $\triangle\exists x\varphi \leftrightarrow \exists x\triangle\varphi$, **(Ax14)** $\triangle\forall x\varphi \leftrightarrow \forall x\triangle\varphi$, **(Ax15)** $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$ if α does not contain free occurrences of x. **Inferences Rules (R3)** $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$, **(R4)** $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$ where x does not occur free in φ.

Let Σ be a first-order signature. A first-order structure over Σ is pair $\mathfrak{U} = \langle A, \cdot^{\mathfrak{U}}\rangle$ where A is a non-empty set and $\cdot^{\mathfrak{U}}$ is a interpretation mapping defined on Σ as follows: for each individual constant symbol c of Σ, $c^{\mathfrak{U}} \in A$; for each function symbol f n-ary of Σ, $f^{\mathfrak{U}} : A^n \to A$; for each predicate symbol P n-ary of Σ, $P^{\mathfrak{U}} : A^n \to B$, where B is a complete $H_3^{\vee,\triangle}$-algebra.

For a given Σ-structure $\langle A, \cdot^{\mathfrak{U}}\rangle$, let us consider the signature $\Sigma' = \Sigma \cup \{c_a\}_{a\in A}$ which is the signature Σ extended by a set with new constants. Let us denote the extended language by $\mathfrak{Fm}(\Sigma')$. We want to define the truth value a closed formula. For this task, we consider the structure $\mathfrak{U}$ and the map $m : CT_{\Sigma'} \to A$, where $CT_{\Sigma'}$ is the set of closed terms (without free variables) of the language $\mathfrak{Fm}_{\Sigma'}$, is defined as follows: if τ is c_a, then $m(\tau) = m(c_a) = a$; if τ is $f(\tau_1, ..., \tau_n)$ and $\tau_i \in CT_{\Sigma'}$, then $m(\tau) = f^{\mathfrak{U}}(m(\tau_1), ..., m(\tau_n))$.

Let φ be a closed formula (sentence) from Σ', then we define $m : \mathfrak{Fm}_{\Sigma'} \to B$ inductively over the complexity of φ as follows: if φ is $P(\tau_1, \cdots, \tau_n)$ with P a n-ary predicate and $\tau_i \in CT_{\Sigma'}$, then $m(\varphi) = P^{\mathfrak{U}}(m(\tau_1), ..., m(\tau_n))$; if φ is $\gamma \vee \psi$ then $m(\varphi) = m(\gamma) \vee m(\psi)$; if φ is $\gamma \to \psi$ then $m(\varphi) = m(\gamma) \to m(\psi)$; if φ is $\triangle\psi$ then $m(\varphi) = \triangle m(\psi)$; let $\psi = \psi(x)$ a formula with x is a unique free variable, we denote $\psi_x^{c_a}$ the formula obtained by replacing x for c_a. Then: if φ is $\exists x\psi$ then $m(\varphi) = \bigvee\limits_{c_a\in\Sigma'} m(\psi_x^{c_a})$; if φ is $\forall x\psi$ then $m(\varphi) = \bigwedge\limits_{c_a\in\Sigma'} m(\psi_x^{c_a})$.

We say that $m : \mathfrak{Fm}_{\Sigma'} \to B$ is $\mathcal{QH}_3^{\vee,\triangle}$-valuation or simply a valuation.

As usual, we can define $\Gamma \vDash \alpha$, that is, for any structure $\mathfrak{U}$, if $\mathfrak{U} \vDash \psi$ for every $\psi \in \Gamma$, then $\mathfrak{U} \vDash \alpha$.

Lemma 9 *Let α be a formula of $\mathcal{QH}_3^{\vee,\triangle}$ and β an instance of α, then there exits $\mathfrak{U}$ such that $\mathfrak{U} \vDash \alpha$ implies $\mathfrak{U} \vDash \beta$.*

Theorem 10 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, if $\Gamma \vdash_\vee \varphi$ then $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$.*

It is important to note that from Theorem 10 and Lemma 9, it is easy to see that every instance of a theorem is valid.

It is clear that $\mathcal{QH}_3^{\vee,\triangle}$ is a tarskian logic. So, we can consider the notion of maximal theories with respect to some formula and the notion of closed theories is defined in the same way. Therefore, we have that Lindenbaum- Łos' Theorem for $\mathcal{QH}_3^{\vee,\triangle}$. Then, we have the following

Now, let us consider the relation $\equiv$ defined by $x \equiv y$ iff $\vdash x \to y$ and $\vdash y \to x$, then we have the algebra $\mathfrak{Fm}_{\Sigma'}/\equiv$ is a $H_3^{\vee,\triangle}$-algebra and the proof is exactly the same as in the propositional case.

Theorem 11 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, with Γ non-trivial maximal respect to φ in $\mathcal{QH}_3^{\vee,\triangle}$. Let $\Gamma/\equiv_\vee = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum*

$\mathfrak{Fm}_\Sigma / \equiv_\vee$, *then:* **1.** *If* $\alpha \in \Gamma$ *and* $\overline{\alpha} = \overline{\beta}$, *then* $\beta \in \Gamma$. *If* $\overline{\alpha} \in \Gamma / \equiv_\vee$, *then* $\overline{\forall x \alpha} \in \Gamma / \equiv_\vee$; *in this case we say that* $\Gamma / \equiv_\vee$ *is monadic.* **2.** $\Gamma / \equiv_\vee$ *is a modal deductive system of* $\mathfrak{Fm}_\Sigma / \equiv_\vee$. *Also, if* $\varphi \notin \Gamma / \equiv_\vee$ *and for any modal deductive system* $\overline{D}$ *being monadic in the sense of 1 and containing properly to* $\Gamma / \equiv_\vee$, *then* $\overline{\varphi} \in \overline{D}$.

Theorem 12 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, *with* Γ *non-trivial maximal respect to* φ *in* $\mathcal{QH}^3_{\vee,\triangle}$. *The map* $v : \mathfrak{Fm}_\Sigma \to \mathbb{C}_3$, *defined by:*

$$v(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Gamma_0 \\ 1/2 & \text{if } \alpha \in \Gamma_{1/2} \\ 1 & \text{if } \alpha \in \Gamma \end{cases}$$

for all $\alpha \in \mathfrak{Fm}_\Sigma$ *it is a valuation for* $\mathcal{H}^3_{\vee,\triangle}$, *where* $\Gamma_{1/2} = \{\alpha \in \mathfrak{Fm}_\Sigma : \alpha \notin \Gamma, \nabla\alpha \in \Gamma\}$ *and* $\Gamma_0 = \{\alpha \in \mathfrak{Fm}_\Sigma : \alpha, \nabla\alpha \notin \Gamma\}$.

Theorem 13 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, *if* $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$ *then* $\Gamma \vdash_\vee \varphi$.

Proof: Let us suppose $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$ and $\Gamma \nvdash_\vee \varphi$. Then, there exists Δ maximal theory such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_\vee \varphi$. From the latter and Theorem 12, there exists a structure $\mathfrak{U}$ such that $\Delta \nvDash_{\mathfrak{U}} \varphi$ but $\Delta \vDash_{\mathcal{A}} \gamma$ for every $\gamma \in \Delta$, which is a contradiction.

□

It is possible to adapt our proof of strong Completeness Theorem in the propositional and first order cases to logics from the certain semisimple varieties of algebras. This is so because the maximal congruences play the same role as the maximal consistent theories in the Lindenbaum-Tarski algebra. From the latter and results of universal algebra, we have the algebra quotient by maximal congruences are isomorphic to semisimple algebras. Therefore, we always have a homomorphism from the Lindenbaum-Tarski algebra to the semisimple algebras. This homomorphism is the same one constructed by Carnielli and Coniglio to prove strong completeness theorems for different logics ([2]). On the other hand, we can observe that A. V. Figallo constructed this homomorphism to study different semisimple varieties. The general presentation of these ideas will be part of a future work.

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From Bunches to Labels and Back in BI Logic

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1 Abstract

The ubiquitous notion of resources is a basic one in many fields but has become more and more central in the design and validation of modern computer systems over the past twenty years. Resource management encompasses various kinds of behaviours and interactions including consumption and production, sharing and separation, spatial distribution and mobility, temporal evolution, sequentiality or non-determinism, ownership and access control. Dealing with various aspects of resource management is mostly in the territory of substructural logics, and more precisely, resource logics such as Linear Logic (LL) [5] with its resource consumption interpretation, the logic of Bunched Implications (BI) [8] with its resource sharing interpretation, or order-aware non-commutative logic (NL) [1]. As specification logics, they allow the modelling of features like interactions, resource distribution and mobility, non-determinism, sequentiality or coordination of entities. Separation Logic and its memory model, of which BI is the logical kernel, has gained momentum and proved itself very successful as an assertion language for verifying programs that handle mutable data structures via pointers [6, 9].

From a semantic point of view, resource interactions such as production and consumption, or separation and sharing are handled in resource models at the level of resource composition. For example, various semantics have been proposed to capture the resource sharing interpretation of BI including monoidal, relational or topological resource semantics [4]. From a proof-theoretic and purely syntactical point of view, the subtleties of a particular resource composition usually leads to the definition of distinct sets of connectives (*e.g.*, additive vs multiplicative, commutative vs non-commutative). Capturing the interaction between various kinds of connectives often results in structures more elaborated than set of multi-sets of formulas. For example, the label-free sequent calculus for BI, which is called LBI, admits sequent the left-hand part of which are structured as bunches [7, 8]. Resource interaction is usually much simpler to handle in labelled proof-systems since labels and label constraints are allowed to reflect and mimic, inside the calculus, the fundamental properties of the resource models they are drawn from. For example, various labelled tableaux calculi, all called TBI, have been proposed for the various semantics of BI [4]. A labelled tableaux calculus has been also developed for Separation Logic and its memory model [3].

Our aim is to study the relationships between labelled and label-free proof-systems in BI logic and, more precisely, with the label-free sequent calculus LBI. The relational, topological and monoidal semantics with a Beth interpretation of the additive disjunction have all been proven sound and complete w.r.t. LBI and TBI in [4, 7, 8]. However, the monoidal semantics in which the additive disjunction has the usual Kripke interpretation and which admits explicitly inconsistent resources together with a total (and not partial) resource composition operator has only been proven complete w.r.t. TBI. Its status w.r.t. LBI is not known and still a difficult open problem. Many attempts at solving the problem from a purely semantic point of view have failed over the past fifteen years. Instead we propose a three-step syntactic approach to proving the completeness of the Kripke monoidal semantics of BI that relies on proof translations.

As a first step, we recently proposed a single-conclusioned sequent-style labelled proof-system called GBI, that can be seen as a kind of intermediate calculus between TBI and LBI. GBI shares with TBI the idea of sets of labels and contraints arranged as a resource graph, but the resource graph is partially constructed on the fly using explicit structural rules on labels and constraints rather than being obtained as the result of a closure operator.

The main result in [2] was the definition of an effective (algorithmic) procedure that systematically translates any LBI-proof into a GBI-proof. This translation is not a one-to-one correspondence sending each LBI-rule occurring in the original proof to its corresponding GBI counterpart in the translated proof. Indeed, most of the translations patterns require several additional structural steps to obtain an actual GBI-proof. However, these patterns are such that the rule-application strategy of the original proof will be contained in the translated proof, making our translation structure preserving in that particular sense.

In [2] we also started to investigate how GBI-proofs could relate to LBI-proofs. Taking advantage of the structure preserving property of the translation we gave a reconstruction algorithm that tries to rebuild a LBI-proof of a formula F, from scratch, knowing only the rule-application strategy followed in a given (normal) GBI-proof of F. The completeness of this reconstruction algorithm, *i.e.*, that it might never get stuck, depends on the completeness of the insertion of semi-distributivity steps in the LBI-proof that are meant to fill in the gaps left by the application of structural rules of GBI (in the given GBI-proof) with no LBI counterpart. The completeness of these intermediate semi-distributivity steps was (and still is) only conjectured and far from obvious.

In this paper, we take a second step and further develop our study of how to translate GBI-proofs into LBI-proofs. We first define a kind of tree-like property for GBI labelled sequents. This tree property allows us to translate the left-hand side of a labelled sequent to a bunch according to the label of the formula on its right-hand side. Refining our analysis of the translation given in [2], we show that every sequent in a GBI-proof obtained by translation of an LBI-proof satisfies our tree property. We also show that all GBI rules preserve the tree property from conclusion to premisses except for the rules of contraction and weakening. The main contribution then follows as we define a restriction of GBI, called $\mathrm{GBI}_{\mathrm{tp}}$, in which the only instances of the weakening and contraction rules that are considered as suitable are the ones preserving the tree property and we show that $\mathrm{GBI}_{\mathrm{tp}}$-proofs can effectively and systematically be translated to LBI-proofs. Let us remark that the main result does not depend on a GBI-proof being built from an assembly of LBI translation patterns, *i.e.*, on the fact that a GBI-proof actually corresponds to some translated image of an LBI-proof. We thus observe that our tree-property can serve as a criterion for defining a notion of normal GBI-proofs for which normality also means LBI-translatability.

Ongoing and future work will focus on making the third and final step of showing that total Kripke monoidal models with explicit inconsistency are complete w.r.t. the label-free sequent calculus LBI. Several directions and approaches can be taken to achieve this final goal. A first interesting direction is to find an effective (algorithmic) procedure of translating TBI-proofs into $\mathrm{GBI}_{\mathrm{tp}}$-proofs since TBI is known to be sound and complete w.r.t. total KRMs. This direction is challenging because TBI is a multi-conclusioned system in which generative rules can be used as many times as needed (which avoids backtracking) to saturate the proof-search space and be able to build a countermodel from Hintikka sets in case of non-provability. A second direction relies on the construction of counter-models in the KRM semantics of BI directly from failed $\mathrm{GBI}_{\mathrm{tp}}$-proof attempts. This direction is also challenging as it requires building countermodels from a single-conclusioned proof-system in which backtracking is allowed.

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Credulous Acceptability, Poison Games and Modal Logic (*extended abstract*)

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1 Introduction

In abstract argumentation theory [3], an argumentation framework is a directed graph $(A, \rightarrow)$ [6]. For $x, y \in A$ such that $x \rightarrow y$ we say that x attacks y. An *admissible* set, of a given attack graph, is a set $X \subseteq A$ such that [6]: (a) no two nodes in X attack one another; and (b) for each node $y \in A \backslash X$ attacking a node in X, there exists a node $z \in X$ attacking y. Such sets are also called *credulously admissible* argument. They form the basis of all main argumentation semantics first developed in [6], and they are central to the influential graph-theoretic systematization of logic programming and default reasoning pursued in [4].

One key reasoning tasks is then to decide whether a given argumentation framework contains at least one non-empty admissible set [7]. Interestingly, the notion has an elegant operationalization in the form of a two-player game, called *Poison Game* [5], or *game for credulous acceptance* [11, 16]. Inspired by it we define a new modal logic, called *Poison Modal Logic* (PML), whose operators capture the strategic abilities of players in the Poison Game, and are therefore fit to express the modal reasoning involved in the notion of credulous admissibility. This explores research lines presented in [9]. The paper also defines a suitable notion of p-bisimulation, which answers another open question [8], namely a notion of structural equivalence tailored for it. More broadly we see the present paper as a contribution to bridging concepts from abstract argumentation theory, games on graphs and modal logic.

This paper is a natural continuation of the line of work interfacing abstract argumentation and modal logic. PML sits at the intersection of two lines of research in modal logic: dynamic logic concerned with the study of operators which transform semantics structures [1, 13, 15]; and game logics analyzing games through logic [2, 14]. To the best of our knowledge, only [10] (private communication) presented a preliminary work on a modal logic inspired by the Poison Game.

2 Poison Modal Logic (PML)

2.1 The Poison Game

The Poison Game [5] is a two-player ($\mathbb{P}$, the proponent, and $\mathbb{O}$, the opponent), win-lose, perfect-information game played on a directed graph (W, R). The game starts by $\mathbb{P}$ selecting a node $w_0 \in W$. After this initial choice, $\mathbb{O}$ selects w_1 a successor of w_0, $\mathbb{P}$ then selects a successor w_1 and so on. However, while $\mathbb{O}$ can choose any successor of the current node, $\mathbb{P}$ can select only successors which have not yet been visited —*poisoned*— by $\mathbb{O}$. $\mathbb{O}$ wins if and only if $\mathbb{P}$ ends up in a position with no available successors. What makes this game interesting for us is that the existence of a winning strategy for $\mathbb{P}$, if (W, R) is finite, can be shown to be equivalent to the existence of a (non-empty) *credulously admissible* argument in the graph [5].

2.2 Syntax and semantics

The poison modal language $\mathcal{L}^{\mathfrak{p}}$ is defined by the following grammar in BNF:

$$\mathcal{L}^{\mathfrak{p}} : \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathbf{P} \cup \{\mathfrak{p}\}$ with $\mathbf{P}$ a countable set of propositional atoms and $\mathfrak{p}$ a distinguished atom called *poison atom*. We will also touch on multi-modal variants of the above language, denoted $\mathcal{L}^{\mathfrak{p}}_n$, where $n \geq 1$ denotes the number of distinct pairs $(\Diamond_i, \blacklozenge_i)$ of modalities, with $1 \leq i \leq n$ and where each $\blacklozenge_i$ comes equipped with a distinct poison atom $\mathfrak{p}_i$.

This language is interpreted on Kripke models $\mathcal{M} = (W, R, V)$. A pointed model is a pair $(\mathcal{M}, w)$ with $w \in \mathcal{M}$. We will call $\mathfrak{M}$ the set of all pointed models and $\mathfrak{M}^{\emptyset}$ the set of pointed models $(\mathcal{M}, w)$ such that $V^{\mathcal{M}}(\mathfrak{p}) = \emptyset$. We define now an operation $\bullet$ on models which modifies valuation V by adding a specific state to $V(\mathfrak{p})$. Formally, for $\mathcal{M} = (W, R, V)$ and $w \in W$:

$$\mathcal{M}^{\bullet}_w = (W, R, V)^{\bullet}_w = (W, R, V'),$$

where $\forall p \in \mathbf{P}, V'(p) = V(p)$ and $V'(\mathfrak{p}) = V(\mathfrak{p}) \cup \{w\}$.

We are now equiped to describe the semantics for the $\blacklozenge$ modality (the other clauses are standard):

$$(\mathcal{M}, w) \models \blacklozenge\varphi \iff \exists v \in W, wRv, (\mathcal{M}^{\bullet}_v, v) \models \varphi.$$

We introduce some auxiliary definitions. The poisoning relation between two pointed models $\overset{\bullet}{\to} \in \mathfrak{M}^2$ is defined as: $(\mathcal{M}, w) \overset{\bullet}{\to} (\mathcal{M}', w') \iff R^{\mathcal{M}}(w, w')$ and $\mathcal{M}' = \mathcal{M}^{\bullet}_{w'}$. Furthermore, we denote $(\mathcal{M}, w)^{\bullet} \subset \mathfrak{M}$ the set of all pointed models accessible from $\mathcal{M}$ via a poisoning relation. Two pointed models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are poison modally equivalent, written $(\mathcal{M}, w) \overset{p}{\leftrightsquigarrow} (\mathcal{M}', w')$, if and only if, $\forall\varphi \in \mathcal{L}^{\mathfrak{p}}$: $(\mathcal{M}, w) \models \varphi \iff (\mathcal{M}', w') \models \varphi$.

2.3 Validities and Expressible Properties

Fact 1. *Let $\varphi, \psi \in \mathcal{L}^{\mathfrak{p}}$ be two formulas, then the following formulas are valid in* PML*:*

$$\blacksquare p \leftrightarrow \Box p$$
$$\Box\mathfrak{p} \to (\blacksquare\varphi \leftrightarrow \Box\varphi)$$
$$\blacksquare(\varphi \to \psi) \to (\blacksquare\varphi \to \blacksquare\psi).$$

To illustrate the expressive power of PML, we show that it is possible to express the existence of cycles in the modal frame, a property not expressible in the standard modal language. Consider the class of formulas δ_n, with $n \in \mathbb{N}_{>0}$, defined inductively as follows, with $i < n$:

$$\delta_1 = \Diamond\mathfrak{p}$$
$$\delta_{i+1} = \Diamond(\neg\mathfrak{p} \wedge \delta_i).$$

Fact 2. *Let $\mathcal{M} = (W, R, V) \in \mathfrak{M}^{\emptyset}$, then for $n \in \mathbb{N}_{>0}$ there exists $w \in W$ such that $(\mathcal{M}, w) \models \blacklozenge\delta_n$ if and only if there exists a cycle of length $i \leq n$ in the frame (W, R).*

A direct consequence of Fact 2 is that PML is not bisimulation invariant. In particular, its formulas are not preserved by tree-unravelings and it does not enjoy the tree model property.

PML (or, more precisely, its infinitary version) can express winning positions in a natural way. Given a frame (W, R), nodes satisfying formulas $\blacklozenge\Box\mathfrak{p}$ are winning for $\mathbb{O}$ as she can move to a dead end for

$\mathbb{P}$. It is also the case for nodes satisfying formula $\blacklozenge\Box\blacklozenge\Box\mathfrak{p}$: she can move to a node in which, no matter which successor $\mathbb{P}$ chooses, she can then push her to a dead end. In general, winning positions for $\mathbb{O}$ are defined by the following infinitary $\mathcal{L}^{\mathfrak{p}}$-formula: $\mathfrak{p} \vee \blacklozenge\Box\mathfrak{p} \vee \blacklozenge\Box\blacklozenge\Box\mathfrak{p} \vee \cdots$. Dually, winning positions for $\mathbb{P}$ are defined by the following infinitary $\mathcal{L}^{\mathfrak{p}}$-formula: $\neg\mathfrak{p} \wedge \Diamond\blacksquare\neg\mathfrak{p} \wedge \Diamond\blacksquare\Diamond\blacksquare\neg\mathfrak{p} \wedge \cdots$.

3 Expressivity of PML

Definition 1 (FOL translation)**.** *Let $p, q, \dots$ in $\mathbf{P}$ be propositional atoms, then their corresponding first-order predicates are called $P, Q, \dots$. The predicate for the poison atom $\mathfrak{p}$ is $\mathfrak{P}$. Let N be a (possibly empty) set of variables, and x a designated variable, then the translation $ST_x^N : \mathcal{L}^{\mathfrak{p}} \to \mathcal{L}$ is defined inductively as follows (where $\mathcal{L}$ is the first-order correspondence language):*

$$\begin{aligned} ST_x^N(p) &= P(x), \forall p \in \mathbf{P} \\ ST_x^N(\neg\varphi) &= \neg ST_x^N(\varphi) \\ ST_x^N(\varphi \wedge \psi) &= ST_x^N(\varphi) \wedge ST_x^N(\psi) \\ ST_x^N(\Diamond\varphi) &= \exists y \left(R(x,y) \wedge ST_y^N(\varphi)\right) \\ ST_x^N(\blacklozenge\varphi) &= \exists y \left(R(x,y) \wedge ST_y^{N\cup\{y\}}(\varphi)\right) \\ ST_x^N(\mathfrak{p}) &= \mathfrak{P}(x) \vee \bigvee_{y\in N} (y = x). \end{aligned}$$

Theorem 1. *Let $(\mathcal{M}, w)$ be a pointed model and $\varphi \in \mathcal{L}^{\mathfrak{p}}$ a formula, we have then:*

$$(\mathcal{M}, w) \models \varphi \iff \mathcal{M} \models ST_x^{\emptyset}(\varphi)[x := w].$$

A relation $Z \subseteq \mathfrak{M} \times \mathfrak{M}$ is a p-bisimulation if, together with the standard clauses for bisimulation:

Zig$_\blacklozenge$: if $(\mathcal{M}_1, w_1)Z(\mathcal{M}_2, w_2)$ and there exists $(\mathcal{M}_1', w_1')$ such that $(\mathcal{M}_1, w_1) \overset{\bullet}{\to} (\mathcal{M}_1', w_1')$, then there exists $(\mathcal{M}_2', w_2')$ such that $(\mathcal{M}_2, w_2) \overset{\bullet}{\to} (\mathcal{M}_2', w_2')$ and $(\mathcal{M}_1', w_1')Z(\mathcal{M}_2', w_2')$.

Zag$_\blacklozenge$: as expected.

Invariance under the existence of a p-bisimulation (in symbols, $\overset{p}{\rightleftharpoons}$) can be proven to characterize the fragment of FOL which is equivalent to PML.

Theorem 2. *For any two pointed models $(\mathcal{M}_1, w_1)$ and $(\mathcal{M}_2, w_2)$, if $(\mathcal{M}_1, w_1) \overset{p}{\rightleftharpoons} (\mathcal{M}_2, w_2)$ then $(\mathcal{M}_1, w_1) \overset{p}{\leftrightsquigarrow} (\mathcal{M}_2, w_2)$.*

Theorem 3. *For any two ω-saturated models $(\mathcal{M}_1, w_1)$ and $(\mathcal{M}_2, w_2)$, if $(\mathcal{M}_1, w_1) \overset{p}{\leftrightsquigarrow} (\mathcal{M}_2, w_2)$ then $(\mathcal{M}_1, w_1) \overset{p}{\rightleftharpoons} (\mathcal{M}_2, w_2)$.*

Theorem 4. *A $\mathcal{L}$ formula is equivalent to the translation of an $\mathcal{L}^{\mathfrak{p}}$ formula if and only if it is p-bisimulation invariant.*

4 Undecidability

In this section we establish the undecidability of PML_3 that corresponds to $\mathcal{L}_3^{\mathfrak{p}}$. We call R, R_1 and R_2 the three accessibility relations of a model of PML_3. In this variant we only consider models whose poison valuation is empty.

We show that the satisfaction problem for PML_3 is undecidable. To do so we reduce the problem of the $\mathbb{N} \times \mathbb{N}$ tilling in a similar way as the undecidability proof for hybrid logic presented in [12].

Theorem 5. *The satisfaction problem for* PML_3 *is undecidable.*

Based on this result we postulate that PML is also undecidable, especially since we can show that:

Theorem 6. PML *does not have the Finite Model Property.*

5 Conclusion

In this article we presented a modal logic to describe the Poison Game which is thus able to detect credulously admissible arguments. This paper is a first exploration of this logic: we gave a first-order translation, a suitable notion of bisimulation and we proved the undecidability of a variant of PML.

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Representation theorems for Grzegorczyk contact algebras

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1 Introduction

In [5] Andrzej Grzegorczyk presented one of the first systems of point-free topology developed in the setting of the so-called *connection structures*. Even today, after a lot has been done and achieved in the area, Grzegorczyk's construction keeps being interesting, especially due to his definition of *point*, which embodies the geometrical intuition of point as diminishing system of regions of space (see Section 3). This talk is devoted to the following standard problems: what kind of topological spaces can be obtained from Grzegorczyk contact algebras and *vice versa*, which topological spaces give rise to Grzegorczyk algebras? The topic of the presentation is located well within the scope of the tradition of Boolean Contact Algebras (see e.g. [1, 6]).

2 Basic concepts

Consider a triple $\mathfrak{B} = \langle R, \leqslant, \mathsf{C}\rangle$, where $\langle R, \leqslant\rangle$ is a boolean lattice and $\mathsf{C} \subseteq R \times R$ satisfies:

$$\neg(0 \mathrel{\mathsf{C}} x), \tag{C0}$$

$$x \leqslant y \Longrightarrow x \mathrel{\mathsf{C}} y, \tag{C1}$$

$$x \mathrel{\mathsf{C}} y \Longrightarrow y \mathrel{\mathsf{C}} x, \tag{C2}$$

$$x \leqslant y \Longrightarrow \forall_{z\in R}(z \mathrel{\mathsf{C}} x \Longrightarrow z \mathrel{\mathsf{C}} y). \tag{C3}$$

Elements of R are called regions and C is a *contact* (*connection*) relation. In $\mathfrak{B}$ we define *non-tangential* inclusion relation:

$$x \ll y \Longrightarrow \neg(x \mathrel{\mathsf{C}} -y)\,,$$

where $-y$ is the boolean complement of y (while $\neg$ is the standard negation operator).

We define $x \bigcirc y$ to mean that $x \cdot y \neq 0$ (with $\cdot$ being the standard meet operation), and take $\perp \subseteq R \times R$ to be the set-theoretical complement of $\bigcirc$.

3 Grzegorczyk contact algebras

A *pre-point* of $\mathfrak{B}$ is a non-empty set X of regions such that:

$$0 \notin X\,, \tag{r0}$$

$$\forall_{u,v\in X}(u = v \vee u \ll v \vee v \ll u)\,, \tag{r1}$$

$$\forall_{u\in X}\exists_{v\in X}\ v \ll u\,, \tag{r2}$$

$$\forall_{x,y\in R}\big(\forall_{u\in X}(u \bigcirc x \wedge u \bigcirc y) \Longrightarrow x \mathrel{\mathsf{C}} y\big)\,. \tag{r3}$$

The purpose of this definition is to formally grasp the intuition of point as the system of diminishing regions determining the unique location in space (see the figures for geometrical intuitions on the Cartesian plane).

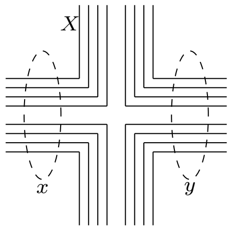

Figure 1: The set X of cross-like regions is not a pre-point, since regions x and y overlap all regions in X but are not in contact.

Let $\mathbf{Q}$ be the set of all pre-points of $\mathfrak{B}$. We extend the set of axioms for $\mathfrak{B}$ with the following postulate:

$$\forall_{x,y\in R}\Big(x \mathrel{\mathsf{C}} y \Longrightarrow \exists_{Q\in\mathbf{Q}_{\mathfrak{R}}}\big((x \perp y\ \vee\ \exists_{z\in Q}\ z \leqslant x \sqcap y)\ \wedge\ \forall_{z\in Q}(z \bigcirc x\ \wedge\ z \bigcirc y)\big)\Big), \tag{G}$$

called *Grzegorczyk axiom*, introduced in [5]. Any $\mathfrak{B}$ which satisfies all the aforementioned axioms is called *Grzegorczyk Contact Algebra* (GCA in short). Every such algebra is a Boolean Contact Algebra in the sense of [1].

A *point* of GCA is any filter generated by a pre-point:

$$\mathfrak{p} \text{ is a point iff } \exists_{Q\in\mathbf{Q}}\, \mathfrak{p} = \{x \in R \mid \exists_{q\in Q}\, q \leqslant x\}\,.$$

In every GCA we can introduce a topology in the set of all points, first by defining the set of all internal points of a region x:

$$\mathbf{Irl}(x) \coloneqq \{\mathfrak{p} \mid x \in \mathfrak{p}\}\,,$$

and second, taking all $\mathbf{Irl}(x)$ as a basis. The natural questions arise: what kind of topological spaces are determined by GCAs and what is the relation between GCAs and the boolean algebras of regular open sets of their topological spaces?

4 Representation theorems and topological duality

Let $\mathscr{T} = \langle S, \mathscr{O}\rangle$ be a topological space. The standard method for obtaining BCAs is via taking RO(X)—the complete algebra of all regular-open subsets of S— as regions, defining the connection relation C by:

$$U \mathrel{\mathsf{C}} V \Longrightarrow \operatorname{Cl} U \cap \operatorname{Cl} V \neq \emptyset\,.$$

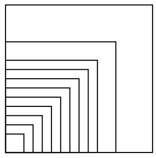

Figure 2: The set of rectangular regions is not a pre-point since the regions are not ordered by non-tangential inclusion.

This practice can be extended to GCAs as well, however one must narrow down the class of spaces, since not all topological spaces give rise to GCAs in the way just described.

In our talk we would like to focus on representation theorems for Grzegorczyk algebras. In particular we would like to show that there is a very strong kinship between GCAs and topological regular spaces. To this end we introduce the class of *concentric* spaces (a subclass of regular spaces), which are T_1-spaces such that in every point p there is a local basis B_p satisfying:

$$\forall_{U,V\in B_p}\big(U = V \vee \mathrm{Cl}\, U \subseteq V \vee \mathrm{Cl}\, V \subseteq U\big)\,. \tag{R1}$$

By an *ω-concentric* space we mean a space which at every point have a local countable basis which satisfies (R1). We prove, among others, the following theorems:

Theorem 1. *Every GCA is isomorphic to a dense subalgebra of a GCA for a concentric topological space.*

Every complete GCA is isomorphic to a GCA for a concentric topological space.

Theorem 2. *Every GCA with c.c.c. (countable chain condition) is isomorphic to a dense subalgebra of a GCA with c.c.c. for an ω-concentric topological space with c.c.c.*

Every complete GCA with c.c.c. is isomorphic to a GCA with c.c.c. for an ω-concentric topological space having c.c.c.

Theorem 3. *Every countable GCA is isomorphic to a dense subalgebra of a GCA for a second-countable regular space.*

Every complete countable GCA is isomorphic to a GCA for a second-countable regular space.

We also demonstrate the following topological duality theorem for a subclass of Grzegorczyk contact algebras:

Theorem 4 (Object duality theorem)**.** *Every complete GCA with c.c.c. is isomorphic to a GCA for a concentric space with c.c.c.; and every concentric c.c.c. space is homeomorphic to a concentric c.c.c. space for some complete GCA with c.c.c.*

The following two problems remain open:

1. full duality for algebras and topological spaces from Theorem 4,
2. duality for the full class of Grzegorczyk algebras (without c.c.c.).

5 Grzegorczyk and de Vries

Last, but not least, we would like to compare GCAs to de Vries constructions from [2]. Among others, we show that every Grzegorczyk point is a maximal concordant filter in the sense of [2] (*maximal round filter* or *end* in contemporary terminology), but not vice versa.

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On Intuitionistic Combinatorial Proofs

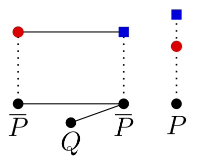
Willem B. Heijltjes, Dominic J. D. Hughes, and Lutz Straßburger

The objective of this presentation is simple to state:

1. Provide the most abstract, syntax-free representation of intuitionistic sequent calculus proofs possible, subject to:
2. Translation from a proof is polynomial-time.

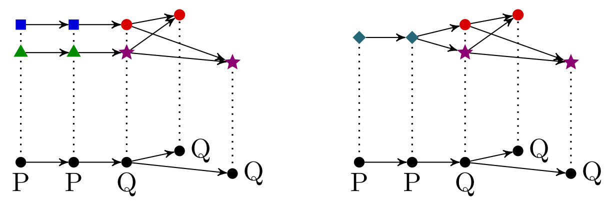
Conventional representations such as lambda calculus or game semantics fail to satisfy 2: by their extensional nature, they identify so many proofs that translation from a proof blows up exponentially in size.

Our solution is to define a notion of *combinatorial proof* for intuitionistic propositional sequent calculus. Combinatorial proofs were introduced as a syntax-free reformulation of classical propositional logic [Hug06a, Hug06b]. For example, here is a combinatorial proof of Peirce's Law $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$:



The lower graph abstracts the formula (one vertex per propositional variable, edges encoding conjunctive relationships); the upper graph has two colour classes, ●● and ■■, each expressing an axiom $P \Rightarrow P$; the dotted lines define a *skew fibration* from the upper graph to the lower graph, a lax notion of graph fibration. The upper graph captures the axioms and logical rules in a proof, the lower graph captures the formula proved, and the skew fibration captures all contraction and weakening, simultaneously and in parallel [Hug06b, Str07].

The intuitionistic setting required reformulating combinatorial proofs with directed edges for implicative relationships. Here are two intuitionistic combinatorial proofs on $(P \Rightarrow P) \Rightarrow Q \vdash Q \wedge Q$,



(1)

Each lower graph, called the *base*, is an abstraction of the formula (akin to a labelled a arena of game semantics [HO00]). Leaving base graphs implicit, we can render the combinatorial proofs compactly:

$(P \Rightarrow P) \Rightarrow Q \ \vdash \ Q \wedge Q$ $\qquad$ $(P \Rightarrow P) \Rightarrow Q \ \vdash \ Q \wedge Q$

Using this compact notation, Figure 1 shows step-by-step translations of intuitionistic sequent calculus proofs into the respective intuitionsitic combinatorial proofs above. Figure 2

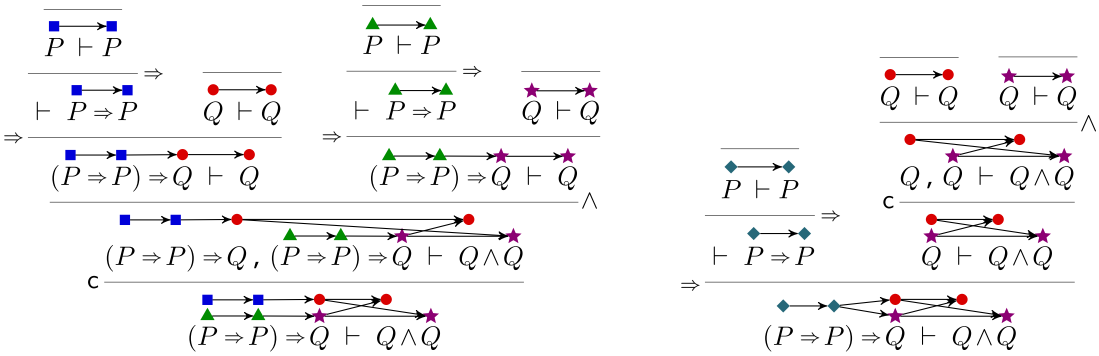


Figure 1: Translating two intuitionstic sequent calculus proofs to intuitionistic combinatorial proofs. The translation is very simple to define: (1) place a pair of tokens atop the propositional variables in each axiom, with a rightward directed edge; (2) trace the tokens down through the proof; (3) each left implication rule and right conjunction rule inserts edges.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:P\vdash x:P}}{\vdash \lambda x.x : P\Rightarrow P} \quad \dfrac{}{w:Q\vdash w:Q}}{f_1 : (P\Rightarrow P)\Rightarrow Q \ \vdash\ f_1(\lambda x.x) : Q} \quad \dfrac{\dfrac{\dfrac{}{y:P\vdash y:P}}{\vdash \lambda y.y : P\Rightarrow P} \quad \dfrac{}{v:Q\vdash v:Q}}{f_2 : (P\Rightarrow P)\Rightarrow Q \ \vdash\ f_2(\lambda y.y) : Q}}{\dfrac{f_1 : (P\Rightarrow P)\Rightarrow Q\ ,\ f_2 : (P\Rightarrow P)\Rightarrow Q \ \vdash\ \langle f_1(\lambda x.x), f_2(\lambda y.y)\rangle \ :\ Q\wedge Q}{f : (P\Rightarrow P)\Rightarrow Q \ \vdash\ \langle f(\lambda x.x), f(\lambda y.y)\rangle \ :\ Q\wedge Q}}$$

$$\dfrac{\dfrac{\dfrac{}{z:P\vdash z:P}}{\vdash \lambda z.z : P\Rightarrow P} \quad \dfrac{\dfrac{\dfrac{}{v_1:Q\vdash v_1:Q} \quad \dfrac{}{v_2:Q\vdash v_2:Q}}{v_1 : Q\ ,\ v_2 : Q \ \vdash\ \langle v_1, v_2\rangle \ :\ Q\wedge Q}}{v : Q \ \vdash\ \langle v, v\rangle \ :\ Q\wedge Q}}{f : (P\Rightarrow P)\Rightarrow Q \ \vdash\ \langle f(\lambda z.z), f(\lambda z.z)\rangle \ :\ Q\wedge Q}$$

Figure 2: Translating the same two intuitionstic sequent calculus proofs into lambda calculus terms. Note that (up to alpha-conversion, renaming bound variables x, y and z) the two terms are the same. On the right, the subterm $\lambda z.z$ from the left sub-proof is duplicated, because of extensionality. In contrast, the translation to a combinatorial proof does not require such a duplication: on the right of Figure 1, the final rule keeps only one pair of tokens over $P \Rightarrow P$, from the left sub-proof.

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shows the corresponding lambda calculus translations. The resulting lambda terms are identical (modulo alpha-conversion), and the right translation duplicates $\lambda z.z$. Because of iterated duplications, translation to a lambda term is exponential-time in the size of the proof. In contrast, translating the right proof to an intuitionistic combinatorial proof involves no duplication. More generally, a proof with n axioms translates to an intuitionistic combinatorial proof with n colour classes. Thus translation to an intuitionistic combinatorial proof is polynomial-time.

Just as the translation to lambda calculus is surjective, we can prove that the translation to intuitionistic combinatorial proofs is surjective. Thus intuitionistic combinatorial proofs are sound and complete for intuitionistic logic. We also prove that if two proofs are equivalent modulo rule commutations which do not involve duplications of entire subproofs, then they translate to the same combinatorial proofs. Taken together, these two theorems formalize the sense in which achieved the two goals stated at the start of this abstract.

In the presentation we will also compare the normalization procedures for classical combinatorial proofs (as presented in [Hug06b, Str17a, Str17b]) and for intuitionistic combinatorial proofs. A surprising observation is that in the intuitionistic case we need to rely on a normalization method for additive linear logic, as presented in [HH15].

If time permits, we will also show how we can translate between syntactic proofs and combinatorial proofs. Here we can observe for the intuitionistic case the same technical subtleties as for the classical case in [Hug06b, AS18].

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From Tableaux to Axiomatic Proofs
A Case of Relating Logic

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Abstract

The aim of our talk consists of three following elements: the first one is to semantically define relating logics, new kind of non-classical logics which enable to express that constituent propositions are related because of some, for instance causal or temporal, relationship; the second one is to present axiomatic and tableaux systems for the logics we study; the third one is connected with an idea of proof of soundness and completeness theorems based on a transition from a tableaux-like proof to an axiomatic one.

1 Introduction

The idea behind relating logic is simple one. Truth-values of compound propositions depend not only on logical values but also whether their component propositions are related. Consider the following examples:

1. Jan arrived in Amsterdam and took part in conference SYSMICS 2019.
2. If you turned off the light, in the room was pretty dark.

In these propositions we want to express more than extensional relations. Propostion 1 might be false even if both component propositions are true, because it is also required that Jan first arrived in Amsterdam and then took part in conference SYSMICS 2019. Similarly in case of proposition 2. This time we require that the fact of turning off the light to be a cause of dark in a room. It is easy to imagine further examples concerning analytic relationship (cf. [2, 115–120]) or content relationship (relevance) in a general sense (cf. [1], [2, 61–72], [5], [6]). Such examples lead to an idea of relating connectives which enable to express standard extensional dependences and non-extensional ones, some kind of intensional relations.

Relating logics are based on an interpretation of propositions which involves two factors: logical value and relation between propositions. In the talk we define them semantically and introduce syntactic approach by means of axiomatic and tableaux systems.

2 Language and Semantics of Relating Logic

Language $\mathcal{L}$ of relating logic consists of propositional variables, Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, relating connectives which are counterparts of Boolean two argument connectives $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$ and brackets.[1] A set of formulas in $\mathcal{L}$ is defined in the standard way:

$$\mathrm{For} \ni A ::= p_n \mid \neg A \mid (A \star A) \mid (A \star^{\mathrm{w}} A),$$

where $n \in \mathbb{N}$ and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

[1] We use letter w for notation of relating connectives because of Polish words wiązać, wiążący, which might be translated as relate, relating.

A model based on relating relation is an ordered pair $\langle v, \mathrm{R}\rangle$ such that $v\colon \mathrm{Prop} \longrightarrow \{0,1\}$ is a valuation of propositional variables and $\mathrm{R} \subseteq \mathrm{For} \times \mathrm{For}$ is a relating relation. Relation R as an element of semantic structure enables to express in metalanguage, in a quite natural way, that some propositions are somehow related. Such relation is supposed to simulate or model various kinds of relationships like temporal or causal one.

An interpretation of propositional variables and formulas build by means of Boolean connectives are defined in the standard way. According to intuitions presented in an introduction we assume the following interpretation in case of formulas build by relating connectives:

$$\langle v, \mathrm{R}\rangle \models A \star^{\mathrm{w}} B \text{ iff } [\langle v, \mathrm{R}\rangle \models A \star B \text{ and } \mathrm{R}(A, B)].$$

We might define notion of a truth with respect to a relating relation R, i.e $\mathrm{R} \models A$ iff $\forall_{v \in \{0,1\}^{\mathrm{Prop}}} \langle v, \mathrm{R}\rangle \models A$. A notion of tautology might be defined in two ways: as a true formula in all models and as a true formula with respect to all relating relations.

A logic is identified with a set of tautologies determined on the ground of some non-empty class of relating relations. The smallest relating logic is logic W (cf. [5]). Its extensions are defined by means of various classes of relations which are determined by some relational conditions. We distinguish three types of classes of relating relations:

- a horizontal class (for short: h-class) — a class of relations satisfying conditions in which we do not refer to complexity of formulas;
- a vertical class (for short: v-class) — a class of relations satisfying some conditions in which we refer to complexity of formulas
- a diagonal class (for short: d-class) — a class which is horizontal and vertical at the same time.

Examples of h-classes are: class of reflexive relations, class of symmetric relations or class of transitive relations (cf. [1], [2, 61–143], [5], [6]).

An example of v-class is a class of relations of eliminations and introductions of binary connectives (cf. [1], [2, 61–143], [5], [6]), i.e. the class of relations determined by the following conditions:

$$\begin{aligned}
&(\circ_1 \Rightarrow \text{or}) \;\; \mathrm{R}(A \circ B, C) \Longrightarrow [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)]\\
&(\text{or} \Rightarrow \circ_1) \;\; [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)] \Longrightarrow \mathrm{R}(A \circ B, C)\\
&(\circ_2 \Rightarrow \text{or}) \;\; \mathrm{R}(A, B \circ C) \Longrightarrow [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)]\\
&(\text{or} \Rightarrow \circ_2) \;\; [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)] \Longrightarrow \mathrm{R}(A, B \circ C).
\end{aligned}$$

Examples of d-classes are intersections of h-classes and v-classes, for instance a class of reflexive relations which are relations of eliminations and introductions of connectives (cf. [1], [2, 61–143], [5], [6]).

Because of three types of classes of relations we distinguish three types of extensions of W, i.e. h-logics (determined by h-classes), v-logics (determined by v-classes) and d-logics (determined by d-classes). In the talk we are going to focus on logics determined by the distinguished classes.

3 Axiomatic and Tableaux Systems of Relating Logics

In many cases relating logics are not difficult to axiomatize. The most important thing we need to know is that relating relation is expressible in language $\mathcal{L}$ by formula $(A \vee^{\mathrm{w}} B) \vee (A \rightarrow^{\mathrm{w}} B)$.

Let $A \looparrowright B := (A \vee^{\mathrm{w}} B) \vee (A \rightarrow^{\mathrm{w}} B)$. Axiomatic system of W consists of the following axiom schemata:

$$(A \star^{\mathrm{w}} B) \leftrightarrow ((A \star B) \wedge (A \looparrowright B)). \qquad (\mathrm{ax}_{\star^{\mathrm{w}}})$$

The only rules of inference is modus ponens:

$$\frac{A, A \rightarrow B}{B}$$

By means of $\looparrowright$ it is quite easy to present axiom schemata which express relational conditions that characterize a class of relation. For instance, we have that:

$A \looparrowright A$	expresses condition of reflexivity
$(A \looparrowright B) \rightarrow (B \looparrowright A)$	expresses condition of symmetry
$((A \looparrowright B) \wedge (B \looparrowright C)) \rightarrow (A \looparrowright C)$	expresses condition of reflexivity.

In turn, tableaux systems might be defined by some methods introduced in [3], [4] and [5]. A set of tableaux expressions is an union of For and the following sets $\{A\mathbf{r}B : A, B \in \mathrm{For}\}$, $\{A\bar{\mathbf{r}}B : A, B \in \mathrm{For}\}$. By expressions $A\mathbf{r}B$ and $A\bar{\mathbf{r}}B$ we say that formulas A, B are related and not related respectively. A tableaux inconsistency is either A and $\neg A$ or $A\mathbf{r}B$ and $A\bar{\mathbf{r}}B$.

In order to define a sound and complete tableaux system for a relating logic we use standard rules for formulas build by Boolean connectives (cf. [3], [4] and [5]). Then it is easy to determine tableaux rules concerning relating connectives:

$$(\mathrm{R}_{\neg\star^{\mathrm{w}}})\ \frac{\neg(A \star^{\mathrm{w}} B)}{\neg(A \star B) \mid A\bar{\mathbf{r}}B} \qquad (\mathrm{R}_{\star^{\mathrm{w}}})\ \frac{A \star^{\mathrm{w}} B}{\begin{array}{c} A \star B \\ A\mathbf{r}B \end{array}}$$

Specific rules which enable to express relational conditions are also not difficult to express. For instance, for reflexivity, transitivity and condition $(\mathrm{or} \Rightarrow \circ_1)$ we have:

$$(\mathrm{R}_{\mathrm{r}})\ \frac{A}{A\mathbf{r}A} \qquad (\mathrm{R}_{\mathrm{t}})\ \frac{A\mathbf{r}B, B\mathbf{r}C}{A\mathbf{r}C} \qquad (\mathrm{R}_{1:\mathrm{or}\Rightarrow\circ_1})\ \frac{A\mathbf{r}C}{(A \circ B)\mathbf{r}C} \qquad (\mathrm{R}_{2:\mathrm{or}\Rightarrow\circ_1})\ \frac{B\mathbf{r}C}{(A \circ B)\mathbf{r}C}$$

where B in rule $(\mathrm{R}_{1:\mathrm{or}\Rightarrow\circ_1})$ and A in rule $(\mathrm{R}_{2:\mathrm{or}\Rightarrow\circ_1})$ already appeared on a branch.

In the talk, we will focus on a problem of passing from tableaux-like proof to axiomatic proof. The main theorem we would like to present will say that if formula A is a branch consequence of set Σ (i.e A has proof from Σ in a tableaux system), then it is also an axiomatic consequence of Σ (i.e. A has proof from Σ in an axiomatic system).

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Below Gödel-Dummett

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The Gödel-Dummett logic LC from [9] is a strengthening of intuitionistic logic IPC with linear Kripke-models. It can be axiomatized by many different axiom schemes:

($\mathcal{L}_1$) $(A \to B) \vee (B \to A)$
($\mathcal{L}_2$) $(A \to B) \vee ((A \to B) \to A)$
($\mathcal{L}_3$) $(A \to B) \vee ((A \to B) \to B)$
($\mathcal{L}_4$) $(A \to B \vee C) \to (A \to B) \vee (A \to C)$
($\mathcal{L}_5$) $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$
($\mathcal{L}_6$) $((A \to B) \to B) \wedge ((B \to A) \to A) \to A \vee B$.

An even larger number of equivalents arises by the fact that in IPC $\vdash A \vee B$ iff $\vdash (A \to C) \wedge (B \to C) \to C$ (DR), and, more generally, $\vdash D \to A \vee B$ iff $\vdash D \wedge (A \to C) \wedge (B \to C) \to C$ (EDR).

For strong completeness of LC see e.g. [13]. In the present research in progress we study logics with linear models originating from logics weaker than IPC. Weaker logics than IPC are the subintuitionistic logics with Kripke models extending F studied by [4, 6] and those with neighborhood models extending WF originated in [7, 12]. Linear extensions of those logics have already been obtained in the case of BPC, the extension of F with transitive persistent models [1, 2, 14]. Our object is to study the character of and the relations between the schemes ($\mathcal{L}_1$), . . .,($\mathcal{L}_6$). Besides syntactic methods we use the construction of neighborhood frames [3] for various logics. We also obtain modal companions for a number of the logics. Hájek's basic fuzzy logic BL [10] compares less well with IPC, and is therefore left out of consideration this time.

Extensions of Corsi's logic F. The logic F is axiomatized by

1. $A \to A \vee B$	*7.* $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
2. $B \to A \vee B$	*8.* $(A \to B) \wedge (B \to C) \to (A \to C)$
3. $A \wedge B \to A$	*9.* $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
4. $A \wedge B \to B$	*10.* $A \to A$
5. $\frac{A \quad B}{A \wedge B}$	*11.* $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
6. $\frac{A \quad A \to B}{B}$	*12.* $\frac{A}{B \to A}$

The axioms 8, 9 and 11 are more descriptively named I, C and D. Corsi [4] proved completeness for Kripke models with an arbitrary relation R without stipulation of persistence of truth.

The axioms needed to obtain IPC from F are

R: $A \wedge (A \to B) \to B$ (defines and is complete for reflexive Kripke frames)
T: $(A \to B) \to ((B \to C) \to (A \to C))$ (defines and is complete for transitive Kripke frames)

P: $p \to (\top \to p)$ (defines and is complete for persistent Kripke models).

Visser's basic logic BPC can be defined as FTP. In the case of Kripke models we mean with linear models of course connected ($\forall xyz\,((xRy \wedge\ xRz) \to (y \neq z \to yRz \vee zRy))$) and transitive models. (Anti-symmetry is covered by persistence.) Visser already proved that over BPC, $\mathcal{L}_2$ is complete with regard to linear models, and that $\mathcal{L}_1$ is not [14], see also [2]. We prove that $\mathcal{L}_1$, $\mathcal{L}_4$ and $\mathcal{L}_5$ are equivalent over F. Moreover, we show that $\mathcal{L}_1$ plus $\mathcal{L}_3$ prove $\mathcal{L}_2$ in F, so $\mathcal{L}_1$ plus $\mathcal{L}_3$ is complete for linear models over BPC. We didn't study DR and EDR in depth yet, but were able to prove that the right-to-left direction of DR can be proved in F, but for DFR one needs FR. The left-to-right direction can be executed in FR in both cases.

Neighborhood models and extensions of the logics WF and $\mathsf{WF_N}$. The logic WF can be obtained by deleting the axioms C, D and I from F, and replacing them by the corresponding rules like concluding $A \to B \wedge C$ from $A \to B$ and $A \to C$ (see [12]).

Neighborhood frames describing the natural basic system WF have been obtained in [12]. These NB-neighborhoods consist of pairs (X, Y) with the X and Y corresponding to the antecedent and consequent of implications.

Definition 1. *$\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ is called an **NB-frame** of subintuitionistic logic if $W \neq \emptyset$ and $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$, and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by $U \to V := \{w \in W \mid (U, V) \in NB(w)\}$, where $NB\colon W \to P(\mathcal{X}^2)$ is such that: $\forall w \in W,\ \forall X, Y \in \mathcal{X},\ (X \subseteq Y\ \Rightarrow\ (X, Y) \in NB(w))$.*

If $\mathfrak{M}$ is a model on such a frame, $\mathfrak{M}, w \Vdash A \to B$ iff $(V(A), V(B)) \in NB(w)$. Also N-neighborhood frames, closer to the neighborhood frames of modal logic, were described. In those frames $\overline{X} \cup Y$ corresponds to implications. An additional rule N [5, 7] axiomatizes them:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad (\mathsf{N})$$

WF plus the rule N is denoted by $\mathsf{WF_N}$. For extensions of $\mathsf{WF_N}$ modal companions can often be found.

Again we can see linearity as the combination of connectedness and transitivity of the neighborhood frames. But, of course, connectedness as well as transitivity now concerns sets of worlds (neighborhoods), not individual worlds. NB-frames are called transitive if, for all $(X, Y) \in NB(w), (Y, Z) \in NB(w)$ we have $(X, Z) \in NB(w)$ as well. The formula I defines this property and is complete for the transitive NB-frames. For the N-frames this becomes, for all $\overline{X} \cup Y \in N(w), \overline{Y} \cup Z \in N(w)$ we have $\overline{X} \cup Z \in N(w)$ as well. This too is defined by I, and $\mathsf{WF_N}$ is complete for the transitive N-frames. We cannot say that the connection between transitivity and connectedness in Kripke and neighborhood frames has completely been cleared up. Note that the axiom I for transitivity of the neighborhood frames, which is provable in F, is weaker than the axiom T for transitivity of the Kripke models. On the other hand, in the canonical models of logics like $\mathsf{I}\mathcal{L}_1$ the worlds are linearly ordered by inclusion. Study of the neighborhood frames for BPC and IPC of [11] may further clarify the matter.

The IPC-equivalents of the introduction all define different connectedness properties. Definability and completeness of these logics is part of the present paper. For example the straightforward

$$\text{for all } X, Y \in \mathcal{X}^2,\ (X, Y) \in NB(w) \text{ or } (Y, X) \in NB(w)$$

is called connected$_1$ by us and is defined by $\mathcal{L}_1$. This formula defines a similar property in the case of N-frames, and is complete for those frames as well.

We can refine the results of the section on F by discussing in which extensions of WF the results are provable.

Proposition 1. *$\mathsf{I}\mathcal{L}_1$, $\mathsf{I}\mathcal{L}_4$ and $\mathsf{I}\mathcal{L}_5$ are equivalent over WF.*

Proposition 2. *$\mathsf{WFI}\mathcal{L}_1\mathcal{L}_3$ proves $\mathsf{I}\mathcal{L}_2$.*

The opposite direction is open.

Proposition 3. *$\mathsf{WF_N IR}\mathcal{L}_1 \Vdash \mathcal{L}_2$.*

Proposition 4. *$\mathsf{WF_N IR}\mathcal{L}_2 \Vdash \mathcal{L}_3$.*

Modal companions. We consider the translation $\Box$ from L, the language of propositional logic, to $L_\Box$, the language of modal propositional logic. It is given by:

$p^\Box = p$;
$(A \wedge B)^\Box = A^\Box \wedge B^\Box$;
$(A \vee B)^\Box = A^\Box \vee B^\Box$;
$(A \to B)^\Box = \Box(A^\Box \to B^\Box)$.

This translation was discussed independently by both [4] and [8] for subintuitionistic logics with Kripke models. We discussed it for extensions of $\mathsf{WF_N}$ in [5, 7]. For example the extension EN of classical modal logic E is a modal companion of $\mathsf{WF_N}$. Here we get modal companions for all of the extensions of $\mathsf{WF_N}$ that we discuss. For example we obtain as a modal companion of the logic $\mathsf{WF_N}\mathcal{L}_1$ the modal logic ENL_1 axiomatized over EN by $\mathsf{L}_1 : \Box(A \to B) \vee \Box(B \to A)$.

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Syntactical approach to Glivenko-like theorems

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Glivenko's theorem [6] asserts that provability in classical logic CL can be translated to that of intuitionistic logic IL by means of double negation. Since then many translations of this kind where proved, e.g. between the Łukasiewicz logic Ł and Hájek's basic fuzzy logic BL [2], or between classical logic CL and the product fuzzy logic Π [1]. It is a common phenomenon that these results are obtained by algebraic (i.e. semantic) means. In this talk we provide a syntactic account of these results. The results presented in this contribution can be found in [8].

Our main result characterizes those extensions of a certain basic substructural logic which are Glivenko-equivalent to classical logic. The proof of this result relies on the notion of an inconsistency lemma (introduced by Raftery [9] and further studied in [8]) and the new notion of an antistructural completion [8]. In particular, it proceeds by identifying sufficient conditions under which classical logic is the antistructural completion of a given substructural logic.

Inconsistency lemmas are equivalences relating inconsistency and validity in a given logic, just like deduction-detachment theorems are equivalences relating theoremhood and validity. For example, it is well known that the following equivalence holds in intuitionistic logic:

$$\Gamma, \varphi \vdash \bot \iff \Gamma \vdash \neg\varphi. \tag{1}$$

This property was first explicitly isolated and systematically studied by Raftery [9], who called such an equivalence an *inconsistency lemma*. He also considered the following dual version of this property, which he called a *dual inconsistency lemma*:

$$\Gamma, \neg\varphi \vdash \bot \iff \Gamma \vdash \varphi. \tag{2}$$

This equivalence, of course, is no longer valid in intuitionistic logic but it does hold for classical logic. (In fact it is the law of excluded middle in disguise.)

We continue the line of research initiated by Raftery and introduce what we call local and parametrized local versions of these properties, by analogy with the so-called local and parametrized local deduction-detachment theorems. This yields a hierarchy of inconsistency lemmas similar to the existing hierarchy of deduction-detachment theorems (see e.g. [4]).

Let us illustrate what the local form of Raftery's inconsistency lemma and dual inconsistency lemma looks like. For example, Hájek's basic logic BL enjoys a *local* inconsistency lemma in the following form:

$$\Gamma, \varphi \vdash \bot \iff \Gamma \vdash \neg\varphi^n \text{ for some } n \in \omega.$$

On the other hand, the infinitary Łukasiewicz logic $Ł_\infty$ (i.e. the infinitary consequence relation of the standard Łukasiewicz algebra on the unit interval $[0, 1]$) enjoys a *dual local* inconsistency lemma in the following form (note the universal rather than existential quantifier here):

$$\Gamma, \neg\varphi^n \vdash \bot \text{ for all } n \in \omega \iff \Gamma \vdash \varphi.$$

We remark that the finitary companion of $Ł_\infty$, i.e. the finitary Łukasiewicz logic Ł, validates the above mentioned dual local inconsistency lemma for finite sets of formulas Γ.

The second component of our proof is the notion of an antistructural completion, which is the natural dual to the notion of a structural completion (see [7]). Recall that the *structural completion* of a logic L is the largest logic $\sigma\mathrm{L}$ which has the same theorems as L. A logic L is then called *structurally complete* if $\sigma\mathrm{L} = \mathrm{L}$. The logic $\sigma\mathrm{L}$ exists for each L and it has a simple description: $\Gamma \vdash_{\sigma\mathrm{L}} \varphi$ if and only if the rule $\Gamma \vdash \varphi$ is *admissible* in L, i.e. for each substitution τ we have $\emptyset \vdash_{\mathrm{L}} \tau\varphi$ whenever $\emptyset \vdash_{\mathrm{L}} \tau\gamma$ for each $\gamma \in \Gamma$.

Dually, the *antistructural completion* of a logic L is defined as the largest logic $\alpha\mathrm{L}$, whenever it exists, which has the same inconsistent (or equivalently, maximally consistent) sets as L. Naturally, a logic L is called *antistructurally complete* if $\alpha\mathrm{L} = \mathrm{L}$. For example, Glivenko's theorem essentially states that $\alpha\mathrm{IL} = \mathrm{CL}$.

Just like $\sigma\mathrm{L}$ can be characterized in terms of admissible rules, the antistructural completion of $\alpha\mathrm{L}$ of L can be characterized in terms of *antiadmissible rules.* These are rules $\Gamma \vdash \varphi$ which for every substitution σ and every set of formulas Δ satisfy the implication:

$$\{\sigma\varphi\} \cup \Delta \text{ is inconsistent in L} \implies \sigma\Gamma \cup \Delta \text{ is inconsistent in L.}$$

Moreover, in many contexts the description of antiadmissible rules can be simplified by omitting the quantification over substitutions. This yields what we call *simply antiadmissible rules*, which for every set of formulas Δ satisfy the implication:

$$\{\varphi\} \cup \Delta \text{ is inconsistent in L} \implies \Gamma \cup \Delta \text{ is inconsistent in L.}$$

Our first main result now ties all these notions together.

Theorem. *Let* L *be a finitary logic with a local inconsistency lemma. Then the following are equivalent:*

1. L *is antistructurally complete.*
2. *Every simply antiadmissible rule is valid in* L.
3. L *enjoys the local* dual *inconsistency lemma.*
4. L *is* semisimple *(i.e. subdirectly irreducible models of* L *are simple).*
5. L *is complete w.r.t. (a subclass of) the class of all simple models of* L.

The above characterization (especially points 4. and 5.) provides a wealth of examples of antistructurally complete logics: e.g. the global modal logic S5, the k-valued Łukasiewicz logics $\mathrm{Ł}_k$, or the infinitary Łukasiewicz logic $\mathrm{Ł}_\infty$.

With the above result in hand, we now proceed to describe the substructural logics Glivenko-equivalent to classical logic. Here we say that a logic L' is *Glivenko-equivalent* to L if

$$\Gamma \vdash_{\mathrm{L}'} \neg\neg\varphi \iff \Gamma \vdash_{\mathrm{L}} \varphi.$$

Our weakest substructural logic is the logic SL (see [3]), which corresponds the bounded nonassociative full Lambek calculus. It is introduced in a standard substructural language

$$\{\wedge, \vee, \&, \to, \rightsquigarrow, \bar{0}, \bar{1}, \top, \bot\}$$

consisting of lattice conjunction and disjunction, strong conjunction & and its right and left residuals $\to$ and $\rightsquigarrow$, and four constants, $\top$ being a lattice top and $\bot$ a lattice bottom. Moreover, we consider the following two defined negations: $\neg\varphi = \varphi \to \bar{0}$ and $\sim\varphi = \varphi \rightsquigarrow \bar{0}$. We identify substructural logics with finitary extensions of SL. (The two implications and negations are equivalent in extensions which validate the axiom of Exchange.)

Theorem. *For every substructural logic* L*, the following are equivalent:*

1. L *is Glivenko-equivalent to classical logic, i.e. for every* $\Gamma \cup \{\varphi\} \subseteq Fm$
$$\Gamma \vdash_{\mathrm{L}} \neg\neg\varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \varphi.$$
2. L *"almost" has the inconsistency lemma of* IL*, i.e.*
$$\Gamma, \varphi \vdash_{\mathrm{L}} \overline{0} \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi, \tag{3}$$
and moreover the following rules are valid in L*:*
$$\neg(\varphi \to \psi) \vdash \neg(\neg\neg\varphi \to \sim\neg\psi) \tag{A}$$
$$\neg(\varphi \mathbin{\&} \neg\psi) \dashv\vdash \neg(\varphi \wedge \neg\psi). \tag{Conj}$$

Furthermore, if $\overline{0}$ *is an inconsistent set in* L *then both properties imply that* $\alpha\mathrm{L} = \mathrm{CL}$*.*

The main idea of a proof of the more interesting direction (2. implies 1.) is the following. Extend L to L_0 by a rule $\overline{0} \vdash \bot$ (i.e. $\overline{0}$ proves everything) and show that the antistructural completion of L_0 is the classical logic. This can be established by purely syntactic means from the assumptions using the following lemma connecting inconsistency lemmas, duals inconsistency lemmas, and antistructural completions.[1]

Lemma. *Let* L *be a substructural logic satisfying the inconsistency lemma of intuitionistic logic* (1)*. Then:*

1. $\alpha\mathrm{L}$ *also satisfies* (1)*.*
2. $\alpha\mathrm{L}$ *satisfies the* dual *inconsistency lemma of classical logic, i.e.* (2)*.*
3. $\alpha\mathrm{L}$ *validates the law of excluded middle, i.e.* $\varphi \vee \neg\varphi$ *is its theorem.*
4. *As a consequence of* (1) *and* (2)*,* $\alpha\mathrm{L}$ *enjoys a deduction-detachment theorem in the form:*
$$\Gamma, \varphi \vdash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \neg(\varphi \wedge \neg\psi).$$

This theorem provides a simple strategy for finding the smallest axiomatic extension of a given substructural logic which is Glivenko-equivalent to classical logic, a problem investigated e.g. in [5]. Namely, given a substructural logic, first extend it by the rules (Conj) and (A) in the form of axioms. Secondly, find axioms which ensure the validity of (3). To this end, use a known deduction-detachment theorem.

Let us for example consider the full Lambek calculus with exchange FL_e (i.e. SL with & being commutative and associative). It is well known that this logic enjoys a local deduction-detachment theorem which in particular yields the equivalence
$$\Gamma, \varphi \vdash_{\mathrm{FL}_e} \overline{0} \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{FL}_e} \neg(\varphi \wedge \overline{1})^k \text{ for some } k \in \omega.$$
Thus we only need to add axioms ensuring that $\neg(\varphi \wedge \overline{1})^k \dashv\vdash \neg\varphi$ in our extension. This can be achieved e.g. by adding the axioms $\neg(\varphi \wedge \overline{1}) \to \neg\varphi$ and $\neg(\varphi \mathbin{\&} \psi) \to \neg(\varphi \wedge \psi)$. Moreover, these axioms can be proved to hold in each extension of FL_e Glivenko-equivalent to CL.

Finally, let us remark that inconsistency lemmas and antistructural completions can be used in a similar fashion to establish the Glivenko-equivalence between Łukasiewicz logic and Hájek's basic fuzzy logic BL.

[1] We remark that the lemma can proved in much grater generality than presented here.

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Countermodels for non-normal modal logics via nested sequents*

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The proof-theoretic framework of *nested sequents* has been very successful in treating normal modal logics. It is used, e.g., for providing modular calculi for all the logics in the so-called modal cube, for tense logics, as well as for modal logics based on propositional intuitionistic logic [2, 5, 9, 11]. The success of this framework might be due to the fact that it provides an ideal meeting point between syntax and semantics: On the one hand, nested sequents can be seen as purely syntactic extensions of the sequent framework with a structural connective corresponding to the modal box. On the other hand, due to the inherent similarity of the underlying tree structure to Kripke models, the nested sequent framework lends itself to very direct methods of countermodel construction from failed proof search by essentially reading off the model from a saturated and unprovable nested sequent. However the full power and flexibility of this framework so far has not yet been harnessed in the context of *non-normal* modal logics. While a first attempt at obtaining nested sequent calculi for non-normal modal logics indeed yielded modular calculi for a reasonably large class of non-normal modal logics by decomposing standard sequent rules [7, 8], the obtained calculi were not shown to inhibit the analogous central spot between syntax and semantics for these logics. In particular, no formula interpretation of the nested sequents was provided, and the calculi were not used to obtain countermodels from failed proof search.

Here we propose an approach to rectify this situation by considering *bimodal* versions of the non-normal modal logics. Such logics seem to have been considered originally in [1] in the form of *ability logics*, but their usefulness extends far beyond this particular interpretation. The main idea is that the neighbourhood semantics of non-normal monotone modal logics naturally gives rise to a second modality, which conveniently is normal. Here we concentrate on one of the most fundamental non-normal modal logics, *monotone modal logic* M [3, 4, 10], and present a nested sequent calculus for its bimodal version. Notably, the nested sequents have a formula interpretation in the bimodal language, and the calculus facilitates the construction of countermodels from failed proof search in a slightly modified version. An additional benefit is that the calculus conservatively extends both the standard nested sequent calculus for normal modal K from [2, 11] and the nested sequent calculus for monotone modal logic M from [7, 8].

The set $\mathcal{F}$ of formulae of bimodal monotone modal logic is given by the following grammar, built over a set $\mathcal{V}$ of propositional variables:

$$\mathcal{F} ::= \bot \mid \mathcal{V} \mid \mathcal{F} \to \mathcal{F} \mid \langle\exists\forall\rangle\mathcal{F} \mid [\forall\forall]\mathcal{F}$$

The remaining propositional connectives are defined by their usual clauses. The semantics are given in terms of *neighbourhood semantics* in the following way, also compare [1, 3, 10].

Definition 1. A *neighbourhood model* is a tuple $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ consisting of a universe W, a *neighbourhood function* $\mathcal{N} : W \to 2^{2^W}$, and a *valuation* $[\![.]\!] : \mathcal{V} \to 2^W$.

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Definition 2. The *truth set of a formula* A in a model $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ is written as $[\![A]\!]$ and extends the valuation $[\![.]\!]$ of the model by the propositional clauses $[\![\bot]\!] = \emptyset$ and $[\![A \to B]\!] = [\![A]\!]^c \cup [\![B]\!]$ together with

- $[\![\langle\exists\forall] A]\!] = \{w \in W \mid \text{ exists } \alpha \in \mathcal{N}(w) \text{ s.t. for all } v \in \alpha : v \in [\![A]\!]\}$
- $[\![[\forall\forall] A]\!] = \{w \in W \mid \text{ for all } \alpha \in \mathcal{N}(w) \text{ and for all } v \in \alpha : v \in [\![A]\!]\}$

If $w \in [\![A]\!]$ we also write $\mathfrak{M}, w \Vdash A$. A formula A is *valid in* M, if for every model $[\![A]\!] = W$.

Hence, the formulation of the truth conditions for the modal operator of monomodal monotone logic in terms of an "exists forall" clause naturally yields the definition of the operator $[\forall\forall]$ in terms of a "forall forall" clause. This can be rewritten into the clause $[\![[\forall\forall]A]\!] = \{w \in W \mid \text{ for all } v \in \bigcup\mathcal{N}(w) : v \in [\![A]\!]\}$ which immediately yields normality of the modality $[\forall\forall]$, since we can take $\bigcup\mathcal{N}(w)$ as the set of successors of w. In particular, it can be seen that the modality $[\forall\forall]$ behaves like a standard K-modality.

In order to capture both modalities $\langle\exists\forall]$ and $[\forall\forall]$ in the nested sequent framework, we introduce the two corresponding structural connectives $\langle . \rangle$ and $[.]$ respectively, with the peculiarity that nested occurrences of these connectives are allowed only in the scope of the latter:

Definition 3. A *nested sequent* has the form

$$\Gamma \Rightarrow \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \ldots, \langle \Sigma_n \Rightarrow \Pi_n \rangle, [\mathcal{S}_1], \ldots, [\mathcal{S}_m] \tag{1}$$

for $n, m \geq 0$, where $\Gamma \Rightarrow \Delta$ as well as the $\Sigma_i \Rightarrow \Pi_i$ are standard sequents, and the $\mathcal{S}_j$ are nested sequents. The *formula interpretation* of the above nested sequent is

$$\bigwedge \Gamma \to \Big(\bigvee \Delta \vee \bigvee\nolimits_{i=1}^{n} \langle\exists\forall] (\bigwedge \Sigma_i \to \bigvee \Pi_i) \vee \bigvee\nolimits_{j=1}^{m} [\forall\forall] \, \iota(\mathcal{S}_j) \Big)$$

where $\iota(\mathcal{S}_j)$ is the formula interpretation of $\mathcal{S}_j$.

In order to obtain a nested sequent calculus for M we need to make sure that applicability of the propositional rules does not enforce normality of the interpretation of the structural connective $\langle . \rangle$. In particular, we cannot permit application of, e.g., the initial sequent rule inside the scope of $\langle\exists\forall]$ – otherwise the formula interpretation of the nested sequent $\Rightarrow \langle p \Rightarrow p \rangle$, i.e., $\langle\exists\forall](p \to p)$ would need to be a theorem, which is not the case in bimodal M.

Definition 4. The nested sequent rules of the calculus $\mathcal{N}_{\mathsf{M}}$ are given in Fig. 1. The rules can be applied anywhere inside a nested sequent except for inside the scope of $\langle . \rangle$.

Soundness of the rules with respect to the formula interpretation can then be shown as usual by obtaining a countermodel for the formula interpretation of the premiss(es) of a rule from a countermodel for the formula interpretation of its conclusion:

Proposition 5. *The rules of Fig. 1 are sound for* M *under the formula interpretation.*

A relatively straightforward proof of completeness for the calculus $\mathcal{N}_{\mathsf{M}}$ can be obtained by using the completeness result for the Hilbert-style axiomatisation of bimodal M in [1] as follows. The axioms given there can be converted into rules of a cut-free standard sequent calculus using, e.g., the methods of [6]. Then, cut-free derivations in the resulting sequent calculus can be converted into cut-free derivations in the nested sequent calculus $\mathcal{N}_{\mathsf{M}}$ along the lines of [7]. Hence together with the previous proposition we obtain:

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \to_R \qquad \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}\ \to_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\forall\forall]A}\ [\forall\forall]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\forall\forall]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\forall\forall]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\exists\forall]A}\ \langle\exists\forall]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle\exists\forall]_L \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\forall\forall]A, \langle\Sigma \Rightarrow \Pi\rangle}\ W$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{ICL} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{ICR} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{W}$$

Figure 1: The nested sequent rules of the calculus $\mathcal{N}_\mathsf{M}$ for the bimodal system.

Theorem 6. *The calculus in Fig. 1 is sound and complete for bimodal monotone modal logic, i.e.: A formula A is a theorem of* M*, if and only if the nested sequent* $\Rightarrow A$ *is derivable in* $\mathcal{N}_\mathsf{M}$.

As an interesting corollary of the sketched completeness proof we even obtain cut-free completeness of the calculus $\mathcal{N}_\mathsf{M}$ restricted to *linear nested sequents*, i.e., the subclass of nested sequents where we restrict m to be at most 1 in 1 along the lines of [7].

Due to the structure of the rules of $\mathcal{N}_\mathsf{M}$, in a derivation of a formula of the $[\forall\forall]$-fragment of M neither the connective $\langle\exists\forall]$ nor its structural version $\langle.\rangle$ occur. Hence, as a further corollary of Thm. 6, the calculus obtained by dropping the rules $\langle\exists\forall]_R, \langle\exists\forall]_L, W$ from $\mathcal{N}_\mathsf{M}$ is complete for this fragment, which is normal modal logic K. Since the rules $[\forall\forall]_R, [\forall\forall]_L$ are exactly the modal right and left rules in the standard nested sequent calculus for modal logic K from [2, 11], this immediately yields a completeness proof for that calculus seen as a fragment of $\mathcal{N}_\mathsf{M}$.

Moreover, by dropping the rules $[\forall\forall]_R, [\forall\forall]_L, W$ from $\mathcal{N}_\mathsf{M}$ we obtain the calculus for monomodal M from [7, 8]. Hence we also obtain a completeness proof for that calculus, together with a formula interpretation, albeit the latter only in the language extended with $[\forall\forall]$. Thus the calculus $\mathcal{N}_\mathsf{M}$ can be seen as a generalisation and combination of both the standard nested sequent calculus for normal modal logic K and the linear nested sequent calculus for monomodal monotone logic M. This seems to support the intuition that bimodal M can be seen as a refinement of modal logic K, where the set of successor states $\bigcup\mathcal{N}(w)$ is further structured by $\mathcal{N}$, a structure which is accessible through the additional connective $\langle\exists\forall]$.

So far the presented nested sequent calculus eliminates one of the shortcomings of the calculi in [7, 8], namely the lack of a formula interpretation. In addition, it facilitates a semantic proof of completeness by constructing a countermodel from failed proof search. The intuition is the same as for normal modal logics: the nodes in a saturated unprovable nested sequent yield the worlds of a Kripke-model. Here the *nodes* of a nested sequent are separated by the $[.]$ operator, so that every node contains a standard sequent and a multiset of structures $\langle\Sigma_i \Rightarrow \Pi_i\rangle$. The successor relation given by $\bigcup\mathcal{N}(w)$ then corresponds to the immediate successor relation between nodes in the nested sequent. The main technical challenge is the construction of the neighbourhood function $\mathcal{N}$ itself. This can be done by adding *annotations* in the form of a set of formulae to every node in the nested sequent, written as $\Gamma \overset{S}{\Rightarrow} \Delta$. Further, to facilitate backwards proof search we absorb contraction into the rules by copying the principal formulae into the premiss(es). The so modified annotated versions of the interesting rules are given in Fig. 2. In all the other rules the annotations are preserved going from conclusion to premiss(es). In the following we write $\ell(w)$ for the annotation of the component v of a nested sequent.

$$\frac{\Gamma \Rightarrow \Delta, [\forall\forall]A, [\stackrel{\emptyset}{\Rightarrow} A]}{\Gamma \Rightarrow \Delta, [\forall\forall]A} [\forall\forall]^*_R \quad \frac{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, [\Sigma, A \stackrel{\{A\}}{\Rightarrow} \Pi]}{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle} \langle\exists\forall]^*_L \quad \frac{\Gamma \Rightarrow \Delta, [\forall\forall]A, [\Sigma \stackrel{\emptyset}{\Rightarrow} \Pi]}{\Gamma \Rightarrow \Delta, [\forall\forall]A, \langle\Sigma \Rightarrow \Pi\rangle} W^*$$

Figure 2: The interesting rules of the annotated variant $\mathcal{N}^*_{\mathsf{M}}$ of the system

Definition 7. The *model generated by a nested sequent* $\mathcal{S}$ is the model $\mathfrak{M}^{\mathcal{S}} = (W, \mathcal{N}, [\![.]\!])$ where W is the set of components (nodes) of $\mathcal{S}$, the valuation is defined by: if $w \in W$, then $w \in [\![p]\!]$ iff w contains $[\Gamma \stackrel{\Sigma}{\Rightarrow} \Delta]$ and $p \in \Gamma$. Finally, the neighbourhood function $\mathcal{N}(w)$ is defined as follows. Let $\mathcal{C}_w$ be the set of immediate successors of w, and let $\ell[\mathcal{C}_w]$ be the set of labels of nodes in $\mathcal{C}_w$. Then let $\mathcal{L}_w := \{\, \{v \in \mathcal{C}(w) \mid \ell(v) = \Sigma\} \mid \Sigma \in \ell[\mathcal{C}_w]\}$. Now, $\mathcal{N}(w)$ is defined as $(\mathcal{L}_w \cup \{\mathcal{C}_w\}) \smallsetminus \{\emptyset\}$ if there is a formula $\langle\exists\forall]A \in \Delta$, and $\mathcal{L}_w \cup \{\mathcal{C}_w\} \cup \{\emptyset\}$ otherwise.

Thus, disregarding the empty set, the set of neighbourhoods of a node in a nested sequent includes the set of all its children, as well as every set of children labelled with the same label. Whether it contains the empty set or not depends on whether there is a formula of the form $\langle\exists\forall]A$ in its succedent. This construction then yields countermodels from failed proof search:

Theorem 8. *If $\mathcal{S}$ is a saturated nested sequent obtained by backwards proof search from a non-nested sequent $\Gamma \Rightarrow \Delta$, then $\mathfrak{M}^{\mathcal{S}}$ is a neighbourhood model, and the root w of $\mathcal{S}$ satisfies for every formula A: if $A \in \Gamma$, then $w \in [\![A]\!]$, and if $A \in \Delta$, the $w \notin [\![A]\!]$.*

An implementation of the resulting proof search procedure which yields either a derivation or a countermodel is available under http://subsell.logic.at/bprover/nnProver/.

While here we considered only monotone logic M, we expect the calculus $\mathcal{N}_{\mathsf{M}}$ to be extensible to a large class of extensions of M. Hence it should provide the basis for an ideal meeting ground for syntax and semantics in the context of non-normal monotone modal logics.

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Constructing illoyal algebra-valued models of set theory *

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Abstract

An algebra-valued model of set theory is called loyal to its algebra if the model and its algebra have the same propositional logic; it is called faithful if all elements of the algebra are truth values of a sentence of the language of set theory in the model. We observe that non-trivial automorphisms of the algebra result in models that are not faithful and apply this to construct three classes of illoyal models: the tail stretches, the transposition twists, and the maximal twists.

The construction of *algebra-valued models of set theory* starts from an algebra $\mathbb{A}$ and a model V of set theory and forms an $\mathbb{A}$-valued model of set theory that reflects both the set theory of V and the logic of $\mathbb{A}$. This construction is the natural generalisation of Boolean-valued models, Heyting-valued models, lattice-valued models, and orthomodular-valued models (Bell, 2011; Grayson, 1979; Ozawa, 2017; Titani, 1999) and was developed by Löwe and Tarafder (2015).

Recently, Passmann (2018) introduced the terms "loyalty" and "faithfulness" while studying the precise relationship between the logic of the algebra $\mathbb{A}$ and the logical phenomena witnessed in the $\mathbb{A}$-valued model of set theory. The model constructed by Löwe and Tarafder (2015) is both loyal and faithful to $\mathbb{PS}_3$.

In this talk, we shall give elementary constructions to produce illoyal models by stretching and twisting Boolean algebras. After we give the basic definitions, we remind the audience of the construction of algebra-valued models of set theory. We then introduce our main technique: a non-trivial automorphisms of an algebra $\mathbb{A}$ excludes values from being truth values of sentences in the $\mathbb{A}$-valued model of set theory. Finally, we apply this technique to produce three classes of models: tail stretches, transposition twists, and maximal twists. This talk is based on Löwe et al. (2018).

1 Basic definitions

Algebras. Let Λ be a set of logical connectives; we shall assume that $\{\wedge, \vee, \mathbf{0}, \mathbf{1}\} \subseteq \Lambda \subseteq \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$. An algebra $\mathbb{A}$ with underlying set A is called a Λ-*algebra* if it has one operation for each of the logical connectives in Λ such that $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a distributive lattice; we can define $\leq$ on $\mathbb{A}$ by $x \leq y$ if and only if $x \wedge y = x$. An element $a \in A$ is an *atom* if it is $\leq$-minimal in $A \backslash \{\mathbf{0}\}$; we write $\mathrm{At}(\mathbb{A})$ for the set of atoms in $\mathbb{A}$. If $\Lambda = \{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication algebra* and if $\Lambda = \{\wedge, \vee, \rightarrow \neg, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication-negation algebra*.

We call a Λ-algebra $\mathbb{A}$ with underlying set A *complete* if for every $X \subseteq A$, the $\leq$-supremum and $\leq$-infimum exist; in this case, we write $\bigvee X$ and $\bigwedge X$ for these elements of $\mathbb{A}$. A complete Λ-algebra $\mathbb{A}$ is called *atomic* if for every $a \in A$, there is an $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$.

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Boolean algebras $\mathbb{B} = (B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ and a *Heyting algebras* $\mathbb{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ are defined as usual.

On an atomic distributive lattice $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, we have a canonical definition for a negation operation, the *complementation negation*: since $\mathbb{A}$ is atomic, every element $a \in A$ is uniquely represented by a set $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$. Then we define the complementation negation by $\neg_{\mathrm{c}}(\bigvee X) := \bigvee\{t \in \mathrm{At}(\mathbb{A})\,;\, t \notin X\}$.

Homomorphisms, assignments, & translations. For any two Λ-algebras $\mathbb{A}$ and $\mathbb{B}$, a map $f : A \rightarrow B$ is called a Λ-*homomorphism* if it preserves all connectives in Λ; it is called a Λ-*isomorphism* if it is a bijective Λ-homomorphism; isomorphisms from $\mathbb{A}$ to $\mathbb{A}$ are called Λ-*automorphisms*.

Since the propositional formulas $\mathcal{L}_\Lambda$ are generated from the propositional variables P, we can think of any Λ-homomorphism defined on $\mathcal{L}_\Lambda$ as a function on P, homomorphically extended to all of $\mathcal{L}_\Lambda$. If $\mathbb{A}$ is a Λ-algebra with underlying set A, we say that Λ-homomorphisms $\iota : \mathcal{L}_\Lambda \rightarrow A$ are $\mathbb{A}$-*assignments*; if S is a set of non-logical symbols, we say that Λ-homomorphisms $T : \mathcal{L}_\Lambda \rightarrow \mathrm{Sent}_{\Lambda,S}$ are S-*translations*.

Using assignments, we can define the *propositional logic of* $\mathbb{A}$ as

$$\mathbf{L}(\mathbb{A}) := \{\varphi \in \mathcal{L}_\Lambda\,;\, \iota(\varphi) = \mathbf{1} \text{ for all } \mathbb{A}\text{-assignments } \iota\}.$$

Note that if $\mathbb{B}$ is a Boolean algebra, then $\mathbf{L}(\mathbb{B}) = \mathbf{CPC}$.

Algebra-valued structures and their propositional logic. If $\mathbb{A}$ is a Λ-algebra and S is a set of non-logical symbols, then any Λ-homomorphism $[\![\cdot]\!] : \mathrm{Sent}_{\Lambda,S} \rightarrow A$ will be called an $\mathbb{A}$-*valued* S-*structure*. Note that if $S' \subseteq S$ and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $[\![\cdot]\!]{\restriction}\mathrm{Sent}_{\Lambda,S'}$ is an $\mathbb{A}$-valued S'-structure. We define the *propositional logic of* $[\![\cdot]\!]$ as

$$\mathbf{L}([\![\cdot]\!]) := \{\varphi \in \mathcal{L}_\Lambda\,;\, [\![T(\varphi)]\!] = \mathbf{1} \text{ for all } S\text{-translations } T\}.$$

Note that if T is an S-translation and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $\varphi \mapsto [\![T(\varphi)]\!]$ is an $\mathbb{A}$-assignment, so

$$\mathbf{L}(\mathbb{A}) \subseteq \mathbf{L}([\![\cdot]\!]). \tag{†}$$

Clearly, $\mathrm{ran}([\![\cdot]\!]) \subseteq A$ is closed under all operations in Λ (since $[\![\cdot]\!]$ is a homomorphism) and thus defines a sub-Λ-algebra $\mathbb{A}_{[\![\cdot]\!]}$ of $\mathbb{A}$. The $\mathbb{A}$-assignments that are of the form $\varphi \mapsto [\![T(\varphi)]\!]$ are exactly the $\mathbb{A}_{[\![\cdot]\!]}$-assignments, so we obtain $\mathbf{L}([\![\cdot]\!]) = \mathbf{L}(\mathbb{A}_{[\![\cdot]\!]})$.

Loyalty & faithfulness. An $\mathbb{A}$-valued S-structure $[\![\cdot]\!]$ is called *loyal to* $\mathbb{A}$ if the converse of (†) holds, i.e., $\mathbf{L}(\mathbb{A}) = \mathbf{L}([\![\cdot]\!] = \mathbf{1})$; it is called *faithful to* $\mathbb{A}$ if for every $a \in A$, there is a $\varphi \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\varphi]\!] = a$; equivalently, if $\mathbb{A}_{[\![\cdot]\!]} = \mathbb{A}$. The two notions central for our paper were introduced by Passmann (2018).

Lemma 1. *If $[\![\cdot]\!]$ is faithful to $\mathbb{A}$, then it is loyal to $\mathbb{A}$.*

Algebra-valued models of set theory. We will work with the general construction of an algebra-valued model of set theory following Löwe and Tarafder (2015), where the precise definitions can be found.

If V is a model of set theory and A is any set, then we construct a universe of *names* $\mathrm{Name}(V, A)$ by transfinite recursion. We then let $S_{V,A}$ be the set of non-logical symbols consisting of the binary relation symbol $\in$ and a constant symbol for every name in $\mathrm{Name}(V, A)$.

If $\mathbb{A}$ is a Λ-algebra with underlying set A, we can now define a map $[\![\cdot]\!]^{\mathbb{A}}$ assigning to each $\varphi \in \mathcal{L}_{\Lambda,S_{V,A}}$ a truth value in $\mathbb{A}$ by recursion, see Löwe and Tarafder (2015) for the precise definitions. As set theorists are usually interested in the restriction to $\mathrm{Sent}_{\Lambda,S}$, we shall use the notation $[\![\cdot]\!]_{\mathbb{A}}$ to refer to this restricted $\mathbb{A}$-valued $\{\in\}$-structure.

The results for algebra-valued models of set theory were originally proved for Boolean algebras, then extended to Heyting algebras:

Theorem 2. *If V is a model of set theory, $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ is a Boolean algebra or Heyting algebra, and φ is any axiom of* ZF, *then $[\![\varphi]\!]_{\mathbb{B}} = \mathbf{1}$.*

Lemma 3. *Let $\mathbb{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ be a Heyting algebra and V be a model of set theory. Then $[\![\cdot]\!]^{\mathrm{Name}}_{\mathbb{H}}$ is faithful to $\mathbb{H}$ (and hence, loyal to $\mathbb{H}$).*

Automorphisms and algebra-valued models of set theory. Given a model of set theory V and any Λ-algebras $\mathbb{A}$ and $\mathbb{B}$ and a Λ-homomorphism $f : \mathbb{A} \to \mathbb{B}$, we can define a map $\widehat{f}$: $\mathrm{Name}(V, \mathbb{A}) \to \mathrm{Name}(V, \mathbb{B})$ by $\in$-recursion such that $f([\![\varphi(u_1, \ldots, u_n)]\!]_{\mathbb{A}}) = [\![\varphi(\widehat{f}(u_1), \ldots, \widehat{f}(u_n))]\!]_{\mathbb{B}}$ for all $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$ with n free variables and $u_1, \ldots, u_n \in \mathrm{Name}(V, \mathbb{A})$. In particular, if $f : \mathbb{A} \to \mathbb{B}$ is a complete Λ-isomorphism and $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$, then $f([\![\varphi]\!]_{\mathbb{A}}) = [\![\varphi]\!]_{\mathbb{B}}$. Hence, if $f : \mathbb{A} \to \mathbb{A}$ is a complete Λ-automorphism with $f(a) \neq a$, then there is no $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$ such that $[\![\varphi]\!]_{\mathbb{A}} = a$.

Proposition 4. *If $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an atomic distributive lattice and $a \in A \backslash \{\mathbf{0}, \mathbf{1}\}$, then there is a $\{\wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1}\}$-automorphism f of $\mathbb{A}$ such that $f(a) \neq a$.*

Note that every $[\![\cdot]\!]_{\mathbb{B}}$ is loyal but not faithful for any non-trivial atomic Boolean algebra $\mathbb{B}$.

2 Stretching and twisting the loyalty of Boolean algebras

In this section, we start from an atomic, complete Boolean algebra $\mathbb{B}$ and modify it, to get an algebra $\mathbb{A}$ that gives rise to an illoyal $[\![\cdot]\!]_{\mathbb{A}}$. The first construction is the well-known construction of tail extensions of Boolean algebras to obtain a Heyting algebra. The other two constructions are *negation twists*: in these, we interpret $\mathbb{B}$ as a Boolean implication algebra via the definition $a \to b := \neg a \vee b$, and then add a new, twisted negation to it that changes its logic.

What can be considered a negation? When twisting the negation, we need to define a sensible negation. Dunn (1995) lists Hazen's *subminimal negation* as the bottom of his *Kite of Negations*: only the rule of contraposition, i.e., $a \leq b$ implies $\neg b \leq \neg a$, is required. In the following, we shall use this as a necessary requirement to be a reasonable candidate for negation.

Tail stretches Let $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ be a Boolean algebra and let $\mathbf{1}^* \notin B$ be an additional element that we add to the top of $\mathbb{B}$ to form the *tail stretch* $\mathbb{H}$ as follows: $H := B \cup \{\mathbf{1}^*\}$, the complete lattice structure of $\mathbb{H}$ is the order sum of $\mathbb{B}$ and the one element lattice $\{\mathbf{1}^*\}$, and $\to^*$ is defined as follows:[1]

$$a \to^* b := \begin{cases} a \to b & \text{if } a, b \in B \text{ such that } a \not\leq b, \\ \mathbf{1}^* & \text{if } a, b \in B \text{ with } a \leq b \text{ or if } b = \mathbf{1}^*, \\ b & \text{if } a = \mathbf{1}^*. \end{cases}$$

[1] In $\mathbb{H}$, we use the (Heyting algebra) definition $\neg_{\mathbb{H}} h := h \to^* \mathbf{0}$ to define a negation; note that if $\mathbf{0} \neq b \in B$, $\neg_{\mathbb{H}} b = \neg b$, but $\neg_{\mathbb{H}} \mathbf{0} = \mathbf{1}^* \neq \mathbf{1} = \neg \mathbf{0}$.

Transposition twists Let $\mathbb{B}$ be an atomic Boolean algebra, $a, b \in \mathrm{At}(\mathbb{B})$ with $a \neq b$, and π be the transposition that transposes a and b. We now define a twisted negation by

$$\neg_\pi(\bigvee X) := \bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}$$

and let the π-*twist* of $\mathbb{B}$ be $\mathbb{B}_\pi := (B, \wedge, \vee, \rightarrow, \neg_\pi, \mathbf{0}, \mathbf{1})$.[2] We observe that the twisted negation $\neg_\pi$ satisfies the rule of contraposition.

Maximal twists Again, let $\mathbb{B}$ be an atomic Boolean algebra with more than two elements and define the maximal negation by

$$\neg_{\mathrm{m}} b := \begin{cases} \mathbf{1} & \text{if } b \neq \mathbf{1} \text{ and} \\ \mathbf{0} & \text{if } b = \mathbf{1} \end{cases}$$

for every $b \in B$. We let the *maximal twist* of $\mathbb{B}$ be $\mathbb{B}_{\mathrm{m}} := (B, \wedge, \vee, \rightarrow, \neg_{\mathrm{m}}, \mathbf{0}, \mathbf{1})$; once more observe that the maximal negation $\neg_{\mathrm{m}}$ satisfies the rule of contraposition.

The following is our main result, which is proved by providing non-trivial automorphisms for each of the three constructions.

Theorem 5. *If $\mathbb{B}$ is a Boolean algebra, then its tail stretch, its transposition twist and its maximal twist are not loyal. In particular, the logics of the transposition twist and of the maximal twist is* **CPC**.

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[2] Note that we do not twist the implication $\rightarrow$ which remains the implication of the original Boolean algebra $\mathbb{B}$ defined by $x \rightarrow y := \neg_{\mathrm{c}} x \vee y$.

Epimorphisms in varieties of Heyting algebras

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A morphism $f\colon \mathbf{A} \to \mathbf{B}$ in a category is an *epimorphism* [5, 10, 11] provided it is right-cancellative, i.e. that for every pair of morphisms $g, h\colon \mathbf{B} \to \mathbf{C}$,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

We will focus on epimorphisms in varieties K of algebras, which we regard as categories whose objects are the members of K and whose morphisms are the algebraic homomorphisms. It is immediate that in such categories all surjective morphisms are indeed epimorphisms. However the converse need not be true in general: for instance, the embedding of three-element chain into the four-element diamond happens to be a non-surjective epimorphism in the variety of distributive lattices. Accordingly, a variety K of algebras is said to have the *epimorphism surjectivity property* (ES property for short), if its epimorphisms are surjective.

The failure of the ES property in distributive lattices can be explained in logical terms as the observation that complements are *implicitly*, but not *explicitly*, definable in distributive lattices, in the sense that when complements exist they are uniquely determined, even if there is no unary term witnessing their explicit definition. In general, the algebraic counterpart K of an algebraizable logic $\vdash$ [4] has the ES property if and only if $\vdash$ satisfies the so-called *(infinite deductive) Beth definability property*, i.e. the demand that all implicit definitions in $\vdash$ can be turned explicit [3, 9, 17]. This raises the question of determining which varieties of Heyting algebras have the ES property or, equivalently, which intermediate logics have the Beth definability property.

Classical results by Kreisel and Maksimova, respectively, state that *all* varieties of Heyting algebras have a weak form of the ES property [12], while only *finitely many* of them have a strong version of it [6, 14, 15, 16]. Nonethelss the standard ES property in varieties of Heyting algebras seems to defy simple characterizations, and very little is known about it. One of the few general results about the topic states that the ES property holds for all varieties with bounded depth [2, Thm. 5.3], yielding in particular a continuum of varieties with the ES property. Remarkably, this observation has been recently generalized [18, Thm. 13] beyond the setting of *integral* and *distributive* residuated lattices [7] as follows:

Theorem 1 (M., Raftery and Wannenburg)**.** *Let K be a variety of commutative square-increasing (involutive) residuated lattices. If the finitely subdirectly irreducible members of K have finite depth and are generated by their negative cones, then K has the ES property.*

On the other hand, the first (*ad hoc*) example of a variety of Heyting algebras lacking the ES property was discovered in [2, Cor. 6.2]. We enhance that observation by ruling out the ES property for a range of well-known varieties. To this end, let **RN** be the Rieger-Nishimura lattice. Relyng on Esakia duality we establish the following:

Theorem 2.

(i) *For every $2 \leqslant n \in \omega$, the variety of all Heyting algebras of width $\leqslant n$ lacks the ES property.*

(ii) *The variety $\mathbb{V}(\mathbf{RN})$ lacks the ES property, and has a continuum of locally finite subvarieties without the ES property.*

Recall that the Kuznetsov-Gerčiu variety KG is the variety generated by finite sums of cyclic (i.e. one-generated) Heyting algebras [8, 13, 1]. As a case study, we will present a full description of subvarieties of KG with the ES property. This yields an alternative proof of the well-known fact that varieties of Gödel algebras have the ES property.

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Generators and Axiomatizations for Varieties of PBZ*–lattices

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PBZ*–lattices are bounded lattice–ordered algebraic structures arising in the study of quantum logics. By definition, *PBZ*–lattices* are the paraorthomodular Brouwer–Zadeh lattices in which each pair consisting of an element and its Kleene complement fulfills the Strong de Morgan condition. They include orthomodular lattices, which are exactly the PBZ*–lattices without unsharp elements, as well as antiortholattices, which are exactly the PBZ*–lattices whose only sharp elements are 0 and 1. See below the formal definitions. Recall that the *sharp elements* of a bounded involution lattice are the elements having their involutions as bounded lattice complements; more precisely, with the terminology of [2], this is the notion of a *Kleene–sharp element*, and, in Brouwer–Zadeh lattices, we have also the notions of a *Brouwer–sharp* and a $\Diamond$–*sharp element*; however, in the particular case of PBZ*–lattices, Kleene–sharp, Brouwer–sharp and $\Diamond$–sharp elements coincide. All the results in this abstract that are not cited from other papers and not mentioned as being immediate are new and original.

We will designate algebras by their underlying sets and denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We recall the following definitions and immediate properties:

- a *bounded involution lattice* (in brief, *BI–lattice*) is an algebra $(L, \wedge, \vee, \cdot', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with partial order $\leq$, $a'' = a$ for all $a \in L$, and $a \leq b$ implies $b' \leq a'$ for all $a, b \in L$; the operation $\cdot'$ of a BI–lattice is called *involution*;
- if an algebra L has a BI–lattice reduct, then we denote by $S(L)$ the set of the *sharp elements* of L, namely $S(L) = \{x \in L \mid x \wedge x' = 0\}$;
- an *ortholattice* is a BI–lattice L such that $S(L) = L$;
- an *orthomodular lattice* is an ortholattice L such that, for all $a, b \in L$, $a \leq b$ implies $a \vee (a' \wedge b) = b$;
- a *pseudo–Kleene algebra* is a BI–lattice $\mathbf{L}$ satisfying, for all $a, b \in L$: $a \wedge a' \leq b \vee b'$; the involution of a pseudo–Kleene algebra is called *Kleene complement*; recall that distributive pseudo–Kleene algebras are called *Kleene algebras* or *Kleene lattices*; clearly, any ortholattice is a pseudo–Kleene algebra;
- an algebra L having a BI–lattice reduct is said to be *paraorthomodular* iff, for all $a, b \in L$, whenever $a \leq b$ and $a' \wedge b = 0$, it follows that $a = b$; note that any orthomodular lattice is a paraorthomodular pseudo–Kleene algebra, but the converse does not hold; however, if L is an ortholattice, then: L is orthomodular iff L is paraorthomodular;

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- a *Brouwer–Zadeh lattice* (in brief, *BZ–lattice*) is an algebra $(L, \wedge, \vee, \cdot', \cdot^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, \cdot', 0, 1)$ is a pseudo–Kleene algebra and, for all $a, b \in L$: $\begin{cases} a \wedge a^{\sim} = 0; & a \leq a^{\sim\sim}; \\ a^{\sim\prime} = a^{\sim\sim}; & a \leq b \text{ implies } b^{\sim} \leq a^{\sim}; \end{cases}$
- a *BZ*–lattice* is a BZ–lattice that satisfies condition $(*) :\ (a \wedge a')^{\sim} \leq a^{\sim} \vee a'^{\sim}$, written in equivalent form: $(*) :\ (a \wedge a')^{\sim} = a^{\sim} \vee a'^{\sim}$;
- a *PBZ*–lattice* is a paraorthomodular BZ*–lattice;
- if we extend their signature by adding a Brouwer complement equalling their Kleene complement, then ortholattices become BZ–lattices and orthomodular lattices become PBZ*–lattices; in any PBZ*–lattice L, $S(L)$ is the largest orthomodular subalgebra of L;
- an *antiortholattice* is a PBZ*–lattice L such that $S(L) = \{0, 1\}$; antiortholattices are exactly the PBZ*–lattices L whose Brower complement is defined by: $0^{\sim} = 1$ and $x^{\sim} = 0$ for all $x \in L \setminus \{0\}$; this Brower complement is called the *trivial Brower complement*; note, also, that any pseudo–Kleene algebra L with $S(L) = \{0, 1\}$ becomes an antiortholattice when endowed with the trivial Brower complement.

PBZ*–lattices form a variety, which we will denote by $\mathbb{PBZL}^*$. We will also denote by $\mathbb{BA}$, $\mathbb{OML}$, $\mathbb{OL}$ and $\mathbb{PKA}$ the varieties of Boolean algebras, orthomodular lattices, ortholattices and pseudo–Kleene algebras. $\mathbb{BA} \subsetneq \mathbb{OML} \subsetneq \mathbb{OL} \subsetneq \mathbb{PKA}$ and, with the extended signature, $\mathbb{OML} = \{L \in \mathbb{PBZL}^* \mid L \vDash x' \approx x^{\sim}\}$. $\mathbb{AOL}$ will denote the class of antiortholattices, which is a proper universal class, since not only it is not closed with respect to direct products, but, as we have proven, each of its members has the bounded lattice reduct directly indecomposable. We also denote by $\mathbb{DIST}$ the variety of distributive PBZ*–lattices.

We consider the following identities in the language of BZ–lattices, where, for any element x of a BZ–lattice, we denote by $\Box x = x'^{\sim}$ and by $\Diamond x = x^{\sim\sim}$:

SK	$x \wedge \Diamond y \leq \Box x \vee y$
SDM	$(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$ (the *Strong de Morgan law*)
WSDM	$(x \wedge y^{\sim})^{\sim} \approx x^{\sim} \vee \Diamond y$ (*weak SDM*)
S2	$(x \wedge (y \wedge y')^{\sim})^{\sim} \approx x^{\sim} \vee \Diamond(y \wedge y')$
S3	$(x \wedge \Diamond(y \wedge y'))^{\sim} \approx x^{\sim} \vee (y \wedge y')^{\sim}$
J0	$x \approx (x \wedge y^{\sim}) \vee (x \wedge \Diamond y)$
J2	$x \approx (x \wedge (y \wedge y')^{\sim}) \vee (x \wedge \Diamond(y \wedge y'))$

Clearly, **J0** implies **J2** and **SDM** implies **WSDM**, which in turn implies **S2** and **S3**. We have proven that, in what follows, whenever we state that a subvariety of $\mathbb{PBZL}^*$ is axiomatized relative to $\mathbb{PBZL}^*$ by axioms $\gamma_1, \ldots, \gamma_n$ for some $n \in \mathbb{N}^*$, we have that, for each $k \in [1, n]$, γ_k is independent from $\{\gamma_i \mid i \in [1, n] \setminus \{k\}\}$.

For any class $\mathbb{C}$ of similar algebras, the variety generated by $\mathbb{C}$ will be denoted by $V(\mathbb{C})$; so $V(\mathbb{C}) = \mathcal{HSP}(\mathbb{C})$, where $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{P}$ are the usual class operators; for any algebra A, $V(\{A\})$ will be streamlined to $V(A)$. We denote by $\mathbb{SDM}$ the variety of the PBZ*–lattices that satisfy the Strong de Morgan condition, and by $\mathbb{SAOL} = \mathbb{SDM} \cap V(\mathbb{AOL})$. Note that $\mathbb{OML} \cap V(\mathbb{AOL}) = \mathbb{BA}$, hence $\mathbb{DIST} \subseteq V(\mathbb{AOL})$. In the lattice of subvarieties of $\mathbb{PBZL}^*$, $\mathbb{BA}$ is the single atom and it has only two covers: its single orthomodular cover, $V(MO_2)$ [1], where MO_2 is the modular ortholattice with four atoms and length three (see the notation in Section 2 below), and $V(D_3)$ [2], where D_3 is the three–element antiortholattice chain (see Section 1 below); furthermore, D_3 belongs to any subvariety of $\mathbb{PBZL}^*$ which is not included in $\mathbb{OML}$, hence $\mathbb{OML} \vee V(D_3)$ is the single cover of $\mathbb{OML}$ in this subvariety lattice.

1 Ordinal Sums

Let us denote by D_n the n–element chain for any $n \in \mathbb{N}^*$, which clearly becomes an antiortholattice with its dual lattice automorphism as Kleene complement and the trivial Brower complement. Moreover, any pseudo–Kleene algebra with the 0 meet–irreducible becomes an antiortholattice when endowed with the trivial Brower complement; furthermore, if we denote by $L \oplus M$ the *ordinal sum* of a lattice L with largest element and a lattice M with smallest element, obtained by glueing L with M at the largest element of L and the smallest element of M, then, for any pseudo–Kleene algebra K and any non–trivial bounded lattice L, if L^d is the dual of L, then $L \oplus K \oplus L^d$ becomes an antiortholattice, with the clear definition for the Kleene complement and the trivial Brower complement. If $\mathbb{C} \subseteq \mathbb{PKA}$, then we denote by $D_2 \oplus \mathbb{C} \oplus D_2 = \{D_2 \oplus K \oplus D_2 \mid K \in \mathbb{C}\} \subsetneq \mathbb{AOL} \cap \mathbb{SAOL} \subsetneq \mathbb{AOL}$.

Recall from [3] that, for any $n \in \mathbb{N}$ with $n \geq 5$, $V(D_3) = V(D_4) \subsetneq V(D_5) = V(D_n) = \mathbb{DIST} \cap \mathbb{SAOL} \subsetneq \mathbb{DIST} = V(\{D_2^\kappa \oplus D_2^\kappa, D_2^\kappa \oplus D_2 \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\})$ and note that $\mathbb{BA} = V(D_2) \subsetneq V(D_3)$ and, for each $j \in \{0,1\}$, $D_{2j+1} = D_2^j \oplus D_2^j$ and $D_{2j+2} = D_2^j \oplus D_2 \oplus D_2^j$.

We have proven the following:

- $\mathbb{BA} = V(D_2) \subsetneq V(D_3) = V(D_4) \subsetneq \ldots \subsetneq V(D_2^n \oplus D_2^n) \subsetneq V(D_2^n \oplus D_2 \oplus D_2^n) \subsetneq V(D_2^{n+1} \oplus D_2^{n+1}) \subsetneq V(D_2^{n+1} \oplus D_2 \oplus D_2^{n+1}) \subsetneq \ldots \subsetneq V(\{D_2^\kappa \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\}) = V(\{D_2^\kappa \oplus D_2 \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\}) = \mathbb{DIST} \subsetneq \mathbb{DIST} \vee \mathbb{SAOL} \subsetneq V(\mathbb{AOL})$, where n designates an arbitrary natural number with $n \geq 2$;
- $\mathbb{SAOL} \cap \mathbb{DIST} = V(D_5) = V(D_2 \oplus \mathbb{BA} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{OML} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{OL} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{PKA} \oplus D_2) = \mathbb{SAOL} \subsetneq \mathbb{DIST} \vee \mathbb{SAOL} = V((D_2 \oplus \mathbb{PKA} \oplus D_2) \cup \{D_2^\kappa \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\})$, the latter equality following from the above;
- $\mathbb{OML} \vee V(D_3) \subsetneq \mathbb{OML} \vee V(D_5) = \mathbb{OML} \vee (\mathbb{DIST} \cap \mathbb{SAOL}) = (\mathbb{OML} \vee \mathbb{DIST}) \cap (\mathbb{OML} \vee \mathbb{SAOL}) \subsetneq \mathbb{OML} \vee \mathbb{DIST}, \mathbb{OML} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee \mathbb{DIST} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee V(\mathbb{AOL}) \subsetneq \mathbb{SDM} \vee V(\mathbb{AOL}) \supsetneq \mathbb{SDM} \supsetneq \mathbb{OML} \vee \mathbb{SAOL}$, where the second equality follows from the more general fact that:

Theorem 1. *L is a sublattice of the lattice of subvarieties of $\mathbb{PBZL}^*$ such that all elements of L except the largest, if L has a largest element, are either subvarieties of $\mathbb{OML}$ or of $V(\mathbb{AOL})$, and the sublattice of L formed of its elements which are subvarieties of $\mathbb{OML}$ is distributive, then L is distributive.*

We know from the above that $\mathbb{OML} \vee V(\mathbb{AOL})$ is not a cover of $\mathbb{OML}$ in the lattice of subvarieties of $\mathbb{PBZL}^*$. The previous theorem shows that $\mathbb{OML} \vee V(\mathbb{AOL})$ is not a cover of $V(\mathbb{AOL})$, either, because, for any subvariety $\mathbb{V}$ of $\mathbb{OML}$ such that $\mathbb{BA} \subsetneq \mathbb{V} \subsetneq \mathbb{OML}$, $\{\mathbb{BA}, \mathbb{V}, \mathbb{OML}, V(\mathbb{AOL}), \mathbb{OML} \vee V(\mathbb{AOL})\}$ fails to be a sublattice of $\mathbb{PBZL}^*$, which can only happen if $V(\mathbb{AOL}) \subsetneq \mathbb{V} \vee V(\mathbb{AOL}) \subsetneq \mathbb{OML} \vee V(\mathbb{AOL})$. The theorem above also implies:

Corollary 2. *The lattice of subvarieties of $V(\mathbb{AOL})$ is distributive.*

2 Horizontal Sums and Axiomatizations

We denote by $A \boxplus B$ the *horizontal sum* of two non–trivial bounded lattices A and B, obtained by glueing them at their smallest elements, as well as at their largest elements; clearly, the horizontal sum is commutative and has D_2 as a neutral element; note that, in the same way, one defines the horizontal sum of an arbitrary family of non–trivial bounded lattices. Whenever

A is a non–trivial orthomodular lattice and B is a non–trivial PBZ*–lattice, $A \boxplus B$ becomes a PBZ*–lattice having A and B as subalgebras, that is with its Kleene and Brower complement restricting to those of A and B, respectively; similarly, the horizontal sum of an arbitrary family of PBZ*–lattices becomes a PBZ*–lattice whenever all members of that family excepting at most one are orthomodular. If $\mathbb{C} \subseteq \mathbb{OML}$ and $\mathbb{D} \subseteq \mathbb{PBZL}^*$, then we denote by $\mathbb{C} \boxplus \mathbb{D} = \{D_1\} \cup \{A \boxplus B \mid A \in \mathbb{C} \setminus \{D_1\}, B \in \mathbb{D} \setminus \{D_1\}\} \subseteq \mathbb{PBZL}^*$.

For any cardinal number κ, we denote by $MO_\kappa = \boxplus_{i<\kappa} D_2^2 \in \mathbb{OML}$, where, by convention, we let $MO_0 = D_2$. All PBZ*–lattices L having the elements of $L \setminus \{0, 1\}$ join–irreducible are of the form $L = MO_\kappa \boxplus A$ for some cardinal number κ and some antiortholattice chain A, hence they are horizontal sums of families of Boolean algebras with antiortholattice chains, so, by a result in [3], the variety they generate is generated by its finite members, from which, noticing that, for any $A \in \mathbb{OML} \setminus \{D_1, D_2\}$ and any non–trivial $B \in \mathbb{AOL}$, the horizontal sum $A \boxplus B$ is subdirectly irreducible exactly when B is subdirectly irreducible and using and the fact that the only subdirectly irreducible antiortholattice chains are D_1, D_2, D_3, D_4 and D_5, we obtain that $V(\{L \in \mathbb{PBZL}^* \mid L \setminus \{0, 1\} \subseteq Ji(L)\}) = V(\{MO_n \boxplus D_k \mid n \in \mathbb{N}, k \in [2, 5]\})$, where we have denoted by $Ji(L)$ the set of the join–irreducibles of an arbitrary lattice L.

We have also proven that:

- $\mathbb{OML} \vee V(\mathbb{AOL}) \subsetneq V(\mathbb{OML} \boxplus \mathbb{AOL}) \subsetneq V(\mathbb{OML} \boxplus V(\mathbb{AOL})) \subsetneq \mathbb{PBZL}^*$;
- the class of the members of $\mathbb{OML} \boxplus V(\mathbb{AOL})$ that satisfy **J2** is $\mathbb{OML} \boxplus \mathbb{AOL}$, hence $V(\mathbb{OML} \boxplus \mathbb{AOL})$ is included in the variety axiomatized by **J2** relative to $V(\mathbb{OML} \boxplus V(\mathbb{AOL}))$.

We have obtained the following axiomatizations:

Theorem 3. *(i)* $V(\mathbb{AOL})$ *is axiomatized by* **J0** *relative to* $\mathbb{PBZL}^*$*.*

(ii) $\mathbb{OML} \vee V(D_3)$ *is axiomatized by* **SK**, **WSDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$*.*

(iii) $\mathbb{OML} \vee \mathbb{SAOL}$ *is axiomatized by* **SDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$*.*

(iv) $\mathbb{OML} \vee V(\mathbb{AOL})$ *is axiomatized by* **WSDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$*.*

(v) $\mathbb{OML} \vee V(\mathbb{AOL})$ *is axiomatized by* **WSDM** *relative to* $V(\mathbb{OML} \boxplus \mathbb{AOL})$*.*

(vi) $V(\mathbb{OML} \boxplus \mathbb{AOL})$ *is axiomatized by* **S2**, **S3** *and* **J2** *relative to* $\mathbb{PBZL}^*$*.*

In Theorem 3, (i) is a streamlining of the axiomatization of $V(\mathbb{AOL})$ obtained in [2]; we have obtained (iv) both by a direct proof and as a corollary of (v) and (vi).

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Simplified Kripke semantics for K45- and KD45-like Gödel modal logics

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1 Introduction

Possibilistic logic [5, 6]) is a well-known uncertainty logic to reasoning with graded (epistemic) beliefs on classical propositions by means of necessity and possiblity measures. In this setting, epistemic states of an agent are represented by possibility distributions. If W is a set of classical evaluations or possible worlds, for a given propositional language, a normalized possibility distribution on W is a mapping $\pi : W \to [0, 1]$, with $\sup_{w \in W} \pi(w) = 1$. π ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution π induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) = \sup\{\pi(w) \mid w \in W, w(\varphi) = 1\}$$
$$N(\varphi) = \inf\{1 - \pi(w) \mid w \in W, w(\varphi) = 0\} \ .$$

$N(\varphi)$ measures to what degree φ can be considered certain given the given epistemic, while $\Pi(\varphi)$ measures the degree in which φ is plausible or possible. Both measures are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$, so that the degree of possibility of a proposition φ equates the degree in which $\neg\varphi$ is not certain. If the normalized condition over possibility distribution is dropped, then we gain the ability to deal with inconsistency. In [7], a possibility distribution which satisfies $\sup_{w \in W} \pi(w) < 1$ is called sub-normal. In this case, given a set W of classical interpretations, a degree of inconsistency can be defined in the following way:

$$inc(W) = 1 - \sup_{w \in W} \pi(w)$$

When the normalised possibility distribution π is $\{0, 1\}$-valued, i.e. when π is the characteristic function of a subset $\emptyset \neq E \subseteq W$, then the structure (W, π), or better (W, E), can be seen in fact as a KD45 frame. In fact, it is folklore that modal logic KD45, which is sound and complete w.r.t. the class of Kripke frames (W, R) where R is a serial, euclidean and transitive binary relation, also has a simplified semantics given by the subclass of frames (W, E), where E is a non-empty subset of W (understanding E as its corresponding binary relation R_E defined as $R_E(w, w')$ iff $w' \in E$).

When we go beyond the classical framework of Boolean algebras of events to many-valued frameworks, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events [4]. In the setting of many-valued modal frameworks over Gödel logic, in [1] the authors claim a similar result as above, in the sense of providing a simplified possibilistic semantics for the logic $KD45(\mathbf{G})$ defined by the class of many-valued Kripke models with a many-valued accessibility relation satisfying counterparts of the serial, euclidean and transitive relations. However, it has to be noted that the completeness proof in [1] has some flaws, as reported by Tuyt.[1] In this paper we will report on a correct proof, not only for the completeness of $KD45(\mathbf{G})$ w.r.t. to its corresponding class of possibilistic frames, but also for the weaker logic $K45(\mathbf{G})$ accounting for partially inconsistent possibilistic Kripke frames.

2 The logic $K45(\mathbf{G})$

In their paper [3] Caicedo and Rodríguez consider a modal logic over Gödel logic with two operators $\Box$ and $\Diamond$. The language $\mathcal{L}_{\Box\Diamond}(Var)$ is built from a countable set Var of propositional variables, connectives symbols $\vee, \wedge, \rightarrow, \bot$, and the modal operator symbols $\Box$ and $\Diamond$. We will simply write $\mathcal{L}_{\Box\Diamond}$ assuming Var is known and fixed.

In their work, Caicedo and Rodríguez define the logic $K(\mathbf{G})$ as the smallest set of formulas containing some axiomatic version of Gödel-Dummet propositional calculus; that is, Heyting calculus plus the prelinearity law and the following additional axioms:

$(K_\Box)$	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	$(K_\Diamond)$	$\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$
$(F_\Box)$	$\Box\top$	(P)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$
$(FS2)$	$(\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$	(Nec)	from φ infer $\Box\varphi$

The logic $K45(\mathbf{G})$ is defined by adding to $K(\mathbf{G})$ the following axioms:

$(4_\Box)$	$\Box\varphi \rightarrow \Box\Box\varphi$	$(4_\Diamond)$	$\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$
$(5_\Box)$	$\Diamond\Box\varphi \rightarrow \Box\varphi$	$(5_\Diamond)$	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$

Let $\vdash_G$ denote deduction in Gödel fuzzy logic G. Let $\mathcal{L}(X)$ denote the set of formulas built by means of the connectives $\wedge, \rightarrow$, and $\bot$, from a given subset of variables $X \subseteq Var$. For simplicity, the extension of a valuation $v : X \rightarrow [0, 1]$ to $\mathcal{L}(X)$ according to Gödel logic interpretation of the connectives will be denoted v as well. It is well known that G is complete for validity with respect to these valuations. We will need the fact that it is actually sound and complete in the following stronger sense, see [2].

Proposition 2.1. *i) If $T \cup \{\varphi\} \subseteq \mathcal{L}(X)$, then $T \vdash_G \varphi$ implies $\inf v(T) \leq v(\varphi)$ for any valuation $v : X \rightarrow [0, 1]$.*

ii) If T is countable, and $T \nvdash_G \varphi_{i_1} \vee .. \vee \varphi_{i_n}$ for each finite subset of a countable family $\{\varphi_i\}_{i \in I}$ there is an evaluation $v : \mathcal{L}(X) \rightarrow [0, 1]$ such that $v(\theta) = 1$ for all $\theta \in T$ and $v(\varphi_i) < 1$ for all $i \in I$.

The following are some theorems of $K(G)$, see [3]. The first one is an axiom in Fitting's systems in [8], the next two were introduced in [3], the fourth one will be useful in our completeness proof and is the only one depending on prelinearity. The last is known as the first connecting axiom given by Fischer Servi.

$T1.$	$\neg\Diamond\theta \leftrightarrow \Box\neg\theta$	$T4.$	$(\Box\varphi \rightarrow \Diamond\psi) \vee \Box((\varphi \rightarrow \psi) \rightarrow \psi)$
$T2.$	$\neg\neg\Box\theta \rightarrow \Box\neg\neg\theta$	$T5.$	$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$
$T3.$	$\Diamond\neg\neg\varphi \rightarrow \neg\neg\Diamond\varphi$		

[1] Personal communication

Next we show that in $K45(\mathbf{G})$ some iterated modalities can be simplified. This is in accordance with our intended simplified semantics for $K45(\mathbf{G})$ that will be formally introduced in the next section.

Proposition 2.2. *The logic $K45(\mathbf{G})$ proves the following schemes:*

$(F_{\Diamond\Box})$	$\Diamond\Box\top \leftrightarrow \Diamond\top$	$(F_{\Box\Diamond})$	$\Box\Diamond\top \leftrightarrow \Box\top$
$(U_{\Diamond})$	$\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$	$(U_{\Box})$	$\Box\Box\varphi \leftrightarrow \Box\varphi$
$(T4_{\Box})$	$(\Box\varphi \to \Diamond\Box\varphi) \vee \Box\varphi$	$(T4_{\Diamond})$	$(\Box\Diamond\varphi \to \Diamond\varphi) \vee (\Diamond\top \to \Diamond\varphi)$

From now on we will use $ThK45(\mathbf{G})$ to denote the set of theorems of $K45(\mathbf{G})$. We close this section with the following observation: deductions in $K45(\mathbf{G})$ can be reduced to derivations in pure propositional Gödel logic G.

Lemma 2.1. *For any theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, it holds that $T \vdash_{K45(\mathbf{G})} \varphi$ iff $T \cup ThK45(\mathbf{G}) \vdash_G \varphi$.*

It is worth noticing that for any valuation v such that $v(ThK45(\mathbf{G})) = 1$ there is no formula φ such that $v(\Diamond\top) < v(\nabla\varphi) < 1$ with $\nabla \in \{\Box, \Diamond\}$ because both formulae $(\Box\varphi \to \Diamond\varphi) \vee \Box\varphi$ and $\Diamond\varphi \to \Diamond\top$ are in $ThK45(\mathbf{G})$.

3 Simplified Kripke semantics and completeness

In this section we will show that $K45(\mathbf{G})$ is complete with respect to a class of simplified Kripke Gödel frames.

Definition 3.1. *A* (normalised) possibilistic Kripke frame*, or* Π*-frame, is a structure* $\langle W, \pi\rangle$ *where* W is a non-empty set of worlds, and $\pi : W \to [0,1]$ *is a (resp. normalised) possibility distribution over* W*.*

A (resp. normalised) possibilistic Gödel Kripke model *is a triple* $\langle W, \pi, e\rangle$ *where* $\langle W, \pi\rangle$ *is a* Π*-frame frame and* $e : W \times Var \to [0,1]$ *provides a Gödel evaluation of variables in each world. For each* $w \in W$*,* $e(w, -)$ *extends to arbitrary formulas in the usual way for the propositional connectives and for modal operators in the following way:*

$$e(w, \Box\varphi) := \inf_{w' \in W}\{\pi(w') \Rightarrow e(w', \varphi)\}$$
$$e(w, \Diamond\varphi) := \sup_{w' \in W}\{\min(\pi(w'), e(w', \varphi))\}.$$

Observe that the evaluation of formulas beginning with a modal operator is in fact independent from the current world. Also note that the $e(-, \Box\varphi)$ and $e(-, \Diamond\varphi)$ are in fact generalisations for Gödel logic propositions of the necessity and possibility degrees of φ introduced in Section 1 for classical propositions, although now they are not dual (with respect to Gödel negation) any longer.

In the rest of this abstract we briefly sketch a weak completeness proof of the logic $K45(\mathbf{G})$ (resp. $KD45(\mathbf{G})$) with respect to the class $\Pi\mathcal{G}$ (resp. $\Pi^*\mathcal{G}$) of (resp. normalised) possibilistic Gödel Kripke models. In fact one can prove a little more, namely completeness for deductions from finite theories.

In what follows, for any formula φ we denote by $Sub(\varphi) \subseteq \mathcal{L}_{\Box\Diamond}$ the set of subformulas of φ and containing the formula $\bot$. Moreover, let $X := \{\Box\theta, \Diamond\theta : \theta \in \mathcal{L}_{\Box\Diamond}\}$ be the set of formulas in $\mathcal{L}_{\Box\Diamond}$ beginning with a modal operator; then $\mathcal{L}_{\Box\Diamond}(Var) = \mathcal{L}(Var \cup X)$. That is, any formula in $\mathcal{L}_{\Box\Diamond}(Var)$ may be seen as a propositional Gödel formula built from the extended set of propositional variables $Var \cup X$. In addition, for a given formula φ, let $\sim_\varphi$ be equivalence relation in $[0,1]^{Var\cup X} \times [0,1]^{Var\cup X}$ defined as follows:

$$u \sim_\varphi w \text{ iff } \forall \psi \in Sub(\varphi) : u(\Box\psi) = w(\Box\psi) \text{ and } u(\Diamond\psi) = w(\Diamond\psi).$$

Now, assume that a formula φ is not a theorem of $K45(\mathbf{G})$. Hence by completeness of Gödel calculus and Lemma 2.1, there exists a Gödel valuation v such that $v(ThK45(\mathbf{G})) = 1$ and $v(\varphi) < 1$. Following the usual canonical model construction, once fixed the valuation v, we define next a canonical $\Pi\mathcal{G}$-model M^v_φ in which we will show φ is not valid.

The *canonical model* $M^v_\varphi = (W^v, \pi^\varphi, e^\varphi)$ is defined as follows:

- W^v is the set $\{u \in [0,1]^{Var\cup X} \mid u \sim_\varphi v \text{ and } u(ThK45(\mathbf{G})) = 1\}$.
- $\pi^\varphi(u) = \inf_{\psi \in Sub(\varphi)}\{\min(v(\Box\psi) \to u(\psi), u(\psi) \to v(\Diamond\psi))\}$.
- $e^\varphi(u, p) = u(p)$ for any $p \in Var$.

Completeness will follow from the next truth-lemma, whose proof is rather involved.

Lemma 3.1 (Truth-lemma). *$e^\varphi(u, \psi) = u(\psi)$ for any $\psi \in Sub(\varphi)$ and any $u \in W^v$.*

Actually, the same proof for weak completeness easily generalizes to get completeness for deductions from finite theories.

Theorem 3.1 (Finite strong completeness). *For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, we have:*

- $T \models_{\Pi\mathcal{G}} \varphi$ *implies* $T \vdash_{K45(\mathbf{G})} \varphi$
- $T \models_{\Pi^*\mathcal{G}} \varphi$ *implies* $T \vdash_{KD45(\mathbf{G})} \varphi$

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Substructural PDL

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Propositional Dynamic Logic PDL, introduced in [6] following the ideas of [11], is a modal logic with applications in formal verification of programs [7], dynamic epistemic logic [1] and deontic logic [10], for example. More generally, PDL can be seen as a logic for reasoning about *structured actions* modifying various types of objects; examples of such actions include programs modifying states of the computer, information state updates or actions of agents changing the world around them.

In this contribution we study versions of PDL where the underlying propositional logic is a weak substructural logic in the vicinity of the full distributive non-associative Lambek calculus with a weak negation. The motivation is to provide a logic for reasoning about structured actions that modify *situations* in the sense of [2]; the link being the informal interpretation of the Routley–Meyer semantics for substructural logics in terms of situations [9].

In a recent paper [14] we studied versions of PDL based on Kripke frames with a ternary accessibility relation (in the style of [4, 8]). These frames do not contain the inclusion ordering essential for modelling situations, nor the compatibility relation articulating the semantics for a wide range of weak negations [5, 12]. Hence, in this contribution we study PDL based on (partially ordered) Routley–Meyer models with a compatibility relation.

Formulas φ and actions A are defined by mutual induction in the usual way [7]

$$A := a \mid A \cup A \mid A; A \mid A^* \mid \varphi?$$
$$\varphi := p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {\sim}\varphi \mid \varphi \to \varphi \mid \varphi \circ \varphi \mid [A]\varphi$$

where p is an atomic formula and a an atomic action. So far, we have results for the language without existential modalities $\langle A \rangle$ dual to $[A]$; inclusion of these is the focus of ongoing work. The implication $\to$ is the left residual of fusion $\circ$ which is assumed to be commutative for the sake of simplicity.

A Routley-Meyer frame is $\mathfrak{F} = \langle S, \leq, L, C, R \rangle$ where $(S, \leq, L)$ is a partially ordered set with an upwards-closed $L \subseteq S$; C is a symmetric binary relation antitone in both positions, that is

- Cxy, $x' \leq x$ and $y' \leq y$ only if $Cx'y'$;

and R is a ternary relation antitone in the first two positions such that

- $Rxyz$ only if $Ryxz$ and
- $x \leq y$ iff there is $z \in L$ such that $Rzxy$.

A (dynamic) Routley-Meyer model based on $\mathfrak{F}$ is $\mathfrak{M} = \langle \mathfrak{F}, [\![\cdot]\!] \rangle$ where

- $[\![\varphi]\!]$ is a subset of S such that $[\![p]\!]$ is upwards-closed and
- $[\![A]\!]$ is a binary relation on S such that $[\![a]\!]$ is antitone in the first position.

It is assumed that $[\![\varphi \wedge \psi]\!]$ ($[\![\varphi \vee \psi]\!]$) is the intersection (union) of $[\![\varphi]\!]$ and $[\![\psi]\!]$ and

*This contribution is based on joint work with Vít Punčochář and Andrew Tedder.

- $[\![{\sim}\varphi]\!] = \{x \mid \forall y\,(Cxy \Rightarrow y \notin [\![\varphi]\!])\}$,
- $[\![\varphi \to \psi]\!] = \{x \mid \forall yz\,((Rxyz \;\;\&\;\; y \in [\![\varphi]\!]) \Rightarrow z \in [\![\psi]\!])\}$,
- $[\![\varphi \circ \psi]\!] = \{x \mid \exists yz\,(Ryzx \;\;\&\;\; y \in [\![\varphi]\!] \;\;\&\;\; z \in [\![\psi]\!])\}$ and
- $[\![[A]\varphi]\!] = \{x \mid \forall y\,(x[\![A]\!]y \Rightarrow y \in [\![\varphi]\!])\}$.

It is also assumed that $[\![A \cup B]\!]$ ($[\![A; B]\!]$) is the union (composition) of $[\![A]\!]$ and $[\![B]\!]$, that $[\![A^*]\!]$ is the reflexive-transitive closure of $[\![A]\!]$ and that

- $[\![\varphi?]\!] = \{\langle x, y\rangle \mid x \leq y \;\;\&\;\; y \in [\![\varphi]\!]\}$

We say that φ is valid in $\mathfrak{M}$ iff $L \subseteq [\![\varphi]\!]$; a finite Γ entails φ in $\mathfrak{M}$ iff $[\![\bigwedge \Gamma]\!] \subseteq [\![\varphi]\!]$. Validity and entailment in a class of frames are defined as usual.

It can be shown that each $[\![A]\!]$ is antitone in its first position. This, together with the other tonicity conditions, entails that $[\![\varphi]\!]$ is an upwards-closed set for all φ (this is the motivation of the unusual definition of $[\![\varphi?]\!]$) and so we have in turn the consequence that Γ entails φ in $\mathfrak{M}$ iff $\bigwedge \Gamma \to \varphi$ is valid in $\mathfrak{M}$ (unlike the semantics without L and $\leq$ where both directions of the equivalence may fail).

Extending the results of [13], we prove completeness and decidability of the set of formulas valid in all frames using filtration in the style of [3].

A *logic* is any set of formulas Λ containing all formulas of the form ($\leftrightarrows$ indicates that both implications are in Λ)

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- $[A]\varphi \wedge [A]\psi \to [A](\varphi \wedge \psi)$
- $[A \cup B]\varphi \leftrightarrows ([A]\varphi \wedge [B]\varphi)$
- $[A; B]\varphi \leftrightarrows [A][B]\varphi$
- $[A^*]\varphi \leftrightarrows (\varphi \wedge [A][A^*]\varphi)$
- $[\varphi?]\varphi$
- $\psi \to [\varphi?]\psi$
- $(\varphi \wedge [\varphi?]\psi) \to \psi$

and closed under the inference rules ('//' indicates a two-way rule):

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$
- $\varphi \to \psi \,/\, [A]\varphi \to [A]\psi$
- $\varphi \to (\psi \to \chi) \,//\, (\psi \circ \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $\varphi \to {\sim}\psi \,//\, \psi \to {\sim}\varphi$
- $\varphi \to [A]\varphi \,/\, \varphi \to [A^*]\varphi$

We write $\Gamma \vdash_\Lambda \Delta$ iff there are finite Γ', Δ' such that $\bigwedge \Gamma' \to \bigvee \Delta'$ is in Λ (hence, the relation $\vdash_\Lambda$ is finitary by definition). A prime Λ-theory is a set of formulas Γ such that $\varphi, \psi \in \Gamma$ only if $\Gamma \vdash_\Lambda \varphi \wedge \psi$ and $\Gamma \vdash_\Lambda \varphi \vee \psi$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$. For each $\Gamma \nvdash_\Lambda \Delta$ there is a prime theory containing Γ but disjoint from Δ [12, 94]. The *canonical* Λ*-frame* the frame-type structure $\mathfrak{F}^\Lambda$ where S^Λ is the set of prime Λ-theories, $\leq^\Lambda$ is set inclusion, L^Λ is the set of prime theories containing Λ and

- $C^\Lambda\Gamma\Delta$ iff $\sim\varphi \in \Gamma$ only if $\varphi \notin \Delta$
- $R^\Lambda\Gamma\Delta\Sigma$ iff $\varphi \in \Gamma$ and $\psi \in \Delta$ only if $\varphi \circ \psi \in \Sigma$

It is a standard observation that $\mathfrak{F}^\Lambda$ is a Routley–Meyer frame for all Λ [12]. The *canonical* Λ*-structure* $\mathfrak{S}^\Lambda$ is the canonical frame with $[\![\cdot]\!]^\Lambda$ defined as follows: $[\![\varphi]\!]^\Lambda = \{\Gamma \in S^\Lambda \mid \varphi \in \Gamma\}$ and $[\![A]\!]^\Lambda = \{\langle \Gamma, \Delta\rangle \mid \forall\varphi\,([A]\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\}$. It can be shown that $\mathfrak{S}^\Lambda$ is not a dynamic Routley–Meyer model (since $\{[a^n]p \mid n \in \mathbb{N}\} \nvdash_\Lambda [a^*]p$, we may show that $[\![a^*]\!]^\Lambda$ is not the reflexive-transitive closure of $[\![a]\!]^\Lambda$).

Fix a finite set of formulas Φ that is closed under subformulas and satisfies the following conditions: i) $[\varphi?]\psi \in \Phi$ only if $\varphi \in \Phi$, ii) $[A \cup B]\varphi \in \Phi$ only if $[A]\varphi, [B]\varphi \in \Phi$, iii) $[A;B]\varphi \in \Phi$ only if $[A][B]\varphi \in \Phi$ and iv) $[A^*]\varphi \in \Phi$ only if $[A][A^*]\varphi \in \Phi$. We define $\Gamma \preceq_\Phi \Delta$ as $(\Gamma \cap \Phi) \subseteq \Delta$. This relation is obviously a preorder; let $\equiv_\Phi$ be the associated equivalence relation and let $[\Gamma]_\Phi$ be the equivalence class of Γ with respect to this relation.

The Φ*-filtration of* $\mathfrak{S}^\Lambda$ is the model-type structure $\mathfrak{M}^\Lambda_\Phi$ such that S_Φ is the (finite) set of equivalence classes $[\Gamma]$ for $\Gamma \in S^\Lambda$, $[\Gamma] \leq_\Phi [\Delta]$ iff $\Gamma \preceq_\Phi \Delta$ and

- $L_\Phi = \{[\Gamma] \mid \exists \Delta \in S^\Lambda \left(\Delta \preceq_\Phi \Gamma \;\;\&\;\; \Delta \in L^\Lambda\right)\}$
- $C_\Phi = \{\langle[\Gamma_1], [\Gamma_2]\rangle \mid \exists \Delta_1, \Delta_2 \left(\Gamma_1 \preceq_\Phi \Delta_1 \;\;\&\;\; \Gamma_2 \preceq_\Phi \Delta_2 \;\;\&\;\; C^\Lambda\Delta_1\Delta_2\right)\}$
- $R_\Phi = \{\langle[\Gamma_1], [\Gamma_2], [\Gamma_3]\rangle \mid \exists \Delta_1, \Delta_2, \left(\Gamma_1 \preceq_\Phi \Delta_1 \;\;\&\;\; \Gamma_2 \preceq_\Phi \Delta_2 \;\;\&\;\; R^\Lambda\Delta_1\Delta_2\Gamma_3\right)\}$
- $[\![p]\!]_\Phi = \{[\Gamma] \mid p \in \Gamma\}$ for $p \in \Phi$ and $[\![p]\!]_\Phi = \emptyset$ otherwise
- $[\![a]\!]_\Phi = \{\langle[\Gamma_1], [\Gamma_2]\rangle \mid \exists \Delta \left(\Gamma_1 \preceq_\Phi \Delta \;\;\&\;\; \Delta[\![a]\!]^\Lambda\Gamma_2\right)\}$ if $[a]\chi \in \Phi$; $[\![a]\!]_\Phi = \emptyset$ otherwise.

The values of $[\![\cdot]\!]_\Phi$ on complex formulas and actions are defined exactly as in dynamic Routley–Meyer models. It can be shown that $\mathfrak{M}^\Lambda_\Phi$ is a dynamic Routley–Meyer model such that if $\varphi \in (\Phi \setminus \Lambda)$, then φ is not valid in $\mathfrak{M}^\Lambda_\Phi$. This implies completeness of the minimal logic Λ_0 with respect to (the set of formulas valid in) all Routley–Meyer frames.

In general, assume that we have $\mathsf{Log}(\mathbf{F})$, the set of formulas valid in all Routley–Meyer frames $\mathfrak{F} \in \mathbf{F}$. If $\mathcal{F}^\Lambda_\Phi \in \mathbf{F}$ for all Φ, then Λ is complete with respect to $\mathbf{F}$ (for instance, this is the case where $\mathbf{F}$ is the class of frames satisfying $Rxxx$ for all x). If $\mathcal{F}^\Lambda_\Phi \notin \mathbf{F}$, then one has to either modify the requirements concerning Φ (while keeping it finite; our argument showing that if $\varphi \in (\Phi \setminus \Lambda)$, then φ is not valid in $\mathfrak{M}^\Lambda_\Phi$ does not work if Φ is infinite) or devise an alternative definition of R_Φ and C_Φ. Such modifications for specific classes of frames (e.g. associative ones) is the focus of ongoing work.

A topic for future work is a modification of our argument not requiring that Φ be finite. An argument based on finite filtration does not go through in case of logics that are known not to have the finite model property, such as the relevant logic R for instance.

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On one-variable fragments of modal predicate logics.

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We consider normal 1-modal predicate logics as they were defined in [2]. I.e., the signature contains predicate letters of different arities, but no constants or functions letters; also we do not specify the equality symbol. A logic is a set of formulas containing all classical validities and the axioms of **K** and closed under Modus Ponens, $\forall$- and $\Box$- introduction, and predicate substitution.

We also consider (normal) modal propositional logics. For a propositional modal logic $\mathbf{\Lambda}$, its minimal predicate extension is denoted by $\mathbf{Q\Lambda}$.

For modal predicate logics several different semantics are known. The most popular is Kripke semantics (with expanding domains). Recall that a *predicate Kripke frame* is a pair (F, D), where $F = (W, R)$ is a propositional Kripke frame (i.e., $W \neq \varnothing,\ R \subseteq W^2$) and D is a family of non-empty sets $(D_u)_{u \in W}$ such that $D_u \subseteq D_v$ whenever uRv. A *predicate Kripke model* over a frame $\mathbf{F} = (F, D)$ is a pair $(\mathbf{F}, \xi)$, where ξ (a valuation) is a family $(\xi_u)_{u\in W}$, ξ_u sends every n-ary predicate letter P to an n-ary relation on D_u (0-ary relation is a fixed value 0 or 1).

The truth predicate $M, u \vDash A$ is defined for any world $u \in W$ and D_u*-sentence* A (a sentence with individuals from D_u as constants). In particular,

$$M, u \vDash P(a_1, \dots, a_n) \text{ iff } (a_1, \dots, a_n) \in \xi_u(P),$$

$$M, u \vDash \Box A \text{ iff } \forall v \in R(u)\, M, v \vDash A,$$

$$M, u \vDash \forall x A \text{ iff } \forall a \in D_u\, M, u \vDash [a/x]A.$$

A formula A is *valid* in $\mathbf{F} = (F, D)$ (in symbols, $\mathbf{F} \vDash A$) if its universal closure is true at any world in any Kripke model over $\mathbf{F}$. The set $\mathbf{ML}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\}$ is a modal predicate logic (the *logic of* $\mathbf{F}$). The *logic of a class* of frames $\mathcal{C}$ is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\mathbf{F}) \mid F \in \mathcal{C}\}$. Logics of this form are called *Kripke complete*.

Similar definitions for propositional logics are well-known, so we skip them.

Formulas constructed from a single variable x and monadic predicate letters are called *1-variable*. Every such formula A translates into a bimodal propositional formula A_* with modalities $\Box$ and $\blacksquare$ if every atom $P_i(x)$ is replaced with the proposition letter p_i and every quantifier $\forall x$ with $\blacksquare$. The *1-variable fragment* of a predicate logic L is the set

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is 1-variable}\}.$$

Then

Lemma 1. *$L{-}1$ is a bimodal propositional logic.*

Recall that for monomodal propositional logics $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ the *fusion* $\mathbf{\Lambda}_1 * \mathbf{\Lambda}_2$ is the smallest bimodal logic containing $\mathbf{\Lambda}_1 \cup \mathbf{\Lambda}_2$ (if the modal connectives in $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ are distinct).

The *semicommutative join* of a monomodal logic $\mathbf{\Lambda}$ (in the language with $\Box$) with $\mathbf{S5}$ (in the language with $\blacksquare$) is obtained from $\mathbf{\Lambda} * \mathbf{S5}$ by adding the axiom

$$(com^l) \qquad \Box\blacksquare p \to \blacksquare\Box p.$$

This logic is denoted by $\mathbf{\Lambda} \rfloor \mathbf{S5}$. The following properties are easily checked.

Lemma 2. *(1)* $\mathbf{\Lambda} \rfloor \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1$.

(2) A propositional Kripke frame $F = (W, R_1, R_2)$ *validates* $\mathbf{\Lambda} \rfloor \mathbf{S5}$ *iff*

$$(W, R_1) \vDash \mathbf{\Lambda} \,\&\, R_2 \textit{ is an equivalence } \&\, R_2 \circ R_1 \subseteq R_1 \circ R_2.$$

Definition 1. *Let* $F_1 = (U_1, R_1)$, $F_2 = (U_2, R_2)$ *be propositional frames. Their* product *is the frame* $F_1 \times F_2 = (U_1 \times U_2, R_h, R_v)$, *where*

- $(u, v)R_h(u', v')$ *iff* uR_1u' *and* $v = v'$,
- $(u, v)R_v(u', v')$ *iff* $u = u'$ *and* vR_2v'.

A semiproduct *(or an* expanding product*) of* F_1 *and* F_2 *is a subframe* (W, S_1, S_2) *of* $F_1 \times F_2$ *such that* $R_h(W) \subseteq W$.

Definition 2. *A* semiproduct $\mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ *of a monomodal propositional logic* $\mathbf{\Lambda}$ *with* $\mathbf{S5}$ *is the logic of the class of all semiproducts of* $\mathbf{\Lambda}$*-frames (i.e., frames validating* $\mathbf{\Lambda}$*) with clusters (i.e., frames with a universal relation).*

From Lemma 2 we readily have

Lemma 3. $\mathbf{\Lambda} \rfloor \mathbf{S5} \subseteq \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.

Definition 3. *The logics* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are called* semiproduct matching *if* $\mathbf{\Lambda} \rfloor \mathbf{S5} = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. $\mathbf{\Lambda}$ *is called* quantifier-friendly *if* $\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \rfloor \mathbf{S5}$.

Definition 4. *The* Kripke-completion $C_{\mathcal{K}}(L)$ *of a predicate logic* L *is the logic of the class of all frames validating* L.

$C_{\mathcal{K}}(L)$ is the smallest Kripke-complete extension of L. From definitions we obtain

Lemma 4. $C_{\mathcal{K}}(\mathbf{Q\Lambda}){-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. *Hence*

$$\mathbf{Q\Lambda}{-}1 \subseteq \mathbf{\Lambda} \leftthreetimes \mathbf{S5}.$$

Also

$$\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$$

if $\mathbf{Q\Lambda}$ *is Kripke-complete, and*

$$\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \rfloor \mathbf{S5}$$

if $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching.*

The following result is proved in [1] (Theorem 9.10).

Theorem 5. $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching*[1] *for* $\mathbf{\Lambda} = \mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}$.

[1]The book [1] uses a different terminology and notation: semiproducts are called 'expanding products', $\mathbf{\Lambda} \rfloor \mathbf{S5}$ is denoted by $[\mathbf{\Lambda}, \mathbf{S5}]^{EX}$, $\mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ by $(\mathbf{\Lambda} \times \mathbf{S5})^{EX}$, and there is no term for 'semiproduct matching'.

This result extends to a slightly larger class.

Definition 5. *A* one-way PTC-logic *is a modal propositional logic axiomatized by formulas of the form* $\Box p \to \Box^n p$ *and closed (i.e., variable-free) formulas.*

Theorem 6. $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching for any one-way PTC-logic* $\mathbf{\Lambda}$.

Corollary 7. *Every one-way PTC-logic is quantifier-friendly.*

Also we have

Theorem 8. *([2], theorem 6.1.29)* $\mathbf{Q\Lambda}$ *is Kripke complete for any one-way PTC-logic* $\mathbf{\Lambda}$.

It is not clear how far theorems 6, 8 can be generalized. Anyway, unlike the case of products and predicate logics with constant domains (cf. [1], [2]), they do not hold for arbitrary Horn axiomatizable logics

Many counterexamples are given by the next theorem.

Let

$$\Box ref := \Box(\Box p \to p),\ \ \Box\mathbf{T} := \mathbf{K} + \Box ref,$$
$$\mathbf{SL4} := \mathbf{K4} + \Diamond p \leftrightarrow \Box p.$$

Theorem 9. *Let* $\mathbf{\Lambda}$ *be a propositional logic such that* $\Box\mathbf{T} \subseteq \mathbf{\Lambda} \subseteq \mathbf{SL4}$. *Then*

- $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are not semiproduct matching.*
- $\mathbf{Q\Lambda}$ *is Kripke incomplete.*

The crucial formula for Kripke incompleteness is

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)),$$

and to disprove semiproduct matching one can use

$$\Box\blacksquare ref := (\Box\forall ref)_* = \Box\blacksquare(\Box p \to p).$$

However, by applying the Kripke bundle semantics (cf. [2]) we can prove the following

Theorem 10. *The logics* $\Box\mathbf{T}$, $\mathbf{SL4}$ *are quantifier-friendly.*

Conjecture *Every Horn axiomatizable logic is quantifier-friendly.*

For some logics $\mathbf{\Lambda}$ between $\Box\mathbf{T}$ and $\mathbf{SL4}$ completeness of $\mathbf{Q\Lambda}$ can be restored by adding $\Box\forall ref$. Let

$$5 := \Diamond\Box p \to \Box p,\ \ \mathbf{K5} := \mathbf{K} + 5,\ \ \mathbf{K45} := \mathbf{K4} + 5,$$
$$\Box\mathbf{S5} := \Box\mathbf{T} + \Box((\Box p \to \Box\Box p) \wedge (\Diamond\Box p \to p)).$$

Theorem 11. *For the logics* $\mathbf{\Lambda} = \Box\mathbf{T}, \mathbf{SL4}, \mathbf{K5}, \Box\mathbf{S5}$:

- $C_{\mathcal{K}}(\mathbf{Q\Lambda}) = \mathbf{Q\Lambda} + \Box\forall ref$,
- $\mathbf{\Lambda} \leftthreetimes \mathbf{S5} = \mathbf{\Lambda} \lrcorner\, \mathbf{S5} + \Box\blacksquare ref$.

Corollary 12. *For logics from Theorem 11*

$$(\mathbf{Q\Lambda} + \Box\forall ref){-}1 = \mathbf{\Lambda} \lrcorner\, \mathbf{S5} + \Box\blacksquare ref.$$

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Proper Convex Functors

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Abstract

In this work we deal with algebraic categories and deterministic weighted automata functors on them. Such categories are the target of generalized determinization [1, 2, 4] and enable coalgebraic modelling beyond sets; such automata are the result of generalized determinization. For example, "determinized" non-deterministic automata, weighted, or probabilistic ones are coalgebraically modelled over the categories of join-semilattices, semimodules for a semiring, and convex algebras, respectively. Moreover, expressions for axiomatizing behavior semantics often live in algebraic categories.

In order to prove completeness of such axiomatizations, the common approach [8, 7, 2] is to prove finality of a certain object in a category of coalgebras over an algebraic category. Proofs are significantly simplified if it suffices to verify finality only w.r.t. coalgebras carried by free finitely generated algebras, as those are the coalgebras that result from generalized determinization. In recent work, Milius [9] proposed the notion of a proper functor. If the functor describing determinized systems in an algebraic category (where also the expressions live) is proper, then it suffices to verify finality only w.r.t. coalgebras carried by free finitely generated algebras in completeness proof of axiomatizations. This was completeness proofs are significantly simplified. However, proving properness is hard, i.e., the notion of properness extracts the essence of difficulty in completeness proofs.

Recalling Milius' definition [9], a functor is proper if and only if for any two states that are behaviourally equivalent in coalgebras with free finitely generated carriers, there is a zig-zag of homomorphisms (called a chain of simulations in the original works on weighted automata and proper semirings) that identifies the two states and whose nodes are *all carried by free finitely generated algebras*.

This notion is a generalization of the notion of a proper semiring introduced by Esik and Maletti [10]: A semiring is proper if and only if its "cubic" functor is proper. A cubic functor is a functor $\mathbb{S} \times (-)^A$ where A is a finite alphabet and $\mathbb{S}$ is a free algebra with a single generator in the algebraic category. Cubic functors model deterministic weighted automata which are models of determinized non-deterministic and probabilistic transition systems. The underlying `Set` functors of cubic functors are also sometimes called deterministic-automata functors, see e.g. [4], as their coalgebras are deterministic weighted automata with output in the semiring/algebra. Having properness of a semiring (cubic functor), together with the property of the semiring being finitely and effectively presentable, yields decidability of the equivalence problem (decidability of trace equivalence, i.e., language equivalence) for weighted automata.

In our work on proper semirings and proper convex functors, recently published at FoSSaCS 2018 [12], see [11] for the full version, motivated by the wish to prove properness of a certain functor $\widehat{F}$ on positive convex algebras (`PCA`) used for axiomatizing trace semantics of probabilistic systems in [2], as well as by the open questions stated in [9, Example 3.19], we provide a framework for proving properness and prove:

- The functor $[0,1] \times (-)^A$ on `PCA` is proper, and the required zig-zag is a span.
- The functor $\widehat{F}$ on `PCA` is proper. This proof is quite involved, and interestingly, provides the only case that we encountered where the zig-zag is not a span: it contains three intermediate nodes of which the middle one forms a span.

Along the way, we instantiate our framework on some known cases like Noetherian semirings and $\mathbb{N}$ (with a zig-zag that is a span), and prove new semirings proper: The semirings $\mathbb{Q}_+$ and $\mathbb{R}_+$ of non-negative rationals and reals, respectively. The shape of these zig-zags is a span as well. It is an interesting question for future work whether these new properness results may lead to new complete axiomatizations of expressions for certain weighted automata.

Our framework requires a proof of so-called *extension lemma* and *reduction lemma* in each case. While the extension lemma is a generic result that covers all cubic functors of interest, the reduction lemma is in all cases a nontrivial property intrinsic to the algebras under consideration. For the semiring of natural numbers it is a consequence of a result that we trace back to Hilbert [16]; for the case of convex algebra $[0,1]$ the result is due to Minkowski [17]. In the case of $\widehat{F}$, we use Kakutani's set-valued fixpoint theorem [6].

All base categories in this work are algebraic categories, i.e., categories $\mathtt{Set}^T$ of Eilenberg-Moore algebras of a finitary monad T on `Set`.

The main category of interest to us is the category `PCA` of positively convex algebras, the Eilenberg-Moore algebras of the monad of finitely supported subprobability distributions, see, e.g., [13, 14] and [15].

Concretely, a positive convex algebra $\mathbb{A}$ in `PCA` is a carrier set A together with infinitely many finitary operations denoting sub-convex sums, i.e., for each tuple $(p_i \mid 1 \le i \le n)$ with $p_i \in [0,1]$ and $\sum_i p_i \le 1$ we have a corresponding n-ary operation, the sub-convex combination with coefficients p_i. (Positive) Convex algebras algebras satisfy two axioms: the projection axiom stating that $\sum p_i x_i = x_k$ if $p_k = 1$; and the barycentre axiom

$$\sum_i p_i (\sum_j p_{ij} x_j) = \sum_j (\sum_i p_i \cdot p_{ij}) x_j.$$

These axioms are precisely the unit and multiplication law required from an Eilenberg-Moore algebra when instantiated to the probability subdistribution monad, and enable working with abstract convex combinations (formal sums) in the usual way as with convex combinations / sums in $\mathbb{R}$.

For the proofs of proper semirings, we work in the category $\mathbb{S}$-`SMOD` of semimodules over a semiring $\mathbb{S}$ which are the Eilenberg-Moore algebras of the monad $T_{\mathbb{S}}$ of finitely supported maps into $\mathbb{S}$.

For $n \in \mathbb{N}$, the free algebra with n generators in $\mathbb{S}$-`SMOD` is the direct product $\mathbb{S}^n$, and in `PCA` it is the n-simplex $\Delta^n = \{(\xi_1, \dots, \xi_n) \mid \xi_j \ge 0, \sum_{j=1}^n \xi_j \le 1\}$.

For the semirings $\mathbb{N}$, $\mathbb{Q}_+$, and $\mathbb{R}_+$ that we deal with (with ring completions $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively), the categories of $\mathbb{S}$-semimodules are:

- `CMON` of commutative monoids for $\mathbb{N}$,
- `AB` of abelian groups for $\mathbb{Z}$,
- `CONE` of convex cones for $\mathbb{R}_+$, and
- $\mathbb{Q}$-`VEC` and $\mathbb{R}$-`VEC` of vector spaces over the field of rational and real numbers, respectively, for $\mathbb{Q}$ and $\mathbb{R}$.

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Semi-analytic Rules and Craig Interpolation

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Abstract

In [1], Iemhoff introduced the notion of a centered axiom and a centered rule as the building blocks for a certain form of sequent calculus which she calls a centered proof system. She then showed how the existence of a terminating centered system implies the uniform interpolation property for the logic that the calculus captures. In this paper we first generalize her centered rules to semi-analytic rules, a dramatically powerful generalization, and then we will show how the semi-analytic calculi consisting of these rules together with our generalization of her centered axioms, lead to the feasible Craig interpolation property. Using this relationship, we first present a uniform method to prove interpolation for different logics from sub-structural logics $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_{ew}}$ and **IPC** to their appropriate classical and modal extensions, including the intuitionistic and classical linear logics. Then we will use our theorem negatively, first to show that so many sub-structural logics including $Ł_n$, G_n, BL, R and RM^e and almost all super-intutionistic logics (except at most seven of them) do not have a semi-analytic calculus.

Let us begin with some preliminaries. First fix a propositional language extending the language of $\mathbf{FL_e}$. By the meta-language of this language we mean the language with which we define the sequent calculi. It extends our given language with the formula symbols (variables) such as ϕ and ψ. A meta-formula is defined as the following: Atomic formulas and formula symbols are meta-formulas and if $\bar{\phi}$ is a set of meta-formulas, then $C(\bar{\phi})$ is also a meta-formula, where $C \in \mathcal{L}$ is a logical connective of the language. Moreover, we have infinitely many variables for meta-multisets and we use capital Greek letters again for them, whenever it is clear from the context whether it is a multiset or a meta-multiset variable. A meta-multiset is a multiset of meta-formulas and meta-multiset variables. By a meta-sequent we mean a sequent where the antecedent and the succedent are both meta-multisets. We use meta-multiset variable and context, interchangeably.

For a meta-formula ϕ, by $V(\phi)$ we mean the meta-formula variables and atomic constants in ϕ. A meta-formula ϕ is called p-free, for an atomic formula or meta-formula variable p, when $p \notin V(\phi)$.

And finally note that by $\mathbf{FL_e}^-$ we mean the system $\mathbf{FL_e}$ minus the following axioms:

$$\frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

And $\mathbf{CFL_e}^-$ has the same rules as $\mathbf{FL_e}^-$, this time in their full multi-conclusion version, where $+$ is added to the language and also the usual left and right rules for $+$ are added to the system.

Now let us define some specific forms of the sequent-style rules:

Definition 1. A rule is called a *left semi-analytic rule* if it is of the form

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where Π_j, Γ_i and Δ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

and it is called a *right semi-analytic rule* if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

where Γ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,r} V(\bar{\psi}_{ir}) \subseteq V(\phi)$$

For the multi-conclusion case, we define a rule to be *left multi-conclusion semi-analytic* if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

with the same variable condition as above and the same condition that all Γ_i's and Δ_i's are meta-multiset variables. A rule is defined to be a *right multi-conclusion semi-analytic* rule if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

again with the similar variable condition and the same condition that all Γ_i's and Δ_i's are meta-multiset variables.
Moreover, the usual modal rules in the cut-free Gentzen calculus for the logics **K**, **K4**, **KD** and **S4** are considered as *semi-analytic modal rules.*

Definition 2. A sequent is called a *centered axiom* if it has the following form:

(1) Identity axiom: $(\phi \Rightarrow \phi)$

(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$

(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$

(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and the variables in any pair of elements in $\bar{\alpha}$ or in $\bar{\beta}$ or in $\bar{\phi}$ are equal.

The main theorem of the paper is the following:

Theorem 3. *(i) If* $\mathbf{FL_e} \subseteq L$*, (*$\mathbf{FL_e}^- \subseteq L$*) and* L *has a single-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then* L *has Craig interpolation.*

(*ii*) *If* $\mathbf{CFL_e} \subseteq L$, $(\mathbf{CFL_e}^- \subseteq L)$ *and* L *has a multi-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then* L *has Craig interpolation.*

Proof. Call the centered sequent system G. Use the Maehara technique to prove that for any derivable sequent $S = (\Sigma, \Lambda \Rightarrow \Delta)$ in G there exists a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \Delta)$ are provable in G and $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, where $V(A)$ is the set of the atoms of A. □

As a positive result, our method provides a uniform way to prove the Craig interpolation property for substructural logics. For instance we have:

Corollary 4. *The logics* $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$, **ILL**, **CLL**, **IPC**, **CPC** *and their* **K**, **KD** *and* **S4** *versions have the Craig interpolation property. The same also goes for* **K4** *and* **K4D** *extensions of* **IPC** *and* **CPC**.

Proof. The usual cut-free sequent calculus for all of these logics consists of semi-analytic rules and centered axioms. Now, use Corollary 3. □

As a much more interesting negative result, which is also the main contribution of our investigation, we show that many different sub-structural logics do not have a complete sequent calculus consisting of semi-analytic rules and cenetered axioms. Our proof is based on the prior works (for instance [4] and [2]) that established some negative results on the Craig interpolation of some sub-structural logics. Considering the naturalness and the prevalence of these rules, our negative results expel so many logics from the elegant realm of natural sequent calculi.

Corollary 5. *None of the logics R, BL and* L_∞, L_n *for* $n \geqslant 3$ *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms.*

Corollary 6. *Except G, G*3 *and* **CPC**, *none of the consistent BL-extensions have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free centered axioms.*

Corollary 7. *Except eight specific logics, none of the consistent extensions of* RM^e *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms.*

Corollary 8. *Except seven specific logics, none of the consistent super-intuitionistic logics have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules, context-sharing semi-analytic rules and centered axioms.*

A more detailed version of the Corollaries 7 and 8 can be found in [3].

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Propositional Union Closed Team Logics: Expressive Power and Axiomatizations

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In this paper, we prove the expressive completeness of some propositional *union closed team logics*, and introduce sound and complete systems of natural deduction for these logics. These logics are variants of *dependence logic*, which is a non-classical first-order logic, introduced by Väänänen, for reasoning about dependencies. This framework extends the classical logic by adding new atomic formulas for charactering dependence and independence between variables. Examples of such atoms are *dependence atoms* (giving rise to dependence logic), and *inclusion atoms* (giving rise to inclusion logic [1]). Hodges [3, 4] observed that dependency properties can only manifest themselves in *multitudes*, and he thus introduced the so-called *team semantics* that dependence logic and its variants adopt. Formulas of these logics are evaluated under *teams*, which in the propositional context are *sets* of valuations.

Logics based on team semantics (also called *team logics*) can have interesting closure properties. For example, dependence logic is *closed downwards*, meaning that the truth of a formula on a team is preserved under taking subteams. In this paper, we consider propositional team logics that are *closed under unions*, meaning that if two teams both satisfy a formula, then their union also satisfies the formula. Inclusion logic is closed under unions. Other known union closed logics are classical logic extended with *anonymity atoms* (introduced very recently by Väänänen [6] to characterize anonymity in the context of privacy), or with the *relevant disjunction* $\veebar$ (introduced by Rönnholm, see [5], and also named *nonempty disjunction* by some other authors [2, 8]).

While propositional downwards closed team logics are well studied (e.g., [7]), propositional union closed team logics are not understood very well yet. It follows from [2] that propositional inclusion logic (**PInc**) with extended inclusion atoms is expressively complete, and **PInc** is thus expressively equivalent to classical logic extended with relevant disjunction (**PU**), which is shown to be also expressively complete in [8]. We show in this paper that classical logic extended with anonymity atoms (**PAm**) is also expressively complete, and **PInc** with slightly less general inclusion atoms is already expressively complete. From the expressive completeness, we will derive the interpolation theorem for these logics. We also provide axiomatizations for **PInc**, **PU** and **PAm**, which are lacking in the literature. We define sound and complete systems of natural deduction for these logics. As with other team logics, these systems do not admit uniform substitution. Another interesting feature of the systems is that the usual disjunction introduction rule ($\phi/\phi \veebar \psi$) is not sound for the relevant disjunction.

1 Propositional union closed team logics

Fix a set Prop of propositional variables, whose elements are denoted by $p, q, r, \ldots$ (with or without subscripts). We first define the *team semantics* for *classical propositional logic* (**CPL**), whose well-formed formulas (called *classical formulas*), in the context of the present paper, are given by the grammar:

$$\alpha ::= p \mid \bot \mid \top \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

Let $\mathsf{N} \subseteq \mathsf{Prop}$ be a set of propositional variables. An *(N-)team* is a set of valuations $v : \mathsf{N} \cup \{\bot, \top\} \to \{0, 1\}$ with $v(\bot) = 0$ and $v(\top) = 1$. Note that the empty set $\varnothing$ is a team. The notion of a classical formula α being *true* on a team X, denoted by $X \vDash \alpha$, is defined inductively as follows:

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- $X \models p$ iff for all $v \in X$, $v(p) = 1$.
- $X \models \bot$ iff $X = \varnothing$.
- $X \models \top$ always holds
- $X \models \neg\alpha$ iff for all $v \in X$, $\{v\} \not\models \alpha$.
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.
- $X \models \alpha \vee \beta$ iff there are $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \alpha$ and $Z \models \beta$.

Clearly, **CPL** has the *empty team property*, *union closure property* and *downwards closure property*:

Empty Team Property: $\varnothing \models \alpha$ holds for all α;

Union Closure: $X \models \alpha$ and $Y \models \alpha$ imply $X \cup Y \models \alpha$;

Downwards Closure: $X \models \alpha$ and $Y \subseteq X$ imply $Y \models \alpha$.

The union closure and downwards closure property together are equivalent to the *flatness property*:

Flatness $X \models \alpha$ if and only if $\{v\} \models \alpha$ for all $v \in X$.

The flatness of classical formulas shows that team semantics is conservative over classical formulas. We now extend **CPL** to three non-flat but union closed team-based logics. Consider a new disjunction $\mathbin{\veebar}$, called *relevant disjunction*, and new atomic formulas of the form $a_1 \dots a_k \subseteq b_1 \dots b_k$ with each $a_i, b_i \in \mathsf{Prop} \cup \{\bot, \top\}$, called *inclusion atoms*, and of the form $\neq(p_1, \dots, p_k; q)$, called *anonymity atoms*. The team semantics of these new connective and atoms are defined as:

- $X \models \phi \mathbin{\veebar} \psi$ iff $X = \varnothing$ or there are nonempty Y and Z such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$.
- $X \models a_1 \dots a_k \subseteq b_1 \dots b_k$ iff for all $v \in X$, there exists $v' \in X$ such that
$$\langle v(a_1), \dots, v(a_k) \rangle = \langle v'(b_1), \dots, v'(b_k) \rangle.$$
- $X \models \neq(p_1, \dots, p_k; q)$ iff for all $v \in X$, there exists $v' \in X$ such that
$$\langle v(p_1), \dots, v(p_k) \rangle = \langle v'(p_1), \dots, v'(p_k) \rangle \text{ and } v(q) \neq v'(q).$$

Note the similarity and difference between the semantics clauses of $\vee$ and $\mathbin{\veebar}$. In particular, we write $\neq(p)$ for $\neq(\langle\rangle; p)$, and clearly its semantics clause is reduced to

- $X \models \neq(p)$ iff either $X = \varnothing$ or there exist $v, v' \in X$ such that $v(p) \neq v'(p)$.

We define the syntax of *propositional union closed logic* (**PU**) as the syntax of **CPL** expanded by adding $\mathbin{\veebar}$, and negation $\neg$ is allowed to occur only in front of classical formulas, that is,

$$\phi ::= p \mid \bot \mid \top \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbin{\veebar} \phi,$$

where α is an arbitrary classical formula. Similarly, *propositional inclusion logic* (**PInc**) is **CPL** extended with inclusion atoms $a_1 \dots a_k \subseteq b_1 \dots b_k$ (and negation occurs only in front of classical formulas), and *propositional anonymity logic* (**PAm**) is **CPL** extended with anonymity atoms $\neq(p_1, \dots, p_k; q)$ (and negation occurs only in front of classical formulas).

For any formula ϕ in $\mathsf{N} \subseteq \mathsf{Prop}$, we write $[\![\phi]\!] = \{X \text{ an } \mathsf{N}\text{-team} : X \models \phi\}$. It is easy to verify that for any formula ϕ in the language of **PU** or **PInc** or **PAm**, the set $[\![\phi]\!]$ contains the empty team $\varnothing$, and is closed under unions, i.e., $X, Y \in [\![\phi]\!]$ implies $X \cup Y \in [\![\phi]\!]$.

2 Expressive completeness

It was proved in [8] that **PU** is *expressively complete* with respect to the set of all union closed team properties which contain the empty team, in the sense that for any set $\mathsf{N} \subseteq \mathsf{Prop}$, for any set P of N-teams that is closed under unions and contains the empty team, we have $\mathsf{P} = [\![\phi]\!]$ for some **PU**-formula ϕ in N. The proof in [8] first defines for any N-team X with $\mathsf{N} = \{p_1, \dots, p_n\}$ a **PU**-formula

$$\Psi_X := \mathop{\bigveebar}_{v \in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}),$$

where $v(i)$ is short for $v(p_i)$, $p_i^1 := p_i$, and $p_i^0 = \neg p_i$. Observing that $Y \models \Psi_X \iff Y = X$ holds for any N-team Y, one then easily establishes that $\mathsf{P} = [\![\bigvee_{X\in\mathsf{P}} \Psi_X]\!]$. Generalizing this argument, we can now show that **PInc** and **PAm** are both expressively complete in the same sense, and in particular, all these three union closed team logics we introduced are equivalent in expressive power.

Theorem 1. **PU** ≡ **PInc** ≡ **PAm**.

Proof. (sketch) We first show that the **PU**-formula Ψ_X is expressible in **PInc**. Define **PAm**-formulas

$$\Theta_X := \bigvee_{v\in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}), \qquad \text{and} \qquad \Phi_X := \bigwedge_{v\in X} \underline{v(1)} \ldots \underline{v(n)} \subseteq p_1 \ldots p_n,$$

where $\underline{0} := \bot$ and $\underline{1} := \top$. Observe that for any N-team Y,

$$Y \models \Theta_X \iff Y \subseteq X, \qquad \text{and} \qquad Y \models \Phi_X \iff X \subseteq Y.$$

Thus, $\Psi_X \equiv \Theta_X \wedge \Phi_X$[1].

To show that Ψ_X is expressible in **PAm**, we show that for any N-team X and any $\mathsf{K} = \{p_{i_1}, \ldots, p_{i_k}\} \subseteq \{p_1, \ldots, p_n\} = \mathsf{N}$, the formula $\Psi_X^{\mathsf{K}} = \dot{\bigvee}_{v\in X} (p_{i_1}^{v(i_1)} \wedge \cdots \wedge p_{i_k}^{v(i_k)})$ is expressible in **PAm** as some ψ_X^{K} by induction on $|\mathsf{K}| \leq n$. If $|\mathsf{K}| = 1$, then $\Psi_X^{\mathsf{K}} \equiv p_{i_1}$ or $\neg p_{i_1}$ or $\neq(p_{i_1})$. If $|\mathsf{K}| = m + 1$, let $\mathsf{K} = \mathsf{K}_0 \cup \{p_{i_{m+1}}\}$, $Y = \{v \in X \mid v(i_{m+1}) = 1\}$ and $Z = \{v \in X \mid v(i_{m+1}) = 0\}$. If $Y = \varnothing$, then by induction hypothesis,

$$\Psi_X^{\mathsf{K}} = \dot{\bigvee}_{v\in Z} (p_{i_1}^{v(i_1)} \wedge \cdots \wedge p_{i_m}^{v(i_m)} \wedge \neg p_{i_{m+1}}) \equiv \Big(\dot{\bigvee}_{v\in Z} (p_{i_1}^{v(i_1)} \wedge \cdots \wedge p_{i_m}^{v(i_m)}) \Big) \wedge \neg p_{i_{m+1}} \equiv \psi_Z^{\mathsf{K}_0} \wedge \neg p_{i_{m+1}}.$$

Similarly, if $Z = \varnothing$, then $\Psi_X^{\mathsf{K}} \equiv \psi_Y^{\mathsf{K}_0} \wedge p_{i_{m+1}}$. Now, if $Y, Z \neq \varnothing$, we have by induction hypothesis that

$$\Psi_X^{\mathsf{K}} \equiv (\psi_Y^{\mathsf{K}_0} \wedge p_{i_{m+1}}) \,\dot\vee\, (\psi_Z^{\mathsf{K}_0} \wedge \neg p_{i_{m+1}}) \equiv \Big((\psi_Y^{\mathsf{K}_0} \wedge p_{i_{m+1}}) \vee (\psi_Z^{\mathsf{K}_0} \wedge \neg p_{i_{m+1}}) \Big) \wedge \neq(p_{i_{m+1}}). \qquad \square$$

We show next that the interpolation property of a team logic is a consequence of the expressive completeness and the *locality property*, which is defined as:

Locality: For any formula ϕ in $\mathsf{N} \subseteq \mathsf{Prop}$, if X is an N_0-team and Y an N_1-team such that $\mathsf{N} \subseteq \mathsf{N}_0, \mathsf{N}_1$ and $X \upharpoonright \mathsf{N} = Y \upharpoonright \mathsf{N}$, then $X \models \phi \iff Y \models \phi$

The team logics **PU**, **PInc** and **PAm** all have the locality property. But let us emphasize here that in the team semantics setting, locality is not a trivial property. Especially, if in the semantics clause of disjunction $\vee$ the two subteams $Y, Z \subseteq X$ are required to be disjoint, then the logic **PInc** is not local any more, as, e.g., the formula $pq \subseteq rs \vee tu \subseteq rs$ (with the modified semantics for $\vee$) is not local.

Theorem 2 (Interpolation)**.** *If a team logic* L *is expressively complete and has the locality property, then it enjoys Craig's Interpolation. In particular,* **PU**, **PInc** *and* **PAm** *enjoy Craig's interpolation.*

Proof. (sketch) Suppose ϕ is an L-formula in $\mathsf{N} \cup \mathsf{N}_0 \subseteq \mathsf{Prop}$, and ψ an L-formula in $\mathsf{N} \cup \mathsf{N}_1 \subseteq \mathsf{Prop}$. Since L is expressively complete, there is an L-formula θ in N such that $[\![\theta]\!] = [\![\phi]\!]|_{\mathsf{N}} = \{X \upharpoonright \mathsf{N} : X \models \phi\}$. It follows from the locality property of L that θ is the desired interpolant, i.e., $\phi \models \theta$ and $\theta \models \psi$. $\square$

3 Axiomatizations

The proof of Theorem 1 and also results in [8] show that every formula in the language of **PU**, **PInc** or **PAm** can be turned into an equivalent formula in a certain normal form, .e.g, the form $\bigvee_{X\in\mathsf{P}} \Psi_X$ for **PU**. Making use of these normal forms, we can axiomatize these union closed team logics.

[1] This **PInc**-formula is essentially adapted from a very similar modal formula in [2], but our formula Φ_X is slightly simpler than the one in [2], which uses slightly more general inclusion atoms. In this sense, the result that our version of **PInc** is expressively complete is a slight refinement of the expressive completeness of another version of **PInc** that follows from [2].

We present in this abstract only the system of natural deduction for **PU**, and the systems for **PInc** or **PAm** have rules for inclusion and anonymity atoms in addition to the following ones. In the following rules, α ranges over classical formulas only:

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \bot\end{matrix}}{\neg\alpha}\,\neg\mathsf{I} \qquad \dfrac{\neg\neg\alpha}{\alpha}\,\neg\neg\mathsf{E} \qquad \dfrac{\alpha \qquad \neg\alpha}{\phi}\,\neg\mathsf{E} \qquad \dfrac{\bot}{\phi}\,\text{ex falso} \qquad \dfrac{\phi \qquad \psi}{\phi\wedge\psi}\,\wedge\mathsf{I} \qquad \dfrac{\phi\wedge\psi}{\phi}\,\wedge\mathsf{E} \qquad \dfrac{\phi\wedge\psi}{\psi}\,\wedge\mathsf{E}$$

$$\dfrac{\phi}{\phi\vee\psi}\,\vee\mathsf{I} \qquad \text{For } \circ\in\{\vee,\dot\vee\}\text{:}\ \dfrac{\begin{matrix}\\ D_0\\ \phi\circ\psi\end{matrix} \qquad \begin{matrix}[\phi]\\ D_1\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ D_2\\ \chi\end{matrix}}{\chi}\,\circ\mathsf{E} \qquad \dfrac{\phi\circ\phi}{\phi}\,\circ\mathsf{Ctr} \qquad \dfrac{\psi\,\dot\vee\,\bot}{\phi}\,\dot\vee\mathsf{E} \qquad \dfrac{\phi\,\dot\vee\,\psi}{\phi\vee\psi}\,\dot\vee>\vee$$

The undischarged assumptions in D_0 contains classical formulas only

$$\dfrac{\phi\,\dot\vee\,(\psi\vee\chi)}{(\phi\,\dot\vee\,\psi)\vee(\phi\,\dot\vee\,\chi)}\,\mathsf{Dstr}\,\dot\vee\,\vee \qquad \dfrac{\phi\vee(\psi\,\dot\vee\,\chi)}{(\phi\vee\psi)\,\dot\vee\,(\phi\vee\chi)}\,\mathsf{Dstr}\vee\dot\vee \qquad \dfrac{\Big(\bigvee_{X\in\mathcal{X}}\Psi_X\Big)\wedge\Big(\bigvee_{Y\in\mathcal{Y}}\Psi_Y\Big)}{\bigvee_{Z\in\mathcal{Z}}\Psi_Z}\,\mathsf{Dstr}\vee\wedge\vee$$

where $\mathcal{Z}=\{Z=\bigcup\mathcal{X}'=\bigcup\mathcal{Y}' \mid \mathcal{X}'\subseteq\mathcal{X}\ \&\ \mathcal{Y}'\subseteq\mathcal{Y}\}$

As other systems for team logics (see e.g., [7, 8]), the above system does not admit *uniform substitution*, as, e.g., the negation rules apply to classical formulas only. Restricted to classical formulas, the above system contains all the usual rules for disjunction $\vee$.

Theorem 3 (Sound and Completeness)**.** *For any* **PU***-formulas ϕ and ψ, we have* $\psi\models\phi \iff \psi\vdash\phi$.

Proof. (idea) Use the normal form of **PU**, and the equivalence of the following clauses:

(i) $\bigvee_{X\in\mathcal{X}}\Psi_X \models \bigvee_{Y\in\mathcal{Y}}\Psi_Y$.

(ii) for each $X\in\mathcal{X}$, there exists $\mathcal{Y}_X\subseteq\mathcal{Y}$ such that $X=\bigcup\mathcal{Y}_X$. □

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