# On one-variable fragments of modal predicate logics.

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We consider normal 1-modal predicate logics as they were defined in [2]. I.e., the signature contains predicate letters of different arities, but no constants or functions letters; also we do not specify the equality symbol. A logic is a set of formulas containing all classical validities and the axioms of $\mathbf{K}$ and closed under Modus Ponens, $\forall$- and $\Box$- introduction, and predicate substitution.

We also consider (normal) modal propositional logics. For a propositional modal logic $\mathbf{\Lambda}$, its minimal predicate extension is denoted by $\mathbf{Q\Lambda}$.

For modal predicate logics several different semantics are known. The most popular is Kripke semantics (with expanding domains). Recall that a *predicate Kripke frame* is a pair $(F, D)$, where $F = (W, R)$ is a propositional Kripke frame (i.e., $W \neq \varnothing,\ R \subseteq W^2$) and $D$ is a family of non-empty sets $(D_u)_{u \in W}$ such that $D_u \subseteq D_v$ whenever $uRv$. A *predicate Kripke model* over a frame $\mathbf{F} = (F, D)$ is a pair $(\mathbf{F}, \xi)$, where $\xi$ (a valuation) is a family $(\xi_u)_{u \in W}$, $\xi_u$ sends every $n$-ary predicate letter $P$ to an $n$-ary relation on $D_u$ (0-ary relation is a fixed value 0 or 1).

The truth predicate $M, u \vDash A$ is defined for any world $u \in W$ and $D_u$*-sentence* $A$ (a sentence with individuals from $D_u$ as constants). In particular,

$$M, u \vDash P(a_1, \dots, a_n) \text{ iff } (a_1, \dots, a_n) \in \xi_u(P),$$

$$M, u \vDash \Box A \text{ iff } \forall v \in R(u)\, M, v \vDash A,$$

$$M, u \vDash \forall x A \text{ iff } \forall a \in D_u\, M, u \vDash [a/x]A.$$

A formula $A$ is *valid* in $\mathbf{F} = (F, D)$ (in symbols, $\mathbf{F} \vDash A$) if its universal closure is true at any world in any Kripke model over $\mathbf{F}$. The set $\mathbf{ML}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\}$ is a modal predicate logic (the *logic of* $\mathbf{F}$). The *logic of a class* of frames $\mathcal{C}$ is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\mathbf{F}) \mid F \in \mathcal{C}\}$. Logics of this form are called *Kripke complete*.

Similar definitions for propositional logics are well-known, so we skip them.

Formulas constructed from a single variable $x$ and monadic predicate letters are called *1-variable*. Every such formula $A$ translates into a bimodal propositional formula $A_*$ with modalities $\Box$ and $\blacksquare$ if every atom $P_i(x)$ is replaced with the proposition letter $p_i$ and every quantifier $\forall x$ with $\blacksquare$. The *1-variable fragment* of a predicate logic $L$ is the set

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is 1-variable}\}.$$

Then

**Lemma 1.** *$L{-}1$ is a bimodal propositional logic.*

Recall that for monomodal propositional logics $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ the *fusion* $\mathbf{\Lambda}_1 * \mathbf{\Lambda}_2$ is the smallest bimodal logic containing $\mathbf{\Lambda}_1 \cup \mathbf{\Lambda}_2$ (if the modal connectives in $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ are distinct).

The *semicommutative join* of a monomodal logic $\mathbf{\Lambda}$ (in the language with $\Box$) with $\mathbf{S5}$ (in the language with $\blacksquare$) is obtained from $\mathbf{\Lambda} * \mathbf{S5}$ by adding the axiom

$$(com^l) \qquad \Box\blacksquare p \to \blacksquare\Box p.$$

This logic is denoted by $\mathbf{\Lambda} \rfloor \mathbf{S5}$. The following properties are easily checked.

**Lemma 2.** *(1)* $\mathbf{\Lambda} \rfloor \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1$.

*(2) A propositional Kripke frame* $F = (W, R_1, R_2)$ *validates* $\mathbf{\Lambda} \rfloor \mathbf{S5}$ *iff*

$$(W, R_1) \vDash \mathbf{\Lambda} \,\&\, R_2 \textit{ is an equivalence } \,\&\, R_2 \circ R_1 \subseteq R_1 \circ R_2.$$

**Definition 1.** *Let* $F_1 = (U_1, R_1)$, $F_2 = (U_2, R_2)$ *be propositional frames. Their* product *is the frame* $F_1 \times F_2 = (U_1 \times U_2, R_h, R_v)$, *where*

- $(u, v)R_h(u', v')$ *iff* $uR_1u'$ *and* $v = v'$,
- $(u, v)R_v(u', v')$ *iff* $u = u'$ *and* $vR_2v'$.

*A* semiproduct *(or an* expanding product*) of* $F_1$ *and* $F_2$ *is a subframe* $(W, S_1, S_2)$ *of* $F_1 \times F_2$ *such that* $R_h(W) \subseteq W$.

**Definition 2.** *A* semiproduct $\mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ *of a monomodal propositional logic* $\mathbf{\Lambda}$ *with* $\mathbf{S5}$ *is the logic of the class of all semiproducts of* $\mathbf{\Lambda}$*-frames (i.e., frames validating* $\mathbf{\Lambda}$*) with clusters (i.e., frames with a universal relation).*

From Lemma 2 we readily have

**Lemma 3.** $\mathbf{\Lambda} \rfloor \mathbf{S5} \subseteq \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$.

**Definition 3.** *The logics* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are called* semiproduct matching *if*
$\mathbf{\Lambda} \rfloor \mathbf{S5} = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. $\mathbf{\Lambda}$ *is called* quantifier-friendly *if* $\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \rfloor \mathbf{S5}$.

**Definition 4.** *The* Kripke-completion $C_{\mathcal{K}}(L)$ *of a predicate logic* $L$ *is the logic of the class of all frames validating* $L$.

$C_{\mathcal{K}}(L)$ is the smallest Kripke-complete extension of $L$. From definitions we obtain

**Lemma 4.** $C_{\mathcal{K}}(\mathbf{Q\Lambda}){-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$. *Hence*

$$\mathbf{Q\Lambda}{-}1 \subseteq \mathbf{\Lambda} \leftthreetimes \mathbf{S5}.$$

*Also*

$$\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$$

*if* $\mathbf{Q\Lambda}$ *is Kripke-complete, and*

$$\mathbf{Q\Lambda}{-}1 = \mathbf{\Lambda} \rfloor \mathbf{S5}$$

*if* $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching.*

The following result is proved in [1] (Theorem 9.10).

**Theorem 5.** $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching*[1] *for* $\mathbf{\Lambda} = \mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}$.

[1] The book [1] uses a different terminology and notation: semiproducts are called 'expanding products', $\mathbf{\Lambda} \rfloor \mathbf{S5}$ is denoted by $[\mathbf{\Lambda}, \mathbf{S5}]^{EX}$, $\mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ by $(\mathbf{\Lambda} \times \mathbf{S5})^{EX}$, and there is no term for 'semiproduct matching'.

This result extends to a slightly larger class.

**Definition 5.** *A* one-way PTC-logic *is a modal propositional logic axiomatized by formulas of the form* $\Box p \to \Box^n p$ *and closed (i.e., variable-free) formulas.*

**Theorem 6.** $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are semiproduct matching for any one-way PTC-logic* $\mathbf{\Lambda}$*.*

**Corollary 7.** *Every one-way PTC-logic is quantifier-friendly.*

Also we have

**Theorem 8.** *([2], theorem 6.1.29)* $\mathbf{Q\Lambda}$ *is Kripke complete for any one-way PTC-logic* $\mathbf{\Lambda}$*.*

It is not clear how far theorems 6, 8 can be generalized. Anyway, unlike the case of products and predicate logics with constant domains (cf. [1], [2]), they do not hold for arbitrary Horn axiomatizable logics

Many counterexamples are given by the next theorem.

Let

$$\Box ref := \Box(\Box p \to p),\ \Box\mathbf{T} := \mathbf{K} + \Box ref,$$
$$\mathbf{SL4} := \mathbf{K4} + \Diamond p \leftrightarrow \Box p.$$

**Theorem 9.** *Let* $\mathbf{\Lambda}$ *be a propositional logic such that* $\Box\mathbf{T} \subseteq \mathbf{\Lambda} \subseteq \mathbf{SL4}$*. Then*

- $\mathbf{\Lambda}$ *and* $\mathbf{S5}$ *are not semiproduct matching.*
- $\mathbf{Q\Lambda}$ *is Kripke incomplete.*

The crucial formula for Kripke incompleteness is

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)),$$

and to disprove semiproduct matching one can use

$$\Box\blacksquare ref := (\Box\forall ref)_* = \Box\blacksquare(\Box p \to p).$$

However, by applying the Kripke bundle semantics (cf. [2]) we can prove the following

**Theorem 10.** *The logics* $\Box\mathbf{T}$, $\mathbf{SL4}$ *are quantifier-friendly.*

**Conjecture** *Every Horn axiomatizable logic is quantifier-friendly.*

For some logics $\mathbf{\Lambda}$ between $\Box\mathbf{T}$ and $\mathbf{SL4}$ completeness of $\mathbf{Q\Lambda}$ can be restored by adding $\Box\forall ref$. Let

$$5 := \Diamond\Box p \to \Box p,\ \mathbf{K5} := \mathbf{K} + 5,\ \mathbf{K45} := \mathbf{K4} + 5,$$
$$\Box\mathbf{S5} := \Box\mathbf{T} + \Box((\Box p \to \Box\Box p) \wedge (\Diamond\Box p \to p)).$$

**Theorem 11.** *For the logics* $\mathbf{\Lambda} = \Box\mathbf{T}, \mathbf{SL4}, \mathbf{K5}, \Box\mathbf{S5}$*:*

- $C_{\mathcal{K}}(\mathbf{Q\Lambda}) = \mathbf{Q\Lambda} + \Box\forall ref$*,*
- $\mathbf{\Lambda} \leftthreetimes \mathbf{S5} = \mathbf{\Lambda} \rfloor \mathbf{S5} + \Box\blacksquare ref$*.*

**Corollary 12.** *For logics from Theorem 11*

$$(\mathbf{Q\Lambda} + \Box\forall ref){-}1 = \mathbf{\Lambda} \rfloor \mathbf{S5} + \Box\blacksquare ref.$$

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