Proper Convex Functors

Ana Sokolova[1] and Harald Woracek[2]

[1] University of Salzburg, Austria
ana.sokolova@cs.uni-salzburg.at
[2] TU Vienna, Austria
harald.woracek@tuwien.ac.at

Abstract

In this work we deal with algebraic categories and deterministic weighted automata functors on them. Such categories are the target of generalized determinization [1, 2, 4] and enable coalgebraic modelling beyond sets; such automata are the result of generalized determinization. For example, "determinized" non-deterministic automata, weighted, or probabilistic ones are coalgebraically modelled over the categories of join-semilattices, semimodules for a semiring, and convex algebras, respectively. Moreover, expressions for axiomatizing behavior semantics often live in algebraic categories.

In order to prove completeness of such axiomatizations, the common approach [8, 7, 2] is to prove finality of a certain object in a category of coalgebras over an algebraic category. Proofs are significantly simplified if it suffices to verify finality only w.r.t. coalgebras carried by free finitely generated algebras, as those are the coalgebras that result from generalized determinization. In recent work, Milius [9] proposed the notion of a proper functor. If the functor describing determinized systems in an algebraic category (where also the expressions live) is proper, then it suffices to verify finality only w.r.t. coalgebras carried by free finitely generated algebras in completeness proof of axiomatizations. This was completeness proofs are significantly simplified. However, proving properness is hard, i.e., the notion of properness extracts the essence of difficulty in completeness proofs.

Recalling Milius' definition [9], a functor is proper if and only if for any two states that are behaviourally equivalent in coalgebras with free finitely generated carriers, there is a zig-zag of homomorphisms (called a chain of simulations in the original works on weighted automata and proper semirings) that identifies the two states and whose nodes are *all carried by free finitely generated algebras*.

This notion is a generalization of the notion of a proper semiring introduced by Esik and Maletti [10]: A semiring is proper if and only if its "cubic" functor is proper. A cubic functor is a functor $\mathbb{S} \times (-)^A$ where A is a finite alphabet and $\mathbb{S}$ is a free algebra with a single generator in the algebraic category. Cubic functors model deterministic weighted automata which are models of determinized non-deterministic and probabilistic transition systems. The underlying `Set` functors of cubic functors are also sometimes called deterministic-automata functors, see e.g. [4], as their coalgebras are deterministic weighted automata with output in the semiring/algebra. Having properness of a semiring (cubic functor), together with the property of the semiring being finitely and effectively presentable, yields decidability of the equivalence problem (decidability of trace equivalence, i.e., language equivalence) for weighted automata.

In our work on proper semirings and proper convex functors, recently published at FoSSaCS 2018 [12], see [11] for the full version, motivated by the wish to prove properness of a certain functor $\widehat{F}$ on positive convex algebras (`PCA`) used for axiomatizing trace semantics of probabilistic systems in [2], as well as by the open questions stated in [9, Example 3.19], we provide a framework for proving properness and prove:

- The functor $[0,1] \times (-)^A$ on PCA is proper, and the required zig-zag is a span.
- The functor $\widehat{F}$ on PCA is proper. This proof is quite involved, and interestingly, provides the only case that we encountered where the zig-zag is not a span: it contains three intermediate nodes of which the middle one forms a span.

Along the way, we instantiate our framework on some known cases like Noetherian semirings and $\mathbb{N}$ (with a zig-zag that is a span), and prove new semirings proper: The semirings $\mathbb{Q}_+$ and $\mathbb{R}_+$ of non-negative rationals and reals, respectively. The shape of these zig-zags is a span as well. It is an interesting question for future work whether these new properness results may lead to new complete axiomatizations of expressions for certain weighted automata.

Our framework requires a proof of so-called *extension lemma* and *reduction lemma* in each case. While the extension lemma is a generic result that covers all cubic functors of interest, the reduction lemma is in all cases a nontrivial property intrinsic to the algebras under consideration. For the semiring of natural numbers it is a consequence of a result that we trace back to Hilbert [16]; for the case of convex algebra $[0,1]$ the result is due to Minkowski [17]. In the case of $\widehat{F}$, we use Kakutani's set-valued fixpoint theorem [6].

All base categories in this work are algebraic categories, i.e., categories $\mathtt{Set}^T$ of Eilenberg-Moore algebras of a finitary monad T on Set.

The main category of interest to us is the category PCA of positively convex algebras, the Eilenberg-Moore algebras of the monad of finitely supported subprobability distributions, see, e.g., [13, 14] and [15].

Concretely, a positive convex algebra $\mathbb{A}$ in PCA is a carrier set A together with infinitely many finitary operations denoting sub-convex sums, i.e., for each tuple $(p_i \mid 1 \le i \le n)$ with $p_i \in [0,1]$ and $\sum_i p_i \le 1$ we have a corresponding n-ary operation, the sub-convex combination with coefficients p_i. (Positive) Convex algebras algebras satisfy two axioms: the projection axiom stating that $\sum p_i x_i = x_k$ if $p_k = 1$; and the barycentre axiom

$$\sum_i p_i (\sum_j p_{ij} x_j) = \sum_j (\sum_i p_i \cdot p_{ij}) x_j.$$

These axioms are precisely the unit and multiplication law required from an Eilenberg-Moore algebra when instantiated to the probability subdistribution monad, and enable working with abstract convex combinations (formal sums) in the usual way as with convex combinations / sums in $\mathbb{R}$.

For the proofs of proper semirings, we work in the category $\mathbb{S}$-SMOD of semimodules over a semiring $\mathbb{S}$ which are the Eilenberg-Moore algebras of the monad $T_{\mathbb{S}}$ of finitely supported maps into $\mathbb{S}$.

For $n \in \mathbb{N}$, the free algebra with n generators in $\mathbb{S}$-SMOD is the direct product $\mathbb{S}^n$, and in PCA it is the n-simplex $\Delta^n = \{(\xi_1, \ldots, \xi_n) \mid \xi_j \ge 0, \sum_{j=1}^n \xi_j \le 1\}$.

For the semirings $\mathbb{N}$, $\mathbb{Q}_+$, and $\mathbb{R}_+$ that we deal with (with ring completions $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively), the categories of $\mathbb{S}$-semimodules are:

- CMON of commutative monoids for $\mathbb{N}$,
- AB of abelian groups for $\mathbb{Z}$,
- CONE of convex cones for $\mathbb{R}_+$, and
- $\mathbb{Q}$-VEC and $\mathbb{R}$-VEC of vector spaces over the field of rational and real numbers, respectively, for $\mathbb{Q}$ and $\mathbb{R}$.

References

[1] Silva, A., Bonchi, F., Bonsangue, M., Rutten, J.: Generalizing the powerset construction, coalgebraically. In: Proc. FSTTCS 2010. Leibniz International Proceedings in Informatics (LIPIcs), vol. 8, pp. 272–283 (2010)

[2] Silva, A., Sokolova, A.: Sound and complete axiomatization of trace semantics for probabilistic systems. Electr. Notes Theor. Comput. Sci. 276, 291–311 (2011), https://doi.org/10.1016/j.entcs.2011.09.027

[3] Sokolova, A., Woracek, H.: Proper semirings and proper convex functors. arXiv 1802.07830 (2018), https://arxiv.org/abs/1802.07830

[4] Jacobs, B., Silva, A., Sokolova, A.: Trace semantics via determinization. J. Comput. Syst. Sci. 81(5), 859–879 (2015)

[5] Jacobs, B., Rutten, J.: A tutorial on (co)algebras and (co)induction. Bulletin of the EATCS 62, 222–259 (1996)

[6] Kakutani, S.: A generalization of Brouwer's fixed point theorem. Duke Math. J. 8, 457–459 (1941), http://projecteuclid.org/euclid.dmj/1077492791

[7] Bonsangue, M., Milius, S., A., S.: Sound and complete axiomatizations of coalgebraic language equivalence. CoRR abs/1104.2803 (2011)

[8] Silva, A.: Kleene coalgebra. Ph.D. thesis, Radboud University Nijmegen (2010)

[9] Milius, S.: Proper functors and their rational fixed point. In: 7th Conference on Algebra and Coalgebra in Computer Science, CALCO 2017, June 12-16, 2017, Ljubljana, Slovenia. pp. 18:1–18:16 (2017), https://doi.org/10.4230/LIPIcs.CALCO.2017.18

[10] Ésik, Z., Maletti, A.: Simulation vs. equivalence. In: Proceedings of the 2010 International Conference on Foundations of Computer Science, FCS 2010, July 12-15, 2010, Las Vegas, Nevada, USA. pp. 119–124 (2010)

[11] Sokolova, A., Woracek, H.: Proper semirings and proper convex functors. arXiv 1802.07830 (2018), https://arxiv.org/abs/1802.07830

[12] Sokolova, A., Woracek, H.: Proper semirings and proper convex functors. In Proc. FoSSaCS 2018, LNCS 10803. pp 331–347 (2018), https://doi.org/10.1007/978-3-319-89366-2_18

[13] Doberkat, E.E.: Eilenberg-Moore algebras for stochastic relations. Inform. and Comput. 204(12), 1756–1781 (2006), http://dx.doi.org/10.1016/j.ic.2006.09.001

[14] Doberkat, E.E.: Erratum and addendum: Eilenberg-Moore algebras for stochastic relations [mr2277336]. Inform. and Comput. 206(12), 1476–1484 (2008), http://dx.doi.org/10.1016/j.ic.2008.08.002

[15] Pumplün, D.: Regularly ordered Banach spaces and positively convex spaces. Results Math. 7(1), 85–112 (1984), http://dx.doi.org/10.1007/BF03322493

[16] Hilbert, D.: Über die Theorie der algebraischen Formen. Math. Ann. 36(4), 473–534 (1890), http://dx.doi.org/10.1007/BF01208503

[17] Minkowski, H.: Geometrie der Zahlen. In 2 Lieferungen. Lfg. 1. Leipzig: B. G. Teubner. 240 S. 8°. (1896)