A (Co)algebraic Approach to Hennessy-Milner Theorems for Weakly Expressive Logics

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1 Introduction

Coalgebraic modal logic, as in [9, 6], is a framework in which modal logics for specifying coalgebras can be developed parametric in the signature of the modal language and the coalgebra type functor T. Given a base logic (usually classical propositional logic), modalities are interpreted via so-called predicate liftings for the functor T. These are natural transformations that turn a predicate over the state space X into a predicate over TX. Given that T-coalgebras come with general notions of T-bisimilarity [11] and behavioral equivalence [7], coalgebraic modal logics are designed to respect those. In particular, if two states are behaviourally equivalent then they satisfy the same formulas. If the converse holds, then the logic is said to be expressive. and we have a generalisation of the classic Hennessy-Milner theorem [5] which states that over the class of image-finite Kripke models, two states are Kripke bisimilar if and only if they satisfy the same formulas in Hennessy-Milner logic.

General conditions for when an expressive coalgebraic modal logic for T-coalgebras exists have been identified in [10, 2, 12]. A condition that ensures that a coalgebraic logic is expressive is when the set of predicate liftings chosen to interpret the modalities is *separating* [10]. Informally, a collection of predicate liftings is separating if they are able to distinguish non-identical elements from TX. This line of research in coalgebraic modal logic has thus taken as starting point the semantic equivalence notion of behavioral equivalence (or T-bisimilarity), and provided results for how to obtain an expressive logic. However, for some applications, modal logics that are not expressive are of independent interest. Such an example is given by *contingency logic* (see e.g. [3, 8]). We can now turn the question of expressiveness around and ask, given a modal language, what is a suitable notion of semantic equivalence?

This abstract is a modest extension of [1] in which the first two authors proposed a notion of Λ-bisimulation which is parametric in a collection Λ of predicate liftings, and therefore tailored to the expressiveness of a given coalgebraic modal logic. The main result was a finitary Hennessy-Milner theorem (which does not assume Λ is separating): If T is finitary, then two states are Λ-bisimilar if and only if they satisfy the same modal Λ-formulas. The definition of Λ-bisimulation was formulated in terms of so-called Z-coherent pairs, where Z is the Λ-bisimulation relation. It was later observed by the third author that Λ-bisimulations can be characterised as the relations Z between T-coalgebras for which the dual relation (consisting of so-called Z-coherent pairs) is a congruence between the complex algebras. Here we collect those results.

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2 Syntax and semantics of coalgebraic modal logic.

Due to lack of space, we assume the reader is familiar with the basic theory of coalgebras and algebras for a functor, and with coalgebraic modal logic. Here we only introduce a few basic concepts and fix notation. We refer to [6, 11] for more details.

A *similarity type* Λ is a set of modal operators with finite arities. Given such a Λ, the set $\mathcal{L}_\Lambda$ of modal formulas is defined in the usual inductive manner.

We denote by Q the contravariant powerset functor on Set. A T-coalgebraic semantics of $\mathcal{L}_\Lambda$-formulas is given by providing a Λ-*structure* $(T, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$ where T is a functor on Set, and for each n-ary $\heartsuit \in \Lambda$, $\llbracket\heartsuit\rrbracket$ is an n-ary predicate lifting, i.e., $\llbracket\heartsuit\rrbracket : Q^n \Rightarrow QT$ is a natural transformation. Different choices of predicate liftings yield different Λ-structures and consequently different logics.

Given a Λ-structure $(T, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$, and a T-coalgebra $\mathbb{X} = (X, \gamma\colon X \to TX)$, the truth of $\mathcal{L}_\Lambda$-formulas in $\mathbb{X}$ is defined inductively in the usual manner for atoms (i.e., $\top$ and $\bot$) and Boolean connectives, and for modalities: $(\mathbb{X}, \upsilon), x \models \heartsuit(\varphi_1, \ldots, \varphi_n)$ iff $\gamma(x) \in \llbracket\heartsuit\rrbracket_X(\llbracket\varphi_1\rrbracket_\mathbb{X}, \ldots, \llbracket\varphi_n\rrbracket_\mathbb{X})$. (Atomic propositions can be included in the usual way via a valuation.)

In the remainder, we let T be a fixed but arbitrary endofunctor on the category Set of sets and functions, and $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ are T-coalgebras. We write $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$, if $\mathbb{X}, x$ and $\mathbb{Y}, y$ satisfy the same $\mathcal{L}_\Lambda$-formulas,

3 Λ-bisimulations

Let $R \subseteq X \times Y$ be a relation with projections $\pi_l\colon R \to X$ and $\pi_r\colon R \to Y$, and let $U \subseteq X$ and $V \subseteq Y$. The pair (U, V) is *R-coherent* if $R[U] \subseteq V$ and $R^{-1}[V] \subseteq U$. One easily verifies that (U, V) is R-coherent iff (U, V) is in the pullback of $Q\pi_l$ and $Q\pi_r$.

Definition 3.1 (Λ-bisimulation)
A relation $Z \subseteq X \times Y$ is a Λ-bisimulation *between $\mathbb{X}$ and $\mathbb{Y}$, if whenever $(x, y) \in Z$, then for all $\heartsuit \in \Lambda$, n-ary, and all Z-coherent pairs $(U_1, V_1), \ldots, (U_n, V_n)$, we have that*

$$\gamma(x) \in \llbracket\heartsuit\rrbracket_X(U_1, \ldots, U_n) \quad \textit{iff} \quad \delta(y) \in \llbracket\heartsuit\rrbracket_Y(V_1, \ldots, V_n). \qquad (\textbf{Coherence})$$

We write $\mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$, if there is a Λ-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ that contains (x, y). A Λ-bisimulation on a T-coalgebra $\mathbb{X}$ is a Λ-bisimulation between $\mathbb{X}$ and $\mathbb{X}$.

We have the following basic properties.

Lemma 3.2

1. *The set of Λ-bisimulations between two T-coalgebras forms a complete lattice.*
2. *On a single T-coalgebra, the largest Λ-bisimulation is an equivalence relation.*
3. *Λ-bisimulations are closed under converse, but not composition.*

The following proposition compares Λ-bisimulations with the coalgebraic notions of T-bisimulations [11] and the weaker notion of precocongruences [4]. Briefly stated, a relation is a precocongruence of its pushout is a behavioural equivalence [7]).

Proposition 3.3 *Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be T-coalgebras, and Z be a relation between X and Y.*

1. *If Z is a T-bisimulation then Z is a Λ-bisimulation.*

2. If Z is a precocongruence then Z is a Λ-bisimulation.

3. If Λ is separating then Z is a Λ-bisimulation iff Z is a precocongruence.

It was shown in [4, Proposition 3.10] that, in general, T-bisimilarity implies precocongruence equivalence which in turn implies behavioural equivalence [7]. This fact together with Proposition 3.3 tells us that Λ-bisimilarity implies behavioural equivalence, whenever Λ is separating. Moreover, it is well known [11] that if T preserves weak pulbacks, then T-bisimilarity coincides with behavioural equivalence. Hence in this case, by Proposition 3.3, it follows that Λ-bisimilarity coincides with T-bisimilarity and behavioural equivalence.

The main result in [1] is the following.

Theorem 3.4 (Finitary Hennessy-Milner theorem) *If T is a finitary functor, then*

1. *For all states $x, x' \in X$:* $\mathbb{X}, x \equiv_\Lambda \mathbb{X}, x'$ *iff* $\mathbb{X}, x \sim_\Lambda \mathbb{X}, x'.$
2. *For all $x \in X$ and $y \in Y$:* $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$ *iff* $\mathbb{X} + \mathbb{Y}, \mathsf{in}_l(x) \sim_\Lambda \mathbb{X} + \mathbb{Y}, \mathsf{in}_r(y)$.

where $\mathsf{in}_l, \mathsf{in}_r$ are the injections into the coproduct/disjoint union.

4 Λ-Bisimulations as duals of congruences

We now use the fact that the contravariant powerset functor Q can be viewed as one part of the duality between Set and CABA, the category of complete atomic Boolean algebras and their homomorphisms. By duality, Q turns a pushout in Set into a pullback in CABA. So given a relation $Z \subseteq X \times Y$ with projections π_l, π_r (forming a span in Set), and letting (P, p_l, p_r) be its pushout, we have that $(QP, Qp_l, Qp_r) \cong (pb(Q\pi_l, Q\pi_r), Q\pi_l, Q\pi_r)$.

In the context of coalgebraic modal logic, we define complex algebras as follows. This definition coincides with the classic one.

Definition 4.1 (Complex algebras)

- *Let $L\colon \mathsf{CABA} \to \mathsf{CABA}$ be the functor $L(A) = \coprod_{\heartsuit \in \Lambda} A^{ar(\heartsuit)}$, and let $\sigma\colon LQ \Longrightarrow QT$ be the bundling up of $[\![\Lambda]\!]$ into one natural transformation. For example, if Λ consists of one unary modality and one binary modality, then $L(A) = A + A^2$ and $\sigma_X\colon QX + (QX)^2 \Longrightarrow QTX$.*
- *The complex algebra of $\mathbb{X} = (X, \gamma\colon X \to TX)$ is the L-algebra $\mathbb{X}^* = (QX, \gamma^*)$ where $\gamma^* = LQX \xrightarrow{\sigma_X} QTX \xrightarrow{Q\gamma} QX$.*

We can now reformulate the definition of Λ-bisimilarity in terms of the complex algebras associated with the coalgebras (by using $(QP, Qp_l, Qp_r) \cong (pb(Q\pi_l, Q\pi_r), Q\pi_l, Q\pi_r)$).

Lemma 4.2 *Z is Λ-bisimulation if and only if the following diagram commutes:*

$$\begin{tikzcd} LQX \arrow[d, "\gamma^*"'] & LQP \arrow[l, "LQp_l"'] \arrow[r, "LQp_r"] & LQY \arrow[d, "\delta^*"] \\ QX \arrow[r, "Q\pi_l"] & QZ & QY \arrow[l, "Q\pi_r"'] \end{tikzcd}$$

Proposition 4.3 *Z is Λ-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ iff the dual of its pushout is a congruence between the complex algebras $\mathbb{X}^*$ and $\mathbb{Y}^*$ (i.e. a span in the category of L-algebras and L-algebra homomorphisms).*

Proof. ($\Rightarrow$) Since (QP, Qp_l, Qp_r) is a pullback of $(QZ, Q\pi_l, Q\pi_r)$, we get a map $h : LQP \to QP$ such that (QP, Qp_l, Qp_r) is a congruence:

$$\begin{tikzcd}
LQX \arrow[d, "\gamma^*"'] & LQP \arrow[l, "LQp_l"'] \arrow[r, "LQp_r"] \arrow[dd, dashed, bend right] & LQY \arrow[d, "\delta^*"] \\
QX \arrow[r, "Q\pi_l"] & QZ & QY \arrow[l, "Q\pi_r"'] \\
& QP \arrow[ul, bend left, "Qp_l"] \arrow[ur, bend right, "Qp_r"'] &
\end{tikzcd}$$

($\Leftarrow$) Follows from commutativity of pullback square. □

References

[1] Z. Bakhtiari and H.H. Hansen. Bisimulation for Weakly Expressive Coalgebraic Modal Logics. In Filippo Bonchi and Barbara König, editors, *7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017)*, volume 72 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:16, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[2] M. Bílková and M. Dostál. Expressivity of many-valued modal logics, coalgebraically. In *WoLLIC 2016*, volume 9803 of *LNCS*, pages 109–124. Springer, 2016.

[3] J. Fan, Y. Wang, and H. van Ditmarsch. Almost necessary. In *AiML 2014*, pages 178–196. College Publications, 2014.

[4] H.H. Hansen, C. Kupke, and E. Pacuit. Neighbourhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science*, 5(2:2), 2009.

[5] M. Hennessy and R. Milner. Algebraic laws for non-determinism and concurrency. *Journal of the ACM*, 32:137–161, 1985.

[6] C. Kupke and D. Pattinson. Coalgebraic semantics of modal logics: an overview. *Theoretical Computer Science*, 412(38):5070–5094, 2011.

[7] A. Kurz. *Logics for coalgebras and applications to computer science.* PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[8] H. Montgomery and R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 9:318–328, 1966.

[9] D. Pattinson. Coalgebraic modal logic: Soundness, completeness and decidability of local consequence. *Theoretical Computer Science*, 309(1-3):177–193, 2003.

[10] D. Pattinson et al. Expressive logics for coalgebras via terminal sequence induction. *Notre Dame Journal of Formal Logic*, 45(1):19–33, 2004.

[11] J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249(1):3–80, 2000.

[12] L. Schröder. Expressivity of coalgebraic modal logic: The limits and beyond. *Theoretical Computer Science*, 390:230–247, 2008.