Countermodels for non-normal modal logics via nested sequents*

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The proof-theoretic framework of *nested sequents* has been very successful in treating normal modal logics. It is used, e.g., for providing modular calculi for all the logics in the so-called modal cube, for tense logics, as well as for modal logics based on propositional intuitionistic logic [2, 5, 9, 11]. The success of this framework might be due to the fact that it provides an ideal meeting point between syntax and semantics: On the one hand, nested sequents can be seen as purely syntactic extensions of the sequent framework with a structural connective corresponding to the modal box. On the other hand, due to the inherent similarity of the underlying tree structure to Kripke models, the nested sequent framework lends itself to very direct methods of countermodel construction from failed proof search by essentially reading off the model from a saturated and unprovable nested sequent. However the full power and flexibility of this framework so far has not yet been harnessed in the context of *non-normal* modal logics. While a first attempt at obtaining nested sequent calculi for non-normal modal logics indeed yielded modular calculi for a reasonably large class of non-normal modal logics by decomposing standard sequent rules [7, 8], the obtained calculi were not shown to inhibit the analogous central spot between syntax and semantics for these logics. In particular, no formula interpretation of the nested sequents was provided, and the calculi were not used to obtain countermodels from failed proof search.

Here we propose an approach to rectify this situation by considering *bimodal* versions of the non-normal modal logics. Such logics seem to have been considered originally in [1] in the form of *ability logics*, but their usefulness extends far beyond this particular interpretation. The main idea is that the neighbourhood semantics of non-normal monotone modal logics naturally gives rise to a second modality, which conveniently is normal. Here we concentrate on one of the most fundamental non-normal modal logics, *monotone modal logic* M [3, 4, 10], and present a nested sequent calculus for its bimodal version. Notably, the nested sequents have a formula interpretation in the bimodal language, and the calculus facilitates the construction of countermodels from failed proof search in a slightly modified version. An additional benefit is that the calculus conservatively extends both the standard nested sequent calculus for normal modal K from [2, 11] and the nested sequent calculus for monotone modal logic M from [7, 8].

The set $\mathcal{F}$ of formulae of bimodal monotone modal logic is given by the following grammar, built over a set $\mathcal{V}$ of propositional variables:

$$\mathcal{F} ::= \bot \mid \mathcal{V} \mid \mathcal{F} \to \mathcal{F} \mid \langle\exists\forall]\mathcal{F} \mid [\forall\forall]\mathcal{F}$$

The remaining propositional connectives are defined by their usual clauses. The semantics are given in terms of *neighbourhood semantics* in the following way, also compare [1, 3, 10].

Definition 1. A *neighbourhood model* is a tuple $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ consisting of a universe W, a *neighbourhood function* $\mathcal{N} : W \to 2^{2^W}$, and a *valuation* $[\![.]\!] : \mathcal{V} \to 2^W$.

*Work funded by WWTF grant MA16-28.

Definition 2. The *truth set of a formula* A in a model $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ is written as $[\![A]\!]$ and extends the valuation $[\![.]\!]$ of the model by the propositional clauses $[\![\bot]\!] = \emptyset$ and $[\![A \to B]\!] = [\![A]\!]^c \cup [\![B]\!]$ together with

- $[\![\langle\exists\forall] A]\!] = \{w \in W \mid \text{ exists } \alpha \in \mathcal{N}(w) \text{ s.t. for all } v \in \alpha : v \in [\![A]\!]\}$
- $[\![[\forall\forall] A]\!] = \{w \in W \mid \text{ for all } \alpha \in \mathcal{N}(w) \text{ and for all } v \in \alpha : v \in [\![A]\!]\}$

If $w \in [\![A]\!]$ we also write $\mathfrak{M}, w \Vdash A$. A formula A is *valid in* M, if for every model $[\![A]\!] = W$.

Hence, the formulation of the truth conditions for the modal operator of monomodal monotone logic in terms of an "exists forall" clause naturally yields the definition of the operator $[\forall\forall]$ in terms of a "forall forall" clause. This can be rewritten into the clause $[\![[\forall\forall] A]\!] = \{w \in W \mid \text{ for all } v \in \bigcup \mathcal{N}(w) : v \in [\![A]\!]\}$ which immediately yields normality of the modality $[\forall\forall]$, since we can take $\bigcup \mathcal{N}(w)$ as the set of successors of w. In particular, it can be seen that the modality $[\forall\forall]$ behaves like a standard K-modality.

In order to capture both modalities $\langle\exists\forall]$ and $[\forall\forall]$ in the nested sequent framework, we introduce the two corresponding structural connectives $\langle . \rangle$ and $[.]$ respectively, with the peculiarity that nested occurrences of these connectives are allowed only in the scope of the latter:

Definition 3. A *nested sequent* has the form

$$\Gamma \Rightarrow \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \ldots, \langle \Sigma_n \Rightarrow \Pi_n \rangle, [\mathcal{S}_1], \ldots, [\mathcal{S}_m] \tag{1}$$

for $n, m \geq 0$, where $\Gamma \Rightarrow \Delta$ as well as the $\Sigma_i \Rightarrow \Pi_i$ are standard sequents, and the $\mathcal{S}_j$ are nested sequents. The *formula interpretation* of the above nested sequent is

$$\bigwedge \Gamma \to \Big(\bigvee \Delta \vee \bigvee\nolimits_{i=1}^{n} \langle\exists\forall] (\bigwedge \Sigma_i \to \bigvee \Pi_i) \vee \bigvee\nolimits_{j=1}^{m} [\forall\forall]\, \iota(\mathcal{S}_j) \Big)$$

where $\iota(\mathcal{S}_j)$ is the formula interpretation of $\mathcal{S}_j$.

In order to obtain a nested sequent calculus for M we need to make sure that applicability of the propositional rules does not enforce normality of the interpretation of the structural connective $\langle . \rangle$. In particular, we cannot permit application of, e.g., the initial sequent rule inside the scope of $\langle\exists\forall]$ – otherwise the formula interpretation of the nested sequent $\Rightarrow \langle p \Rightarrow p \rangle$, i.e., $\langle\exists\forall](p \to p)$ would need to be a theorem, which is not the case in bimodal M.

Definition 4. The nested sequent rules of the calculus $\mathcal{N}_{\mathsf{M}}$ are given in Fig. 1. The rules can be applied anywhere inside a nested sequent except for inside the scope of $\langle . \rangle$.

Soundness of the rules with respect to the formula interpretation can then be shown as usual by obtaining a countermodel for the formula interpretation of the premiss(es) of a rule from a countermodel for the formula interpretation of its conclusion:

Proposition 5. *The rules of Fig. 1 are sound for* M *under the formula interpretation.*

A relatively straightforward proof of completeness for the calculus $\mathcal{N}_{\mathsf{M}}$ can be obtained by using the completeness result for the Hilbert-style axiomatisation of bimodal M in [1] as follows. The axioms given there can be converted into rules of a cut-free standard sequent calculus using, e.g., the methods of [6]. Then, cut-free derivations in the resulting sequent calculus can be converted into cut-free derivations in the nested sequent calculus $\mathcal{N}_{\mathsf{M}}$ along the lines of [7]. Hence together with the previous proposition we obtain:

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ \mathsf{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \to_R \qquad \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}\ \to_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\forall\forall]A}\ [\forall\forall]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\forall\forall]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\forall\forall]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\exists\forall]A}\ \langle\exists\forall]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle\exists\forall]_L \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\forall\forall]A, \langle\Sigma \Rightarrow \Pi\rangle}\ W$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{ICL} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{ICR} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{W}$$

Figure 1: The nested sequent rules of the calculus $\mathcal{N}_{\mathsf{M}}$ for the bimodal system.

Theorem 6. *The calculus in Fig. 1 is sound and complete for bimodal monotone modal logic, i.e.: A formula A is a theorem of M, if and only if the nested sequent $\Rightarrow A$ is derivable in $\mathcal{N}_{\mathsf{M}}$.*

As an interesting corollary of the sketched completeness proof we even obtain cut-free completeness of the calculus $\mathcal{N}_{\mathsf{M}}$ restricted to *linear nested sequents*, i.e., the subclass of nested sequents where we restrict m to be at most 1 in 1 along the lines of [7].

Due to the structure of the rules of $\mathcal{N}_{\mathsf{M}}$, in a derivation of a formula of the $[\forall\forall]$-fragment of M neither the connective $\langle\exists\forall]$ nor its structural version $\langle . \rangle$ occur. Hence, as a further corollary of Thm. 6, the calculus obtained by dropping the rules $\langle\exists\forall]_R, \langle\exists\forall]_L, W$ from $\mathcal{N}_{\mathsf{M}}$ is complete for this fragment, which is normal modal logic K. Since the rules $[\forall\forall]_R, [\forall\forall]_L$ are exactly the modal right and left rules in the standard nested sequent calculus for modal logic K from [2, 11], this immediately yields a completeness proof for that calculus seen as a fragment of $\mathcal{N}_{\mathsf{M}}$.

Moreover, by dropping the rules $[\forall\forall]_R, [\forall\forall]_L, W$ from $\mathcal{N}_{\mathsf{M}}$ we obtain the calculus for monomodal M from [7, 8]. Hence we also obtain a completeness proof for that calculus, together with a formula interpretation, albeit the latter only in the language extended with $[\forall\forall]$. Thus the calculus $\mathcal{N}_{\mathsf{M}}$ can be seen as a generalisation and combination of both the standard nested sequent calculus for normal modal logic K and the linear nested sequent calculus for monomodal monotone logic M. This seems to support the intuition that bimodal M can be seen as a refinement of modal logic K, where the set of successor states $\bigcup \mathcal{N}(w)$ is further structured by $\mathcal{N}$, a structure which is accessible through the additional connective $\langle\exists\forall]$.

So far the presented nested sequent calculus eliminates one of the shortcomings of the calculi in [7, 8], namely the lack of a formula interpretation. In addition, it facilitates a semantic proof of completeness by constructing a countermodel from failed proof search. The intuition is the same as for normal modal logics: the nodes in a saturated unprovable nested sequent yield the worlds of a Kripke-model. Here the *nodes* of a nested sequent are separated by the $[.]$ operator, so that every node contains a standard sequent and a multiset of structures $\langle \Sigma_i \Rightarrow \Pi_i \rangle$. The successor relation given by $\bigcup \mathcal{N}(w)$ then corresponds to the immediate successor relation between nodes in the nested sequent. The main technical challenge is the construction of the neighbourhood function $\mathcal{N}$ itself. This can be done by adding *annotations* in the form of a set of formulae to every node in the nested sequent, written as $\Gamma \stackrel{S}{\Rightarrow} \Delta$. Further, to facilitate backwards proof search we absorb contraction into the rules by copying the principal formulae into the premiss(es). The so modified annotated versions of the interesting rules are given in Fig. 2. In all the other rules the annotations are preserved going from conclusion to premiss(es). In the following we write $\ell(w)$ for the annotation of the component v of a nested sequent.

$$\frac{\Gamma \Rightarrow \Delta, [\forall\forall]A, [\stackrel{\emptyset}{\Rightarrow} A]}{\Gamma \Rightarrow \Delta, [\forall\forall]A} \; [\forall\forall]^*_R \qquad \frac{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, [\Sigma, A \stackrel{\{A\}}{\Rightarrow} \Pi]}{\Gamma, \langle\exists\forall]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle} \; \langle\exists\forall]^*_L \qquad \frac{\Gamma \Rightarrow \Delta, [\forall\forall]A, [\Sigma \stackrel{\emptyset}{\Rightarrow} \Pi]}{\Gamma \Rightarrow \Delta, [\forall\forall]A, \langle\Sigma \Rightarrow \Pi\rangle} \; W^*$$

Figure 2: The interesting rules of the annotated variant $\mathcal{N}^*_{\mathsf{M}}$ of the system

Definition 7. The *model generated by a nested sequent* $\mathcal{S}$ is the model $\mathfrak{M}^{\mathcal{S}} = (W, \mathcal{N}, [\![.]\!])$ where W is the set of components (nodes) of $\mathcal{S}$, the valuation is defined by: if $w \in W$, then $w \in [\![p]\!]$ iff w contains $[\Gamma \stackrel{\Sigma}{\Rightarrow} \Delta]$ and $p \in \Gamma$. Finally, the neighbourhood function $\mathcal{N}(w)$ is defined as follows. Let $\mathcal{C}_w$ be the set of immediate successors of w, and let $\ell[\mathcal{C}_w]$ be the set of labels of nodes in $\mathcal{C}_w$. Then let $\mathcal{L}_w := \{ \{v \in \mathcal{C}(w) \mid \ell(v) = \Sigma\} \mid \Sigma \in \ell[\mathcal{C}_w]\}$. Now, $\mathcal{N}(w)$ is defined as $(\mathcal{L}_w \cup \{\mathcal{C}_w\}) \smallsetminus \{\emptyset\}$ if there is a formula $\langle\exists\forall]A \in \Delta$, and $\mathcal{L}_w \cup \{\mathcal{C}_w\} \cup \{\emptyset\}$ otherwise.

Thus, disregarding the empty set, the set of neighbourhoods of a node in a nested sequent includes the set of all its children, as well as every set of children labelled with the same label. Whether it contains the empty set or not depends on whether there is a formula of the form $\langle\exists\forall]A$ in its succedent. This construction then yields countermodels from failed proof search:

Theorem 8. *If $\mathcal{S}$ is a saturated nested sequent obtained by backwards proof search from a non-nested sequent $\Gamma \Rightarrow \Delta$, then $\mathfrak{M}^{\mathcal{S}}$ is a neighbourhood model, and the root w of $\mathcal{S}$ satisfies for every formula A: if $A \in \Gamma$, then $w \in [\![A]\!]$, and if $A \in \Delta$, the $w \notin [\![A]\!]$.*

An implementation of the resulting proof search procedure which yields either a derivation or a countermodel is available under http://subsell.logic.at/bprover/nnProver/.

While here we considered only monotone logic M, we expect the calculus $\mathcal{N}_{\mathsf{M}}$ to be extensible to a large class of extensions of M. Hence it should provide the basis for an ideal meeting ground for syntax and semantics in the context of non-normal monotone modal logics.

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