Propositional Union Closed Team Logics: Expressive Power and Axiomatizations

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In this paper, we prove the expressive completeness of some propositional *union closed team logics*, and introduce sound and complete systems of natural deduction for these logics. These logics are variants of *dependence logic*, which is a non-classical first-order logic, introduced by Väänänen, for reasoning about dependencies. This framework extends the classical logic by adding new atomic formulas for charactering dependence and independence between variables. Examples of such atoms are *dependence atoms* (giving rise to dependence logic), and *inclusion atoms* (giving rise to inclusion logic [1]). Hodges [3, 4] observed that dependency properties can only manifest themselves in *multitudes*, and he thus introduced the so-called *team semantics* that dependence logic and its variants adopt. Formulas of these logics are evaluated under *teams*, which in the propositional context are *sets* of valuations.

Logics based on team semantics (also called *team logics*) can have interesting closure properties. For example, dependence logic is *closed downwards*, meaning that the truth of a formula on a team is preserved under taking subteams. In this paper, we consider propositional team logics that are *closed under unions*, meaning that if two teams both satisfy a formula, then their union also satisfies the formula. Inclusion logic is closed under unions. Other known union closed logics are classical logic extended with *anonymity atoms* (introduced very recently by Väänänen [6] to characterize anonymity in the context of privacy), or with the *relevant disjunction* $\veebar$ (introduced by Rönnholm, see [5], and also named *nonempty disjunction* by some other authors [2, 8]).

While propositional downwards closed team logics are well studied (e.g., [7]), propositional union closed team logics are not understood very well yet. It follows from [2] that propositional inclusion logic (**PInc**) with extended inclusion atoms is expressively complete, and **PInc** is thus expressively equivalent to classical logic extended with relevant disjunction (**PU**), which is shown to be also expressively complete in [8]. We show in this paper that classical logic extended with anonymity atoms (**PAm**) is also expressively complete, and **PInc** with slightly less general inclusion atoms is already expressively complete. From the expressive completeness, we will derive the interpolation theorem for these logics. We also provide axiomatizations for **PInc**, **PU** and **PAm**, which are lacking in the literature. We define sound and complete systems of natural deduction for these logics. As with other team logics, these systems do not admit uniform substitution. Another interesting feature of the systems is that the usual disjunction introduction rule ($\phi/\phi \veebar \psi$) is not sound for the relevant disjunction.

1 Propositional union closed team logics

Fix a set Prop of propositional variables, whose elements are denoted by $p,q,r,\dots$ (with or without subscripts). We first define the *team semantics* for *classical propositional logic* (**CPL**), whose well-formed formulas (called *classical formulas*), in the context of the present paper, are given by the grammar:

$$\alpha ::= p \mid \bot \mid \top \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\vee\alpha$$

Let $\mathsf{N}\subseteq\mathsf{Prop}$ be a set of propositional variables. An *(N-)team* is a set of valuations $v:\mathsf{N}\cup\{\bot,\top\}\to\{0,1\}$ with $v(\bot)=0$ and $v(\top)=1$. Note that the empty set $\emptyset$ is a team. The notion of a classical formula α being *true* on a team X, denoted by $X\models\alpha$, is defined inductively as follows:

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- $X \models p$ iff for all $v \in X$, $v(p) = 1$.
- $X \models \bot$ iff $X = \varnothing$.
- $X \models \top$ always holds
- $X \models \neg\alpha$ iff for all $v \in X$, $\{v\} \not\models \alpha$.
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.
- $X \models \alpha \vee \beta$ iff there are $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \alpha$ and $Z \models \beta$.

Clearly, **CPL** has the *empty team property*, *union closure property* and *downwards closure property*:

Empty Team Property: $\varnothing \models \alpha$ holds for all α;

Union Closure: $X \models \alpha$ and $Y \models \alpha$ imply $X \cup Y \models \alpha$;

Downwards Closure: $X \models \alpha$ and $Y \subseteq X$ imply $Y \models \alpha$.

The union closure and downwards closure property together are equivalent to the *flatness property*:

Flatness $X \models \alpha$ if and only if $\{v\} \models \alpha$ for all $v \in X$.

The flatness of classical formulas shows that team semantics is conservative over classical formulas. We now extend **CPL** to three non-flat but union closed team-based logics. Consider a new disjunction $\veebar$, called *relevant disjunction*, and new atomic formulas of the form $a_1 \dots a_k \subseteq b_1 \dots b_k$ with each $a_i, b_i \in \mathsf{Prop} \cup \{\bot, \top\}$, called *inclusion atoms*, and of the form $\neq(p_1, \dots, p_k; q)$, called *anonymity atoms*. The team semantics of these new connective and atoms are defined as:

- $X \models \phi \veebar \psi$ iff $X = \varnothing$ or there are nonempty Y and Z such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$.
- $X \models a_1 \dots a_k \subseteq b_1 \dots b_k$ iff for all $v \in X$, there exists $v' \in X$ such that
$$\langle v(a_1), \dots, v(a_k)\rangle = \langle v'(b_1), \dots, v'(b_k)\rangle.$$
- $X \models \neq(p_1, \dots, p_k; q)$ iff for all $v \in X$, there exists $v' \in X$ such that
$$\langle v(p_1), \dots, v(p_k)\rangle = \langle v'(p_1), \dots, v'(p_k)\rangle \text{ and } v(q) \neq v'(q).$$

Note the similarity and difference between the semantics clauses of $\vee$ and $\veebar$. In particular, we write $\neq(p)$ for $\neq(\langle\rangle; p)$, and clearly its semantics clause is reduced to

- $X \models \neq(p)$ iff either $X = \varnothing$ or there exist $v, v' \in X$ such that $v(p) \neq v'(p)$.

We define the syntax of *propositional union closed logic* (**PU**) as the syntax of **CPL** expanded by adding $\veebar$, and negation $\neg$ is allowed to occur only in front of classical formulas, that is,

$$\phi ::= p \mid \bot \mid \top \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \veebar \phi,$$

where α is an arbitrary classical formula. Similarly, *propositional inclusion logic* (**PInc**) is **CPL** extended with inclusion atoms $a_1 \dots a_k \subseteq b_1 \dots b_k$ (and negation occurs only in front of classical formulas), and *propositional anonymity logic* (**PAm**) is **CPL** extended with anonymity atoms $\neq(p_1, \dots, p_k; q)$ (and negation occurs only in front of classical formulas).

For any formula ϕ in $\mathsf{N} \subseteq \mathsf{Prop}$, we write $[\![\phi]\!] = \{X \text{ an } \mathsf{N}\text{-team} : X \models \phi\}$. It is easy to verify that for any formula ϕ in the language of **PU** or **PInc** or **PAm**, the set $[\![\phi]\!]$ contains the empty team $\varnothing$, and is closed under unions, i.e., $X, Y \in [\![\phi]\!]$ implies $X \cup Y \in [\![\phi]\!]$.

2 Expressive completeness

It was proved in [8] that **PU** is *expressively complete* with respect to the set of all union closed team properties which contain the empty team, in the sense that for any set $\mathsf{N} \subseteq \mathsf{Prop}$, for any set P of N-teams that is closed under unions and contains the empty team, we have $\mathsf{P} = [\![\phi]\!]$ for some **PU**-formula ϕ in N. The proof in [8] first defines for any N-team X with $\mathsf{N} = \{p_1, \dots, p_n\}$ a **PU**-formula

$$\Psi_X := \mathop{\veebar}_{v \in X} (p_1^{v(1)} \wedge \dots \wedge p_n^{v(n)}),$$

where $v(i)$ is short for $v(p_i)$, $p_i^1 := p_i$, and $p_i^0 = \neg p_i$. Observing that $Y \models \Psi_X \iff Y = X$ holds for any N-team Y, one then easily establishes that $\mathsf{P} = [\![\bigvee_{X\in\mathsf{P}}\Psi_X]\!]$. Generalizing this argument, we can now show that **PInc** and **PAm** are both expressively complete in the same sense, and in particular, all these three union closed team logics we introduced are equivalent in expressive power.

Theorem 1. **PU** $\equiv$ **PInc** $\equiv$ **PAm**.

Proof. *(sketch)* We first show that the **PU**-formula Ψ_X is expressible in **PInc**. Define **PAm**-formulas

$$\Theta_X := \bigvee_{v\in X}(p_1^{v(1)}\wedge\dots\wedge p_n^{v(n)}),\qquad\text{and}\qquad \Phi_X := \bigwedge_{v\in X}\underline{v(1)}\dots\underline{v(n)}\subseteq p_1\dots p_n,$$

where $\underline{0} := \bot$ and $\underline{1} := \top$. Observe that for any N-team Y,

$$Y\models\Theta_X \iff Y\subseteq X,\qquad\text{and}\qquad Y\models\Phi_X\iff X\subseteq Y.$$

Thus, $\Psi_X\equiv\Theta_X\wedge\Phi_X$[1].

To show that Ψ_X is expressible in **PAm**, we show that for any N-team X and any $\mathsf{K} = \{p_{i_1},\dots,p_{i_k}\}\subseteq\{p_1,\dots,p_n\} = \mathsf{N}$, the formula $\Psi_X^{\mathsf{K}} = \dot{\bigvee}_{v\in X}(p_{i_1}^{v(i_1)}\wedge\dots\wedge p_{i_k}^{v(i_k)})$ is expressible in **PAm** as some ψ_X^{K} by induction on $|\mathsf{K}|\leq n$. If $|\mathsf{K}| = 1$, then $\Psi_X^{\mathsf{K}}\equiv p_{i_1}$ or $\neg p_{i_1}$ or $\neq(p_{i_1})$. If $|\mathsf{K}| = m+1$, let $\mathsf{K} = \mathsf{K}_0\cup\{p_{i_{m+1}}\}$, $Y = \{v\in X\mid v(i_{m+1}) = 1\}$ and $Z = \{v\in X\mid v(i_{m+1}) = 0\}$. If $Y = \varnothing$, then by induction hypothesis,

$$\Psi_X^{\mathsf{K}} = \dot{\bigvee}_{v\in Z}(p_{i_1}^{v(i_1)}\wedge\dots\wedge p_{i_m}^{v(i_m)}\wedge\neg p_{i_{m+1}})\equiv\Big(\dot{\bigvee}_{v\in Z}(p_{i_1}^{v(i_1)}\wedge\dots\wedge p_{i_m}^{v(i_m)})\Big)\wedge\neg p_{i_{m+1}}\equiv\psi_Z^{\mathsf{K}_0}\wedge\neg p_{i_{m+1}}.$$

Similarly, if $Z = \varnothing$, then $\Psi_X^{\mathsf{K}}\equiv\psi_Y^{\mathsf{K}_0}\wedge p_{i_{m+1}}$. Now, if $Y,Z\neq\varnothing$, we have by induction hypothesis that

$$\Psi_X^{\mathsf{K}}\equiv(\psi_Y^{\mathsf{K}_0}\wedge p_{i_{m+1}})\,\dot\vee\,(\psi_Z^{\mathsf{K}_0}\wedge\neg p_{i_{m+1}})\equiv\Big((\psi_Y^{\mathsf{K}_0}\wedge p_{i_{m+1}})\vee(\psi_Z^{\mathsf{K}_0}\wedge\neg p_{i_{m+1}})\Big)\wedge\neq(p_{i_{m+1}}).\qquad\square$$

We show next that the interpolation property of a team logic is a consequence of the expressive completeness and the *locality property*, which is defined as:

Locality: For any formula ϕ in $\mathsf{N}\subseteq\mathsf{Prop}$, if X is an N_0-team and Y an N_1-team such that $\mathsf{N}\subseteq\mathsf{N}_0,\mathsf{N}_1$ and $X\upharpoonright\mathsf{N} = Y\upharpoonright\mathsf{N}$, then $X\models\phi\iff Y\models\phi$

The team logics **PU**, **PInc** and **PAm** all have the locality property. But let us emphasize here that in the team semantics setting, locality is not a trivial property. Especially, if in the semantics clause of disjunction $\vee$ the two subteams $Y,Z\subseteq X$ are required to be disjoint, then the logic **PInc** is not local any more, as, e.g., the formula $pq\subseteq rs\vee tu\subseteq rs$ (with the modified semantics for $\vee$) is not local.

Theorem 2 (Interpolation)**.** *If a team logic* L *is expressively complete and has the locality property, then it enjoys Craig's Interpolation. In particular,* **PU**, **PInc** *and* **PAm** *enjoy Craig's interpolation.*

Proof. *(sketch)* Suppose ϕ is an L-formula in $\mathsf{N}\cup\mathsf{N}_0\subseteq\mathsf{Prop}$, and ψ an L-formula in $\mathsf{N}\cup\mathsf{N}_1\subseteq\mathsf{Prop}$. Since L is expressively complete, there is an L-formula θ in N such that $[\![\theta]\!] = [\![\phi]\!]|_{\mathsf{N}} = \{X\upharpoonright\mathsf{N} : X\models\phi\}$. It follows from the locality property of L that θ is the desired interpolant, i.e., $\phi\models\theta$ and $\theta\models\psi$. $\square$

3 Axiomatizations

The proof of Theorem 1 and also results in [8] show that every formula in the language of **PU**, **PInc** or **PAm** can be turned into an equivalent formula in a certain normal form, .e.g, the form $\bigvee_{X\in\mathsf{P}}\Psi_X$ for **PU**. Making use of these normal forms, we can axiomatize these union closed team logics.

[1]This **PInc**-formula is essentially adapted from a very similar modal formula in [2], but our formula Φ_X is slightly simpler than the one in [2], which uses slightly more general inclusion atoms. In this sense, the result that our version of **PInc** is expressively complete is a slight refinement of the expressive completeness of another version of **PInc** that follows from [2].

We present in this abstract only the system of natural deduction for **PU**, and the systems for **PInc** or **PAm** have rules for inclusion and anonymity atoms in addition to the following ones. In the following rules, α ranges over classical formulas only:

$$\dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \bot\end{matrix}}{\neg\alpha}\,\neg\mathsf{I} \qquad \dfrac{\neg\neg\alpha}{\alpha}\,\neg\neg\mathsf{E} \qquad \dfrac{\alpha \qquad \neg\alpha}{\phi}\,\neg\mathsf{E} \qquad \dfrac{\bot}{\phi}\,\textsf{ex falso} \qquad \dfrac{\phi \qquad \psi}{\phi\wedge\psi}\,\wedge\mathsf{I} \qquad \dfrac{\phi\wedge\psi}{\phi}\,\wedge\mathsf{E} \qquad \dfrac{\phi\wedge\psi}{\psi}\,\wedge\mathsf{E}$$

$$\dfrac{\phi}{\phi\vee\psi}\,\vee\mathsf{I} \qquad \text{For } \circ\in\{\vee,\dot{\vee}\}\text{:}\ \dfrac{\begin{matrix} \\ D_0\\ \phi\circ\psi\end{matrix} \qquad \begin{matrix}[\phi]\\ D_1\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ D_2\\ \chi\end{matrix}}{\chi}\,\circ\mathsf{E} \qquad \dfrac{\phi\circ\phi}{\phi}\,\circ\mathsf{Ctr} \qquad \dfrac{\psi\,\dot{\vee}\,\bot}{\phi}\,\dot{\vee}\mathsf{E} \qquad \dfrac{\phi\,\dot{\vee}\,\psi}{\phi\vee\psi}\,\dot{\vee}>\vee$$

The undischarged assumptions in D_0 contains classical formulas only

$$\dfrac{\phi\,\dot{\vee}\,(\psi\vee\chi)}{(\phi\,\dot{\vee}\,\psi)\vee(\phi\,\dot{\vee}\,\chi)}\,\mathsf{Dstr}\,\dot{\vee}\,\vee \qquad \dfrac{\phi\vee(\psi\,\dot{\vee}\,\chi)}{(\phi\vee\psi)\,\dot{\vee}\,(\phi\vee\chi)}\,\mathsf{Dstr}\vee\dot{\vee} \qquad \dfrac{\Big(\bigvee_{X\in\mathcal{X}}\Psi_X\Big)\wedge\Big(\bigvee_{Y\in\mathcal{Y}}\Psi_Y\Big)}{\bigvee_{Z\in\mathcal{Z}}\Psi_Z}\,\mathsf{Dstr}\vee\wedge\vee$$

where $\mathcal{Z}=\{Z=\bigcup\mathcal{X}'=\bigcup\mathcal{Y}' \mid \mathcal{X}'\subseteq\mathcal{X}\ \&\ \mathcal{Y}'\subseteq\mathcal{Y}\}$

As other systems for team logics (see e.g., [7, 8]), the above system does not admit *uniform substitution*, as, e.g., the negation rules apply to classical formulas only. Restricted to classical formulas, the above system contains all the usual rules for disjunction $\vee$.

Theorem 3 (Sound and Completeness)**.** *For any* **PU***-formulas* ϕ *and* ψ*, we have* $\psi\models\phi \iff \psi\vdash\phi$.

Proof. *(idea)* Use the normal form of **PU**, and the equivalence of the following clauses:

(i) $\bigvee_{X\in\mathcal{X}}\Psi_X \models \bigvee_{Y\in\mathcal{Y}}\Psi_Y$.

(ii) for each $X\in\mathcal{X}$, there exists $\mathcal{Y}_X\subseteq\mathcal{Y}$ such that $X=\bigcup\mathcal{Y}_X$. □

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