Generators and Axiomatizations for Varieties of PBZ*–lattices

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PBZ*–lattices are bounded lattice–ordered algebraic structures arising in the study of quantum logics. By definition, *PBZ*–lattices* are the paraorthomodular Brouwer–Zadeh lattices in which each pair consisting of an element and its Kleene complement fulfills the Strong de Morgan condition. They include orthomodular lattices, which are exactly the PBZ*–lattices without unsharp elements, as well as antiortholattices, which are exactly the PBZ*–lattices whose only sharp elements are 0 and 1. See below the formal definitions. Recall that the *sharp elements* of a bounded involution lattice are the elements having their involutions as bounded lattice complements; more precisely, with the terminology of [2], this is the notion of a *Kleene–sharp element*, and, in Brouwer–Zadeh lattices, we have also the notions of a *Brouwer–sharp* and a $\diamond$–*sharp element*; however, in the particular case of PBZ*–lattices, Kleene–sharp, Brouwer–sharp and $\diamond$–sharp elements coincide. All the results in this abstract that are not cited from other papers and not mentioned as being immediate are new and original.

We will designate algebras by their underlying sets and denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We recall the following definitions and immediate properties:

- a *bounded involution lattice* (in brief, *BI–lattice*) is an algebra $(L, \wedge, \vee, \cdot', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with partial order $\leq$, $a'' = a$ for all $a \in L$, and $a \leq b$ implies $b' \leq a'$ for all $a, b \in L$; the operation $\cdot'$ of a BI–lattice is called *involution*;
- if an algebra L has a BI–lattice reduct, then we denote by $S(L)$ the set of the *sharp elements* of L, namely $S(L) = \{x \in L \mid x \wedge x' = 0\}$;
- an *ortholattice* is a BI–lattice L such that $S(L) = L$;
- an *orthomodular lattice* is an ortholattice L such that, for all $a, b \in L$, $a \leq b$ implies $a \vee (a' \wedge b) = b$;
- a *pseudo–Kleene algebra* is a BI–lattice $\mathbf{L}$ satisfying, for all $a, b \in L$: $a \wedge a' \leq b \vee b'$; the involution of a pseudo–Kleene algebra is called *Kleene complement*; recall that distributive pseudo–Kleene algebras are called *Kleene algebras* or *Kleene lattices*; clearly, any ortholattice is a pseudo–Kleene algebra;
- an algebra L having a BI–lattice reduct is said to be *paraorthomodular* iff, for all $a, b \in L$, whenever $a \leq b$ and $a' \wedge b = 0$, it follows that $a = b$; note that any orthomodular lattice is a paraorthomodular pseudo–Kleene algebra, but the converse does not hold; however, if L is an ortholattice, then: L is orthomodular iff L is paraorthomodular;

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- a *Brouwer–Zadeh lattice* (in brief, *BZ–lattice*) is an algebra $(L, \wedge, \vee, \cdot', \cdot^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, \cdot', 0, 1)$ is a pseudo–Kleene algebra and, for all $a, b \in L$: $\begin{cases} a \wedge a^{\sim} = 0; & a \leq a^{\sim\sim}; \\ a^{\sim\prime} = a^{\sim\sim}; & a \leq b \text{ implies } b^{\sim} \leq a^{\sim}; \end{cases}$
- a *BZ** *–lattice* is a BZ–lattice that satisfies condition $(*)$: $(a \wedge a')^{\sim} \leq a^{\sim} \vee a'^{\sim}$, written in equivalent form: $(*)$: $(a \wedge a')^{\sim} = a^{\sim} \vee a'^{\sim}$;
- a *PBZ** *–lattice* is a paraorthomodular BZ*–lattice;
- if we extend their signature by adding a Brouwer complement equalling their Kleene complement, then ortholattices become BZ–lattices and orthomodular lattices become PBZ*–lattices; in any PBZ*–lattice L, $S(L)$ is the largest orthomodular subalgebra of L;
- an *antiortholattice* is a PBZ*–lattice L such that $S(L) = \{0, 1\}$; antiortholattices are exactly the PBZ*–lattices L whose Brower complement is defined by: $0^{\sim} = 1$ and $x^{\sim} = 0$ for all $x \in L \setminus \{0\}$; this Brower complement is called the *trivial Brower complement*; note, also, that any pseudo–Kleene algebra L with $S(L) = \{0, 1\}$ becomes an antiortholattice when endowed with the trivial Brower complement.

PBZ*–lattices form a variety, which we will denote by $\mathbb{PBZL}^*$. We will also denote by $\mathbb{BA}$, $\mathbb{OML}$, $\mathbb{OL}$ and $\mathbb{PKA}$ the varieties of Boolean algebras, orthomodular lattices, ortholattices and pseudo–Kleene algebras. $\mathbb{BA} \subsetneq \mathbb{OML} \subsetneq \mathbb{OL} \subsetneq \mathbb{PKA}$ and, with the extended signature, $\mathbb{OML} = \{L \in \mathbb{PBZL}^* \mid L \vDash x' \approx x^{\sim}\}$. $\mathbb{AOL}$ will denote the class of antiortholattices, which is a proper universal class, since not only it is not closed with respect to direct products, but, as we have proven, each of its members has the bounded lattice reduct directly indecomposable. We also denote by $\mathbb{DIST}$ the variety of distributive PBZ*–lattices.

We consider the following identities in the language of BZ–lattices, where, for any element x of a BZ–lattice, we denote by $\Box x = x'^{\sim}$ and by $\Diamond x = x^{\sim\sim}$:

SK	$x \wedge \Diamond y \leq \Box x \vee y$
SDM	$(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$ (the *Strong de Morgan law*)
WSDM	$(x \wedge y^{\sim})^{\sim} \approx x^{\sim} \vee \Diamond y$ (*weak SDM*)
S2	$(x \wedge (y \wedge y')^{\sim})^{\sim} \approx x^{\sim} \vee \Diamond(y \wedge y')$
S3	$(x \wedge \Diamond(y \wedge y'))^{\sim} \approx x^{\sim} \vee (y \wedge y')^{\sim}$
J0	$x \approx (x \wedge y^{\sim}) \vee (x \wedge \Diamond y)$
J2	$x \approx (x \wedge (y \wedge y')^{\sim}) \vee (x \wedge \Diamond(y \wedge y'))$

Clearly, **J0** implies **J2** and **SDM** implies **WSDM**, which in turn implies **S2** and **S3**. We have proven that, in what follows, whenever we state that a subvariety of $\mathbb{PBZL}^*$ is axiomatized relative to $\mathbb{PBZL}^*$ by axioms $\gamma_1, \ldots, \gamma_n$ for some $n \in \mathbb{N}^*$, we have that, for each $k \in [1, n]$, γ_k is independent from $\{\gamma_i \mid i \in [1, n] \setminus \{k\}\}$.

For any class $\mathbb{C}$ of similar algebras, the variety generated by $\mathbb{C}$ will be denoted by $V(\mathbb{C})$; so $V(\mathbb{C}) = \mathcal{HSP}(\mathbb{C})$, where $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{P}$ are the usual class operators; for any algebra A, $V(\{A\})$ will be streamlined to $V(A)$. We denote by $\mathbb{SDM}$ the variety of the PBZ*–lattices that satisfy the Strong de Morgan condition, and by $\mathbb{SAOL} = \mathbb{SDM} \cap V(\mathbb{AOL})$. Note that $\mathbb{OML} \cap V(\mathbb{AOL}) = \mathbb{BA}$, hence $\mathbb{DIST} \subseteq V(\mathbb{AOL})$. In the lattice of subvarieties of $\mathbb{PBZL}^*$, $\mathbb{BA}$ is the single atom and it has only two covers: its single orthomodular cover, $V(MO_2)$ [1], where MO_2 is the modular ortholattice with four atoms and length three (see the notation in Section 2 below), and $V(D_3)$ [2], where D_3 is the three–element antiortholattice chain (see Section 1 below); furthermore, D_3 belongs to any subvariety of $\mathbb{PBZL}^*$ which is not included in $\mathbb{OML}$, hence $\mathbb{OML} \vee V(D_3)$ is the single cover of $\mathbb{OML}$ in this subvariety lattice.

1 Ordinal Sums

Let us denote by D_n the n–element chain for any $n \in \mathbb{N}^*$, which clearly becomes an antiortholattice with its dual lattice automorphism as Kleene complement and the trivial Brower complement. Moreover, any pseudo–Kleene algebra with the 0 meet–irreducible becomes an antiortholattice when endowed with the trivial Brower complement; furthermore, if we denote by $L \oplus M$ the *ordinal sum* of a lattice L with largest element and a lattice M with smallest element, obtained by glueing L with M at the largest element of L and the smallest element of M, then, for any pseudo–Kleene algebra K and any non–trivial bounded lattice L, if L^d is the dual of L, then $L \oplus K \oplus L^d$ becomes an antiortholattice, with the clear definition for the Kleene complement and the trivial Brower complement. If $\mathbb{C} \subseteq \mathbb{PKA}$, then we denote by $D_2 \oplus \mathbb{C} \oplus D_2 = \{D_2 \oplus K \oplus D_2 \mid K \in \mathbb{C}\} \subsetneq \mathbb{AOL} \cap \mathbb{SAOL} \subsetneq \mathbb{AOL}$.

Recall from [3] that, for any $n \in \mathbb{N}$ with $n \geq 5$, $V(D_3) = V(D_4) \subsetneq V(D_5) = V(D_n) = \mathbb{DIST} \cap \mathbb{SAOL} \subsetneq \mathbb{DIST} = V(\{D_2^\kappa \oplus D_2^\kappa, D_2^\kappa \oplus D_2 \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\})$ and note that $\mathbb{BA} = V(D_2) \subsetneq V(D_3)$ and, for each $j \in \{0,1\}$, $D_{2j+1} = D_2^j \oplus D_2^j$ and $D_{2j+2} = D_2^j \oplus D_2 \oplus D_2^j$.

We have proven the following:

- $\mathbb{BA} = V(D_2) \subsetneq V(D_3) = V(D_4) \subsetneq \ldots \subsetneq V(D_2^n \oplus D_2^n) \subsetneq V(D_2^n \oplus D_2 \oplus D_2^n) \subsetneq V(D_2^{n+1} \oplus D_2^{n+1}) \subsetneq V(D_2^{n+1} \oplus D_2 \oplus D_2^{n+1}) \subsetneq \ldots \subsetneq V(\{D_2^\kappa \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\}) = V(\{D_2^\kappa \oplus D_2 \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\}) = \mathbb{DIST} \subsetneq \mathbb{DIST} \vee \mathbb{SAOL} \subsetneq V(\mathbb{AOL})$, where n designates an arbitrary natural number with $n \geq 2$;

- $\mathbb{SAOL} \cap \mathbb{DIST} = V(D_5) = V(D_2 \oplus \mathbb{BA} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{OML} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{OL} \oplus D_2) \subsetneq V(D_2 \oplus \mathbb{PKA} \oplus D_2) = \mathbb{SAOL} \subsetneq \mathbb{DIST} \vee \mathbb{SAOL} = V((D_2 \oplus \mathbb{PKA} \oplus D_2) \cup \{D_2^\kappa \oplus D_2^\kappa \mid \kappa \text{ a cardinal number}\})$, the latter equality following from the above;

- $\mathbb{OML} \vee V(D_3) \subsetneq \mathbb{OML} \vee V(D_5) = \mathbb{OML} \vee (\mathbb{DIST} \cap \mathbb{SAOL}) = (\mathbb{OML} \vee \mathbb{DIST}) \cap (\mathbb{OML} \vee \mathbb{SAOL}) \subsetneq \mathbb{OML} \vee \mathbb{DIST}, \mathbb{OML} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee \mathbb{DIST} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee V(\mathbb{AOL}) \subsetneq \mathbb{SDM} \vee V(\mathbb{AOL}) \supsetneq \mathbb{SDM} \supsetneq \mathbb{OML} \vee \mathbb{SAOL}$, where the second equality follows from the more general fact that:

Theorem 1. *L is a sublattice of the lattice of subvarieties of $\mathbb{PBZL}^*$ such that all elements of L except the largest, if L has a largest element, are either subvarieties of $\mathbb{OML}$ or of $V(\mathbb{AOL})$, and the sublattice of L formed of its elements which are subvarieties of $\mathbb{OML}$ is distributive, then L is distributive.*

We know from the above that $\mathbb{OML} \vee V(\mathbb{AOL})$ is not a cover of $\mathbb{OML}$ in the lattice of subvarieties of $\mathbb{PBZL}^*$. The previous theorem shows that $\mathbb{OML} \vee V(\mathbb{AOL})$ is not a cover of $V(\mathbb{AOL})$, either, because, for any subvariety $\mathbb{V}$ of $\mathbb{OML}$ such that $\mathbb{BA} \subsetneq \mathbb{V} \subsetneq \mathbb{OML}$, $\{\mathbb{BA}, \mathbb{V}, \mathbb{OML}, V(\mathbb{AOL}), \mathbb{OML} \vee V(\mathbb{AOL})\}$ fails to be a sublattice of $\mathbb{PBZL}^*$, which can only happen if $V(\mathbb{AOL}) \subsetneq \mathbb{V} \vee V(\mathbb{AOL}) \subsetneq \mathbb{OML} \vee V(\mathbb{AOL})$. The theorem above also implies:

Corollary 2. *The lattice of subvarieties of $V(\mathbb{AOL})$ is distributive.*

2 Horizontal Sums and Axiomatizations

We denote by $A \boxplus B$ the *horizontal sum* of two non–trivial bounded lattices A and B, obtained by glueing them at their smallest elements, as well as at their largest elements; clearly, the horizontal sum is commutative and has D_2 as a neutral element; note that, in the same way, one defines the horizontal sum of an arbitrary family of non–trivial bounded lattices. Whenever

A is a non–trivial orthomodular lattice and B is a non–trivial PBZ*–lattice, $A \boxplus B$ becomes a PBZ*–lattice having A and B as subalgebras, that is with its Kleene and Brower complement restricting to those of A and B, respectively; similarly, the horizontal sum of an arbitrary family of PBZ*–lattices becomes a PBZ*–lattice whenever all members of that family excepting at most one are orthomodular. If $\mathbb{C} \subseteq \mathbb{OML}$ and $\mathbb{D} \subseteq \mathbb{PBZL}^*$, then we denote by $\mathbb{C} \boxplus \mathbb{D} = \{D_1\} \cup \{A \boxplus B \mid A \in \mathbb{C} \setminus \{D_1\}, B \in \mathbb{D} \setminus \{D_1\}\} \subseteq \mathbb{PBZL}^*$.

For any cardinal number κ, we denote by $MO_\kappa = \boxplus_{i<\kappa} D_2^2 \in \mathbb{OML}$, where, by convention, we let $MO_0 = D_2$. All PBZ*–lattices L having the elements of $L \setminus \{0,1\}$ join–irreducible are of the form $L = MO_\kappa \boxplus A$ for some cardinal number κ and some antiortholattice chain A, hence they are horizontal sums of families of Boolean algebras with antiortholattice chains, so, by a result in [3], the variety they generate is generated by its finite members, from which, noticing that, for any $A \in \mathbb{OML} \setminus \{D_1, D_2\}$ and any non–trivial $B \in \mathbb{AOL}$, the horizontal sum $A \boxplus B$ is subdirectly irreducible exactly when B is subdirectly irreducible and using and the fact that the only subdirectly irreducible antiortholattice chains are D_1, D_2, D_3, D_4 and D_5, we obtain that $V(\{L \in \mathbb{PBZL}^* \mid L \setminus \{0,1\} \subseteq Ji(L)\}) = V(\{MO_n \boxplus D_k \mid n \in \mathbb{N}, k \in [2,5]\})$, where we have denoted by $Ji(L)$ the set of the join–irreducibles of an arbitrary lattice L.

We have also proven that:

- $\mathbb{OML} \vee V(\mathbb{AOL}) \subsetneq V(\mathbb{OML} \boxplus \mathbb{AOL}) \subsetneq V(\mathbb{OML} \boxplus V(\mathbb{AOL})) \subsetneq \mathbb{PBZL}^*$;
- the class of the members of $\mathbb{OML} \boxplus V(\mathbb{AOL})$ that satisfy **J2** is $\mathbb{OML} \boxplus \mathbb{AOL}$, hence $V(\mathbb{OML} \boxplus \mathbb{AOL})$ is included in the variety axiomatized by **J2** relative to $V(\mathbb{OML} \boxplus V(\mathbb{AOL}))$.

We have obtained the following axiomatizations:

Theorem 3. *(i)* $V(\mathbb{AOL})$ *is axiomatized by* **J0** *relative to* $\mathbb{PBZL}^*$.

(ii) $\mathbb{OML} \vee V(D_3)$ *is axiomatized by* **SK**, **WSDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$.

(iii) $\mathbb{OML} \vee \mathbb{SAOL}$ *is axiomatized by* **SDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$.

(iv) $\mathbb{OML} \vee V(\mathbb{AOL})$ *is axiomatized by* **WSDM** *and* **J2** *relative to* $\mathbb{PBZL}^*$.

(v) $\mathbb{OML} \vee V(\mathbb{AOL})$ *is axiomatized by* **WSDM** *relative to* $V(\mathbb{OML} \boxplus \mathbb{AOL})$.

(vi) $V(\mathbb{OML} \boxplus \mathbb{AOL})$ *is axiomatized by* **S2**, **S3** *and* **J2** *relative to* $\mathbb{PBZL}^*$.

In Theorem 3, (i) is a streamlining of the axiomatization of $V(\mathbb{AOL})$ obtained in [2]; we have obtained (iv) both by a direct proof and as a corollary of (v) and (vi).

References

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