Syntactical approach to Glivenko-like theorems

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Glivenko's theorem [6] asserts that provability in classical logic CL can be translated to that of intuitionistic logic IL by means of double negation. Since then many translations of this kind where proved, e.g. between the Łukasiewicz logic Ł and Hájek's basic fuzzy logic BL [2], or between classical logic CL and the product fuzzy logic Π [1]. It is a common phenomenon that these results are obtained by algebraic (i.e. semantic) means. In this talk we provide a syntactic account of these results. The results presented in this contribution can be found in [8].

Our main result characterizes those extensions of a certain basic substructural logic which are Glivenko-equivalent to classical logic. The proof of this result relies on the notion of an inconsistency lemma (introduced by Raftery [9] and further studied in [8]) and the new notion of an antistructural completion [8]. In particular, it proceeds by identifying sufficient conditions under which classical logic is the antistructural completion of a given substructural logic.

Inconsistency lemmas are equivalences relating inconsistency and validity in a given logic, just like deduction-detachment theorems are equivalences relating theoremhood and validity. For example, it is well known that the following equivalence holds in intuitionistic logic:

$$\Gamma, \varphi \vdash \bot \quad \iff \quad \Gamma \vdash \neg\varphi. \tag{1}$$

This property was first explicitly isolated and systematically studied by Raftery [9], who called such an equivalence an *inconsistency lemma*. He also considered the following dual version of this property, which he called a *dual inconsistency lemma*:

$$\Gamma, \neg\varphi \vdash \bot \quad \iff \quad \Gamma \vdash \varphi. \tag{2}$$

This equivalence, of course, is no longer valid in intuitionistic logic but it does hold for classical logic. (In fact it is the law of excluded middle in disguise.)

We continue the line of research initiated by Raftery and introduce what we call local and parametrized local versions of these properties, by analogy with the so-called local and parametrized local deduction-detachment theorems. This yields a hierarchy of inconsistency lemmas similar to the existing hierarchy of deduction-detachment theorems (see e.g. [4]).

Let us illustrate what the local form of Raftery's inconsistency lemma and dual inconsistency lemma looks like. For example, Hájek's basic logic BL enjoys a *local* inconsistency lemma in the following form:

$$\Gamma, \varphi \vdash \bot \quad \iff \quad \Gamma \vdash \neg\varphi^n \text{ for some } n \in \omega.$$

On the other hand, the infinitary Łukasiewicz logic $Ł_\infty$ (i.e. the infinitary consequence relation of the standard Łukasiewicz algebra on the unit interval $[0, 1]$) enjoys a *dual local* inconsistency lemma in the following form (note the universal rather than existential quantifier here):

$$\Gamma, \neg\varphi^n \vdash \bot \text{ for all } n \in \omega \quad \iff \quad \Gamma \vdash \varphi.$$

We remark that the finitary companion of $Ł_\infty$, i.e. the finitary Łukasiewicz logic Ł, validates the above mentioned dual local inconsistency lemma for finite sets of formulas Γ.

The second component of our proof is the notion of an antistructural completion, which is the natural dual to the notion of a structural completion (see [7]). Recall that the *structural completion* of a logic L is the largest logic σL which has the same theorems as L. A logic L is then called *structurally complete* if $\sigma\mathrm{L} = \mathrm{L}$. The logic σL exists for each L and it has a simple description: $\Gamma \vdash_{\sigma\mathrm{L}} \varphi$ if and only if the rule $\Gamma \vdash \varphi$ is *admissible* in L, i.e. for each substitution τ we have $\emptyset \vdash_{\mathrm{L}} \tau\varphi$ whenever $\emptyset \vdash_{\mathrm{L}} \tau\gamma$ for each $\gamma \in \Gamma$.

Dually, the *antistructural completion* of a logic L is defined as the largest logic αL, whenever it exists, which has the same inconsistent (or equivalently, maximally consistent) sets as L. Naturally, a logic L is called *antistructurally complete* if $\alpha\mathrm{L} = \mathrm{L}$. For example, Glivenko's theorem essentially states that $\alpha\mathrm{IL} = \mathrm{CL}$.

Just like σL can be characterized in terms of admissible rules, the antistructural completion of αL of L can be characterized in terms of *antiadmissible rules*. These are rules $\Gamma \vdash \varphi$ which for every substitution σ and every set of formulas Δ satisfy the implication:

$$\{\sigma\varphi\} \cup \Delta \text{ is inconsistent in L} \implies \sigma\Gamma \cup \Delta \text{ is inconsistent in L.}$$

Moreover, in many contexts the description of antiadmissible rules can be simplified by omitting the quantification over substitutions. This yields what we call *simply antiadmissible rules*, which for every set of formulas Δ satisfy the implication:

$$\{\varphi\} \cup \Delta \text{ is inconsistent in L} \implies \Gamma \cup \Delta \text{ is inconsistent in L.}$$

Our first main result now ties all these notions together.

Theorem. *Let* L *be a finitary logic with a local inconsistency lemma. Then the following are equivalent:*

1. L *is antistructurally complete.*
2. *Every simply antiadmissible rule is valid in* L*.*
3. L *enjoys the local* dual *inconsistency lemma.*
4. L *is* semisimple *(i.e. subdirectly irreducible models of* L *are simple).*
5. L *is complete w.r.t. (a subclass of) the class of all simple models of* L*.*

The above characterization (especially points 4. and 5.) provides a wealth of examples of antistructurally complete logics: e.g. the global modal logic S5, the k-valued Łukasiewicz logics Ł_k, or the infinitary Łukasiewicz logic Ł_∞.

With the above result in hand, we now proceed to describe the substructural logics Glivenko-equivalent to classical logic. Here we say that a logic L′ is *Glivenko-equivalent* to L if

$$\Gamma \vdash_{\mathrm{L}'} \neg\neg\varphi \iff \Gamma \vdash_{\mathrm{L}} \varphi.$$

Our weakest substructural logic is the logic SL (see [3]), which corresponds the bounded nonassociative full Lambek calculus. It is introduced in a standard substructural language

$$\{\wedge, \vee, \&, \to, \rightsquigarrow, \bar{0}, \bar{1}, \top, \bot\}$$

consisting of lattice conjunction and disjunction, strong conjunction & and its right and left residuals $\to$ and $\rightsquigarrow$, and four constants, $\top$ being a lattice top and $\bot$ a lattice bottom. Moreover, we consider the following two defined negations: $\neg\varphi = \varphi \to \bar{0}$ and ${\sim}\varphi = \varphi \rightsquigarrow \bar{0}$. We identify substructural logics with finitary extensions of SL. (The two implications and negations are equivalent in extensions which validate the axiom of Exchange.)

Theorem. *For every substructural logic* L*, the following are equivalent:*

1. L *is Glivenko-equivalent to classical logic, i.e. for every* $\Gamma \cup \{\varphi\} \subseteq Fm$

$$\Gamma \vdash_{\mathrm{L}} \neg\neg\varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \varphi.$$

2. L *"almost" has the inconsistency lemma of* IL*, i.e.*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \overline{0} \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi, \tag{3}$$

and moreover the following rules are valid in L*:*

$$\neg(\varphi \to \psi) \vdash \neg(\neg\neg\varphi \to \sim\neg\psi) \tag{A}$$
$$\neg(\varphi \,\&\, \neg\psi) \dashv\vdash \neg(\varphi \wedge \neg\psi). \tag{Conj}$$

Furthermore, if $\overline{0}$ *is an inconsistent set in* L *then both properties imply that* $\alpha\mathrm{L} = \mathrm{CL}$*.*

The main idea of a proof of the more interesting direction (2. implies 1.) is the following. Extend L to L_0 by a rule $\overline{0} \vdash \bot$ (i.e. $\overline{0}$ proves everything) and show that the antistructural completion of L_0 is the classical logic. This can be established by purely syntactic means from the assumptions using the following lemma connecting inconsistency lemmas, duals inconsistency lemmas, and antistructural completions.[1]

Lemma. *Let* L *be a substructural logic satisfying the inconsistency lemma of intuitionistic logic* (1)*. Then:*

1. $\alpha\mathrm{L}$ *also satisfies* (1)*.*
2. $\alpha\mathrm{L}$ *satisfies the* dual *inconsistency lemma of classical logic, i.e.* (2)*.*
3. $\alpha\mathrm{L}$ *validates the law of excluded middle, i.e.* $\varphi \vee \neg\varphi$ *is its theorem.*
4. *As a consequence of* (1) *and* (2)*,* $\alpha\mathrm{L}$ *enjoys a deduction-detachment theorem in the form:*

$$\Gamma, \varphi \vdash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \neg(\varphi \wedge \neg\psi).$$

This theorem provides a simple strategy for finding the smallest axiomatic extension of a given substructural logic which is Glivenko-equivalent to classical logic, a problem investigated e.g. in [5]. Namely, given a substructural logic, first extend it by the rules (Conj) and (A) in the form of axioms. Secondly, find axioms which ensure the validity of (3). To this end, use a known deduction-detachment theorem.

Let us for example consider the full Lambek calculus with exchange FL_e (i.e. SL with & being commutative and associative). It is well known that this logic enjoys a local deduction-detachment theorem which in particular yields the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{FL}_e} \overline{0} \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{FL}_e} \neg(\varphi \wedge \overline{1})^k \text{ for some } k \in \omega.$$

Thus we only need to add axioms ensuring that $\neg(\varphi \wedge \overline{1})^k \dashv\vdash \neg\varphi$ in our extension. This can be achieved e.g. by adding the axioms $\neg(\varphi \wedge \overline{1}) \to \neg\varphi$ and $\neg(\varphi \,\&\, \psi) \to \neg(\varphi \wedge \psi)$. Moreover, these axioms can be proved to hold in each extension of FL_e Glivenko-equivalent to CL.

Finally, let us remark that inconsistency lemmas and antistructural completions can be used in a similar fashion to establish the Glivenko-equivalence between Łukasiewicz logic and Hájek's basic fuzzy logic BL.

[1] We remark that the lemma can proved in much grater generality than presented here.

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