Substructural PDL

Igor Sedlár*

The Czech Academy of Sciences, Institute of Computer Science
`sedlar@cs.cas.cz`

Propositional Dynamic Logic PDL, introduced in [6] following the ideas of [11], is a modal logic with applications in formal verification of programs [7], dynamic epistemic logic [1] and deontic logic [10], for example. More generally, PDL can be seen as a logic for reasoning about *structured actions* modifying various types of objects; examples of such actions include programs modifying states of the computer, information state updates or actions of agents changing the world around them.

In this contribution we study versions of PDL where the underlying propositional logic is a weak substructural logic in the vicinity of the full distributive non-associative Lambek calculus with a weak negation. The motivation is to provide a logic for reasoning about structured actions that modify *situations* in the sense of [2]; the link being the informal interpretation of the Routley–Meyer semantics for substructural logics in terms of situations [9].

In a recent paper [14] we studied versions of PDL based on Kripke frames with a ternary accessibility relation (in the style of [4, 8]). These frames do not contain the inclusion ordering essential for modelling situations, nor the compatibility relation articulating the semantics for a wide range of weak negations [5, 12]. Hence, in this contribution we study PDL based on (partially ordered) Routley–Meyer models with a compatibility relation.

Formulas φ and actions A are defined by mutual induction in the usual way [7]

$$
\begin{aligned}
A &:= a \mid A \cup A \mid A;A \mid A^* \mid \varphi?\\
\varphi &:= p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid {\sim}\varphi \mid \varphi \to \varphi \mid \varphi \circ \varphi \mid [A]\varphi
\end{aligned}
$$

where p is an atomic formula and a an atomic action. So far, we have results for the language without existential modalities $\langle A \rangle$ dual to $[A]$; inclusion of these is the focus of ongoing work. The implication $\to$ is the left residual of fusion $\circ$ which is assumed to be commutative for the sake of simplicity.

A Routley-Meyer frame is $\mathfrak{F} = \langle S, \leq, L, C, R \rangle$ where $(S, \leq, L)$ is a partially ordered set with an upwards-closed $L \subseteq S$; C is a symmetric binary relation antitone in both positions, that is

- Cxy, $x' \leq x$ and $y' \leq y$ only if $Cx'y'$;

and R is a ternary relation antitone in the first two positions such that

- $Rxyz$ only if $Ryxz$ and
- $x \leq y$ iff there is $z \in L$ such that $Rzxy$.

A (dynamic) Routley-Meyer model based on $\mathfrak{F}$ is $\mathfrak{M} = \langle \mathfrak{F}, [\![\cdot]\!] \rangle$ where

- $[\![\varphi]\!]$ is a subset of S such that $[\![p]\!]$ is upwards-closed and
- $[\![A]\!]$ is a binary relation on S such that $[\![a]\!]$ is antitone in the first position.

It is assumed that $[\![\varphi \land \psi]\!]$ ($[\![\varphi \lor \psi]\!]$) is the intersection (union) of $[\![\varphi]\!]$ and $[\![\psi]\!]$ and

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- $[\![\sim\varphi]\!] = \{x \mid \forall y\,(Cxy \Rightarrow y \notin [\![\varphi]\!])\}$,
- $[\![\varphi \to \psi]\!] = \{x \mid \forall yz\,((Rxyz \ \ \& \ \ y \in [\![\varphi]\!]) \Rightarrow z \in [\![\psi]\!])\}$,
- $[\![\varphi \circ \psi]\!] = \{x \mid \exists yz\,(Ryzx \ \ \& \ \ y \in [\![\varphi]\!] \ \ \& \ \ z \in [\![\psi]\!])\}$ and
- $[\![[A]\varphi]\!] = \{x \mid \forall y\,(x[\![A]\!]y \Rightarrow y \in [\![\varphi]\!])\}$.

It is also assumed that $[\![A \cup B]\!]$ ($[\![A;B]\!]$) is the union (composition) of $[\![A]\!]$ and $[\![B]\!]$, that $[\![A^*]\!]$ is the reflexive-transitive closure of $[\![A]\!]$ and that

- $[\![\varphi?]\!] = \{\langle x, y\rangle \mid x \leq y \ \ \& \ \ y \in [\![\varphi]\!]\}$

We say that φ is valid in $\mathfrak{M}$ iff $L \subseteq [\![\varphi]\!]$; a finite Γ entails φ in $\mathfrak{M}$ iff $[\![\bigwedge \Gamma]\!] \subseteq [\![\varphi]\!]$. Validity and entailment in a class of frames are defined as usual.

It can be shown that each $[\![A]\!]$ is antitone in its first position. This, together with the other tonicity conditions, entails that $[\![\varphi]\!]$ is an upwards-closed set for all φ (this is the motivation of the unusual definition of $[\![\varphi?]\!]$) and so we have in turn the consequence that Γ entails φ in $\mathfrak{M}$ iff $\bigwedge \Gamma \to \varphi$ is valid in $\mathfrak{M}$ (unlike the semantics without L and $\leq$ where both directions of the equivalence may fail).

Extending the results of [13], we prove completeness and decidability of the set of formulas valid in all frames using filtration in the style of [3].

A *logic* is any set of formulas Λ containing all formulas of the form ($\leftrightarrows$ indicates that both implications are in Λ)

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- $[A]\varphi \wedge [A]\psi \to [A](\varphi \wedge \psi)$
- $[A \cup B]\varphi \leftrightarrows ([A]\varphi \wedge [B]\varphi)$
- $[A;B]\varphi \leftrightarrows [A][B]\varphi$
- $[A^*]\varphi \leftrightarrows (\varphi \wedge [A][A^*]\varphi)$
- $[\varphi?]\varphi$
- $\psi \to [\varphi?]\psi$
- $(\varphi \wedge [\varphi?]\psi) \to \psi$

and closed under the inference rules ('//' indicates a two-way rule):

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$
- $\varphi \to \psi \,/\, [A]\varphi \to [A]\psi$
- $\varphi \to (\psi \to \chi) \,//\, (\psi \circ \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $\varphi \to {\sim}\psi \,//\, \psi \to {\sim}\varphi$
- $\varphi \to [A]\varphi \,/\, \varphi \to [A^*]\varphi$

We write $\Gamma \vdash_\Lambda \Delta$ iff there are finite Γ', Δ' such that $\bigwedge \Gamma' \to \bigvee \Delta'$ is in Λ (hence, the relation $\vdash_\Lambda$ is finitary by definition). A prime Λ-theory is a set of formulas Γ such that $\varphi, \psi \in \Gamma$ only if $\Gamma \vdash_\Lambda \varphi \wedge \psi$ and $\Gamma \vdash_\Lambda \varphi \vee \psi$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$. For each $\Gamma \nvdash_\Lambda \Delta$ there is a prime theory containing Γ but disjoint from Δ [12, 94]. The *canonical* Λ*-frame* the frame-type structure $\mathfrak{F}^\Lambda$ where S^Λ is the set of prime Λ-theories, $\leq^\Lambda$ is set inclusion, L^Λ is the set of prime theories containing Λ and

- $C^\Lambda\Gamma\Delta$ iff $\sim\varphi \in \Gamma$ only if $\varphi \notin \Delta$
- $R^\Lambda\Gamma\Delta\Sigma$ iff $\varphi \in \Gamma$ and $\psi \in \Delta$ only if $\varphi \circ \psi \in \Sigma$

It is a standard observation that $\mathfrak{F}^\Lambda$ is a Routley–Meyer frame for all Λ [12]. The *canonical* Λ*-structure* $\mathfrak{S}^\Lambda$ is the canonical frame with $[\![\cdot]\!]^\Lambda$ defined as follows: $[\![\varphi]\!]^\Lambda = \{\Gamma \in S^\Lambda \mid \varphi \in \Gamma\}$ and $[\![A]\!]^\Lambda = \{\langle\Gamma, \Delta\rangle \mid \forall\varphi\,([A]\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\}$. It can be shown that $\mathfrak{S}^\Lambda$ is not a dynamic Routley–Meyer model (since $\{[a^n]p \mid n \in \mathbb{N}\} \not\vdash_\Lambda [a^*]p$, we may show that $[\![a^*]\!]^\Lambda$ is not the reflexive-transitive closure of $[\![a]\!]^\Lambda$).

Fix a finite set of formulas Φ that is closed under subformulas and satisfies the following conditions: i) $[\varphi?]\psi \in \Phi$ only if $\varphi \in \Phi$, ii) $[A \cup B]\varphi \in \Phi$ only if $[A]\varphi, [B]\varphi \in \Phi$, iii) $[A; B]\varphi \in \Phi$ only if $[A][B]\varphi \in \Phi$ and iv) $[A^*]\varphi \in \Phi$ only if $[A][A^*]\varphi \in \Phi$. We define $\Gamma \preceq_\Phi \Delta$ as $(\Gamma \cap \Phi) \subseteq \Delta$. This relation is obviously a preorder; let $\equiv_\Phi$ be the associated equivalence relation and let $[\Gamma]_\Phi$ be the equivalence class of Γ with respect to this relation.

The Φ*-filtration of* $\mathfrak{S}^\Lambda$ is the model-type structure $\mathfrak{M}^\Lambda_\Phi$ such that S_Φ is the (finite) set of equivalence classes $[\Gamma]$ for $\Gamma \in S^\Lambda$, $[\Gamma] \leq_\Phi [\Delta]$ iff $\Gamma \preceq_\Phi \Delta$ and

- $L_\Phi = \{[\Gamma] \mid \exists\Delta \in S^\Lambda\,(\Delta \preceq_\Phi \Gamma \;\&\; \Delta \in L^\Lambda)\}$
- $C_\Phi = \{\langle[\Gamma_1], [\Gamma_2]\rangle \mid \exists\Delta_1, \Delta_2\,(\Gamma_1 \preceq_\Phi \Delta_1 \;\&\; \Gamma_2 \preceq_\Phi \Delta_2 \;\&\; C^\Lambda\Delta_1\Delta_2)\}$
- $R_\Phi = \{\langle[\Gamma_1], [\Gamma_2], [\Gamma_3]\rangle \mid \exists\Delta_1, \Delta_2, (\Gamma_1 \preceq_\Phi \Delta_1 \;\&\; \Gamma_2 \preceq_\Phi \Delta_2 \;\&\; R^\Lambda\Delta_1\Delta_2\Gamma_3)\}$
- $[\![p]\!]_\Phi = \{[\Gamma] \mid p \in \Gamma\}$ for $p \in \Phi$ and $[\![p]\!]_\Phi = \emptyset$ otherwise
- $[\![a]\!]_\Phi = \{\langle[\Gamma_1], [\Gamma_2]\rangle \mid \exists\Delta\,(\Gamma_1 \preceq_\Phi \Delta \;\&\; \Delta[\![a]\!]^\Lambda\Gamma_2)\}$ if $[a]\chi \in \Phi$; $[\![a]\!]_\Phi = \emptyset$ otherwise.

The values of $[\![\cdot]\!]_\Phi$ on complex formulas and actions are defined exactly as in dynamic Routley–Meyer models. It can be shown that $\mathfrak{M}^\Lambda_\Phi$ is a dynamic Routley–Meyer model such that if $\varphi \in (\Phi \setminus \Lambda)$, then φ is not valid in $\mathfrak{M}^\Lambda_\Phi$. This implies completeness of the minimal logic Λ_0 with respect to (the set of formulas valid in) all Routley–Meyer frames.

In general, assume that we have $\mathsf{Log}(\mathbf{F})$, the set of formulas valid in all Routley–Meyer frames $\mathfrak{F} \in \mathbf{F}$. If $\mathcal{F}^\Lambda_\Phi \in \mathbf{F}$ for all Φ, then Λ is complete with respect to $\mathbf{F}$ (for instance, this is the case where $\mathbf{F}$ is the class of frames satisfying $Rxxx$ for all x). If $\mathcal{F}^\Lambda_\Phi \notin \mathbf{F}$, then one has to either modify the requirements concerning Φ (while keeping it finite; our argument showing that if $\varphi \in (\Phi \setminus \Lambda)$, then φ is not valid in $\mathfrak{M}^\Lambda_\Phi$ does not work if Φ is infinite) or devise an alternative definition of R_Φ and C_Φ. Such modifications for specific classes of frames (e.g. associative ones) is the focus of ongoing work.

A topic for future work is a modification of our argument not requiring that Φ be finite. An argument based on finite filtration does not go through in case of logics that are known not to have the finite model property, such as the relevant logic R for instance.

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