Below Gödel-Dummett

Dick de Jongh[1] and Fatemeh Shirmohammadzadeh Maleki[2]

[1] Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands
D.H.J.deJongh@uva.nl
[2] School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
f.shmaleki2012@yahoo.com

The Gödel-Dummett logic LC from [9] is a strengthening of intuitionistic logic IPC with linear Kripke-models. It can be axiomatized by many different axiom schemes:

($\mathcal{L}_1$) $(A \to B) \vee (B \to A)$
($\mathcal{L}_2$) $(A \to B) \vee ((A \to B) \to A)$
($\mathcal{L}_3$) $(A \to B) \vee ((A \to B) \to B)$
($\mathcal{L}_4$) $(A \to B \vee C) \to (A \to B) \vee (A \to C)$
($\mathcal{L}_5$) $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$
($\mathcal{L}_6$) $((A \to B) \to B) \wedge ((B \to A) \to A) \to A \vee B.$

An even larger number of equivalents arises by the fact that in IPC $\vdash A \vee B$ iff $\vdash (A \to C) \wedge (B \to C) \to C$ (DR), and, more generally, $\vdash D \to A \vee B$ iff $\vdash D \wedge (A \to C) \wedge (B \to C) \to C$ (EDR).

For strong completeness of LC see e.g. [13]. In the present research in progress we study logics with linear models originating from logics weaker than IPC. Weaker logics than IPC are the subintuitionistic logics with Kripke models extending F studied by [4, 6] and those with neighborhood models extending WF originated in [7, 12]. Linear extensions of those logics have already been obtained in the case of BPC, the extension of F with transitive persistent models [1, 2, 14]. Our object is to study the character of and the relations between the schemes ($\mathcal{L}_1$), . . .,($\mathcal{L}_6$). Besides syntactic methods we use the construction of neighborhood frames [3] for various logics. We also obtain modal companions for a number of the logics. Hájek's basic fuzzy logic BL [10] compares less well with IPC, and is therefore left out of consideration this time.

Extensions of Corsi's logic F. The logic F is axiomatized by

1. $A \to A \vee B$
2. $B \to A \vee B$
3. $A \wedge B \to A$
4. $A \wedge B \to B$
5. $\frac{A \quad B}{A \wedge B}$
6. $\frac{A \quad A \to B}{B}$
7. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
8. $(A \to B) \wedge (B \to C) \to (A \to C)$
9. $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
10. $A \to A$
11. $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
12. $\frac{A}{B \to A}$

The axioms 8, 9 and 11 are more descriptively named I, C and D. Corsi [4] proved completeness for Kripke models with an arbitrary relation R without stipulation of persistence of truth.

The axioms needed to obtain IPC from F are

R: $A \wedge (A \to B) \to B$ (defines and is complete for reflexive Kripke frames)
T: $(A \to B) \to ((B \to C) \to (A \to C))$ (defines and is complete for transitive Kripke frames)

P: $p \to (\top \to p)$ (defines and is complete for persistent Kripke models).

Visser's basic logic BPC can be defined as FTP. In the case of Kripke models we mean with linear models of course connected ($\forall xyz\,((xRy \wedge\ xRz) \to (y \neq z \to yRz \vee zRy)))$ and transitive models. (Anti-symmetry is covered by persistence.) Visser already proved that over BPC, $\mathcal{L}_2$ is complete with regard to linear models, and that $\mathcal{L}_1$ is not [14], see also [2]. We prove that $\mathcal{L}_1$, $\mathcal{L}_4$ and $\mathcal{L}_5$ are equivalent over F. Moreover, we show that $\mathcal{L}_1$ plus $\mathcal{L}_3$ prove $\mathcal{L}_2$ in F, so $\mathcal{L}_1$ plus $\mathcal{L}_3$ is complete for linear models over BPC. We didn't study DR and EDR in depth yet, but were able to prove that the right-to-left direction of DR can be proved in F, but for DFR one needs FR. The left-to-right direction can be executed in FR in both cases.

Neighborhood models and extensions of the logics WF and $\mathsf{WF_N}$. The logic WF can be obtained by deleting the axioms C, D and I from F, and replacing them by the corresponding rules like concluding $A \to B \wedge C$ from $A \to B$ and $A \to C$ (see [12]).

Neighborhood frames describing the natural basic system WF have been obtained in [12]. These NB-neighborhoods consist of pairs (X, Y) with the X and Y corresponding to the antecedent and consequent of implications.

Definition 1. *$\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ is called an **NB-frame** of subintuitionistic logic if $W \neq \emptyset$ and $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$, and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by $U \to V := \{w \in W \mid (U, V) \in NB(w)\}$, where $NB\colon W \to P(\mathcal{X}^2)$ is such that: $\forall w \in W,\ \forall X, Y \in \mathcal{X},\ (X \subseteq Y\ \Rightarrow\ (X, Y) \in NB(w))$.*

If $\mathfrak{M}$ is a model on such a frame, $\mathfrak{M}, w \Vdash A \to B$ iff $(V(A), V(B)) \in NB(w)$. Also N-neighborhood frames, closer to the neighborhood frames of modal logic, were described. In those frames $\overline{X} \cup Y$ corresponds to implications. An additional rule N [5, 7] axiomatizes them:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad (\mathsf{N})$$

WF plus the rule N is denoted by $\mathsf{WF_N}$. For extensions of $\mathsf{WF_N}$ modal companions can often be found.

Again we can see linearity as the combination of connectedness and transitivity of the neighborhood frames. But, of course, connectedness as well as transitivity now concerns sets of worlds (neighborhoods), not individual worlds. NB-frames are called transitive if, for all $(X, Y) \in NB(w), (Y, Z) \in NB(w)$ we have $(X, Z) \in NB(w)$ as well. The formula I defines this property and is complete for the transitive NB-frames. For the N-frames this becomes, for all $\overline{X} \cup Y \in N(w), \overline{Y} \cup Z \in N(w)$ we have $\overline{X} \cup Z \in N(w)$ as well. This too is defined by I, and $\mathsf{WF_N}$ is complete for the transitive N-frames. We cannot say that the connection between transitivity and connectedness in Kripke and neighborhood frames has completely been cleared up. Note that the axiom I for transitivity of the neighborhood frames, which is provable in F, is weaker than the axiom T for transitivity of the Kripke models. On the other hand, in the canonical models of logics like $\mathsf{I}\mathcal{L}_1$ the worlds are linearly ordered by inclusion. Study of the neighborhood frames for BPC and IPC of [11] may further clarify the matter.

The IPC-equivalents of the introduction all define different connectedness properties. Definability and completeness of these logics is part of the present paper. For example the straightforward

$$\text{for all } X, Y \in \mathcal{X}^2,\ (X, Y) \in NB(w) \text{ or } (Y, X) \in NB(w)$$

is called connected$_1$ by us and is defined by $\mathcal{L}_1$. This formula defines a similar property in the case of N-frames, and is complete for those frames as well.

We can refine the results of the section on F by discussing in which extensions of WF the results are provable.

Proposition 1. *$\mathsf{I}\mathcal{L}_1$, $\mathsf{I}\mathcal{L}_4$ and $\mathsf{I}\mathcal{L}_5$ are equivalent over* WF.

Proposition 2. *$\mathsf{WFI}\mathcal{L}_1\mathcal{L}_3$ proves $\mathsf{I}\mathcal{L}_2$.*

The opposite direction is open.

Proposition 3. $\mathsf{WF_N IR}\mathcal{L}_1 \Vdash \mathcal{L}_2$.

Proposition 4. $\mathsf{WF_N IR}\mathcal{L}_2 \Vdash \mathcal{L}_3$.

Modal companions. We consider the translation $\Box$ from L, the language of propositional logic, to $L_\Box$, the language of modal propositional logic. It is given by:

$p^\Box = p$;
$(A \wedge B)^\Box = A^\Box \wedge B^\Box$;
$(A \vee B)^\Box = A^\Box \vee B^\Box$;
$(A \to B)^\Box = \Box(A^\Box \to B^\Box)$.

This translation was discussed independently by both [4] and [8] for subintuitionistic logics with Kripke models. We discussed it for extensions of $\mathsf{WF_N}$ in [5, 7]. For example the extension EN of classical modal logic E is a modal companion of $\mathsf{WF_N}$. Here we get modal companions for all of the extensions of $\mathsf{WF_N}$ that we discuss. For example we obtain as a modal companion of the logic $\mathsf{WF_N}\mathcal{L}_1$ the modal logic ENL_1 axiomatized over EN by $\mathsf{L}_1 : \Box(A \to B) \vee \Box(B \to A)$.

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