The McKinsey-Tarski Theorem for Topological Evidence Models*

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1 Introduction

Epistemic logics (i.e. the family of modal logics concerned with that which an epistemic agent *believes* or *knows*) found a modelisation in [Hin62] in the form of Kripke frames. [Hin62] reasonably claims that the accessibility relation encoding knowledge must be minimally reflexive and transitive, which on the syntactic level translates to the corresponding logic of knowledge containing the axioms of $\mathsf{S4}$. This, paired with the fact (proven by [MT44]) that $\mathsf{S4}$ is the logic of topological spaces under the *interior semantics*, lays the ground for a topological treatment of knowledge. Moreover, treating the K modality as the topological interior operator, and the open sets as "pieces of evidence" adds an evidential dimension to the notion of knowledge that one cannot get within the framework of Kripke frames.

Reading epistemic sentences using the interior semantics might be too simplistic: it equates "knowing" and "having evidence", plus attempts to bring a notion of belief into this framework have not been very felicitous.

Following the precepts of [Sta06], a logic that allows us to talk about knowledge, belief and the relation thereof, about evidence (both basic and combined) and justification is introduced in [BBÖS16]. This is the framework of *topological evidence models* and this paper builds on it.

1.1 The Interior Semantics: the McKinsey-Tarski Theorem

Let Prop be a countable set of propositional variables and let us consider a modal language $\mathcal{L}_\Box$ defined as follows: $\phi ::= p \,|\, \phi \wedge \phi \,|\, \neg\phi \,|\, \Box\phi$, with $p \in \mathsf{Prop}$.

A *topological model* is a topological space (X, τ) along with a valuation $V : \mathsf{Prop} \to 2^X$. The semantics of a formula ϕ is defined recursively as follows: $\|p\| = V(p)$; $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$, $\|\neg\phi\| = X \backslash \|\phi\|$, $\|\Box\phi\| = \text{Int}\,\|\phi\|$.

Theorem 1 ([MT44])**.** *The logic of topological spaces under the interior semantics is* $\mathsf{S4}$*.*

As mentioned above, reading epistemic sentences via the interior semantics has some issues For details, see Section 1.2 of [FG18], and Chapters 3 and 4 of [Ö17]. A new semantics devoid of these issues is proposed in [BBÖS16]: the *dense interior semantics.*

1.2 The Dense Interior Semantics

Our language is now $\mathcal{L}_{\forall KB\Box\Box_0}$, which includes the modalities K (knowledge), B (belief), $[\forall]$ (infallible knowledge), $\Box_0$ (basic evidence), $\Box$ (combined evidence).

*This paper compiles the results contained in the first two chapters of Saúl Fernández González's Master's thesis [FG18]. The authors wish to thank Guram Bezhanishvili for his input.

Definition 2 (The dense interior semantics)**.** We read sentences on *topological evidence models* (i.e. tuples (X, τ, E_0, V) where (X, τ, V) is a topological model and E_0 is a designated subbasis) as follows: $x \in \|K\phi\|$ iff $x \in \text{Int}\,\|\phi\|$ and $\text{Int}\,\|\phi\|$ is *dense*[1]; $x \in \|B\phi\|$ iff $\text{Int}\,\|\phi\|$ is dense; $x \in \|[\forall]\phi\|$ iff $\|\phi\| = X$; $x \in \|\Box_0\phi\|$ iff there is $e \in E_0$ with $x \in e \subseteq \|\phi\|$; $x \in \|\Box\phi\|$ iff $x \in \text{Int}\,\|\phi\|$. Validity is defined in the standard way.

Fragments of the logic. The following logics are obtained by considering certain fragments of the language (i.e. certain subsets of the modalities above).

"K-only", $\mathcal{L}_K$	S4.2.
"Knowledge", $\mathcal{L}_{\forall K}$	S5 axioms and rules for $[\forall]$, plus S4.2 for K, plus axioms $[\forall]\phi \to K\phi$ and $\neg[\forall]\neg K\phi \to [\forall]\neg K\neg\phi$.
"Combined evidence", $\mathcal{L}_{\forall\Box}$	S5 for $[\forall]$, S4 for $\Box$, plus $[\forall]\phi \to \Box\phi$.
"Evidence", $\mathcal{L}_{\forall\Box\Box_0}$	S5 for $[\forall]$, S4 for $\Box$, plus the axioms $\Box_0\phi \to \Box_0\Box_0\phi$, $[\forall]\phi \to \Box_0\phi$, $\Box_0\phi \to \Box\phi$, $(\Box_0\phi \wedge [\forall]\psi) \to \Box_0(\phi \wedge [\forall]\psi)$.

K and B are definable in the evidence fragments, thus we can think of the logic of $\mathcal{L}_{\forall\Box\Box_0}$ as the "full logic".

2 Generic Models

McKinsey and Tarski also proved the following:

Theorem 3 ([MT44])**.** *The logic of a single dense-in-itself metrisable space*[2] *under the interior semantics is* S4*.*

Within the framework of the interior semantics, this tells us that there exist "natural" spaces, such as the real line, which are "generic" enough to capture the logic of the whole class of topological spaces. The main aim of this paper is to translate this idea to the framework of topological evidence models, i.e., finding topo-e-models which are "generic". Formally:

Definition 4 (Generic models)**.** Let $\mathcal{L}$ be a language and (X, τ) a topological space. We will say that (X, τ) is a *generic model for* $\mathcal{L}$ if the sound and complete $\mathcal{L}$-logic over the class of all topological evidence models is sound and complete with respect to the family

$$\{(X, \tau, E_0) : E_0 \text{ is a subbasis of } \tau\}.$$

If $\Box_0$ is not in the language, then a generic model is simply a topological space for which the corresponding $\mathcal{L}$-logic is sound and complete.

2.1 The K-only Fragment

Recall that the logic of the "K-only" fragment of our language is S4.2. The following is our main result:

Theorem 5. S4.2 *is the logic of any d-i-i metrisable space under the dense interior semantics.*

[1] A set $U \subseteq X$ is dense whenever $\text{Cl}\, U = X$.

[2] A space is *dense-in-itself* (d-i-i) if it has no open singletons and *metrisable* if there is a metric d generating the topology. The real line $\mathbb{R}$, the rational line $\mathbb{Q}$, and the Cantor space are examples of d-i-i metrisable spaces.

Proof sketch. Let (X, τ) be such a space. The proof of completeness relies on the following:
Lemma. S4.2 is sound and complete with respect to finite cofinal rooted preorders. Each of these can be written as a disjoint union $W = A \cup B$, where B is a finite rooted preorder and A is a final cluster (i.e. $x \leq y$ for all $x \in W, y \in A$).
Partition lemma [BBLBvM18]. Any d-i-i metrisable space admits a partition $\{G, U_1, ..., U_n\}$, where G is a d-i-i subspace with dense complement and each U_i is open, for every $n \geq 1$.
Theorem [BBLBvM18]. Given a rooted preorder B, and a d-i-i metrisable space G, there exists a continuous, open and surjective map $f : G \to B$.

Now, let $W = A \cup B$ be a finite cofinal rooted preorder, with $A = \{a_1, ..., a_n\}$ its final cluster. We partition X into $\{G, U_1, ..., U_n\}$ as per the partition lemma and we extend the open, continuous and surjective map $f : G \to B$ to a map $\bar{f} : X \to W$ by mapping each $x \in U_i$ to a_i. We can see that under $\bar{f}$: (i) the image of a dense open set is an upset ($\bar{f}$ is *dense-open*); (ii) the preimage of an upset is a dense open set ($\bar{f}$ is *dense-continuous*).

Moreover, we have:
Lemma. Given a dense-open and dense-continous onto map $\bar{f} : X \to W$, and given a formula ϕ and a valuation V such that $W, V, \bar{f}x \nvDash \phi$ under the Kripke semantics, we have that $X, V^f, x \nvDash \phi$ under the dense interior semantics, where $V^f(p) = \{x \in X : \bar{f}x \in V(p)\}$.

Completeness follows. ■

Corollary 6. *$\mathbb{R}$, $\mathbb{Q}$ and the Cantor space are generic models for the knowledge fragment $\mathcal{L}_K$.*

2.2 Universal Modality and the Logic of $\mathbb{Q}$

As a connected space, $\mathbb{R}$ is not a generic model for the fragments $\mathcal{L}_{\forall K}$, $\mathcal{L}_{\forall \Box}$ and $\mathcal{L}_{\forall \Box \Box_0}$. We can however see that there are d-i-i, metrisable yet disconnected spaces (such as $\mathbb{Q}$) which are generic models for these fragments.

Theorem 7. *$\mathbb{Q}$ is a generic model for $\mathcal{L}_{\forall K}$ and $\mathcal{L}_{\forall \Box}$.*

Proof sketch for $\mathcal{L}_{\forall K}$. We use: (i) the logic of the $\mathcal{L}_{\forall K}$ fragment is sound and complete with respect to finite cofinal preorders under the Kripke semantics; (ii) any finite cofinal preorder W is a p-morphic image via a dense-open dense-continuous p-morphism of a disjoint finite union of finite *rooted* cofinal preorders, $p : W_1 \uplus ... \uplus W_n \to W$.

Take $a_1 < ... < a_{n-1} \in \mathbb{R} \backslash \mathbb{Q}$ and let $A_1 = (-\infty, a_1)$, $A_n = (a_{n-1}, \infty)$ and $A_i = (a_{i-1}, a_i)$ for $1 < i < n$. We have that $\{A_1, ..., A_n\}$ partitions $\mathbb{Q}$ in n open sets each isomorphic to $\mathbb{Q}$. As per Theorem 5, there exists a dense-open dense-continuous onto map $f_i : A_i \to W_i$. By taking $f = f_1 \cup ... \cup f_n$ and composing it with p above we obtain a dense-open, dense-continuous onto map $\mathbb{Q} \to W$. Completeness then follows as in Theorem 5. ■

Theorem 8. *$\mathbb{Q}$ is a generic model for $\mathcal{L}_{\forall \Box \Box_0}$.*

Proof sketch. This proof uses the fact that the logic is complete with respect to quasi-models of the form $(X, \leq, E_0, V)$, where $\leq$ is a preorder and E_0 is a collection of $\leq$-upsets. Given a continuous, open and surjective map $f : \mathbb{Q} \to (X, \leq)$, we can define a valuation $V^f(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ and a subbasis of $\mathbb{Q}$, $E_0^f = \{e \subseteq \mathbb{Q} : f[e] \in E_0\}$ such that

$$(\mathbb{Q}, E_0^f, V^f), x \vDash \phi \text{ iff } (X, \leq, E_0, V), fx \vDash \phi,$$

whence the result follows. ■

Completeness with respect to a single topo-e-model. The logic of the fragment $\mathcal{L}_{\forall\Box\Box_0}$ is sound and complete with respect to the class of topo-e-models based on $\mathbb{Q}$ with arbitrary subbases. Could we get completeness with respect to a designated subbasis? An obvious candidate would be perhaps the most paradigmatic case of subbasis-which-isn't-a-basis, namely $\mathcal{S} = \{(-\infty, a), (b, \infty) : a, b \in \mathbb{Q}\}$. As it turns out, the logic is not complete with respect to the class of topo-e-models based on $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S})$. Let $\mathsf{Prop} = \{p_1, p_2, p_3\}$ and consider the formula

$$\gamma = \bigwedge_{i=1,2,3} (\Box_0 p_i \wedge \neg[\forall]\neg\Box_0\neg p_i) \bigwedge_{i\neq j\in\{1,2,3\}} \neg[\forall]\neg(\Box_0 p_i \wedge \neg\Box_0 p_j).$$

as it turns out, γ is consistent in the logic yet $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S}) \vDash \neg\gamma$.

Generalising the results. We finish by outlining a class of generic models for all the fragments we are working with. The only part in the previous proofs that makes $\mathbb{Q}$ a generic model for these fragments but not other d-i-i metrisable spaces like $\mathbb{R}$ is the possibility to partition $\mathbb{Q}$ in n open sets which are homeomorphic to $\mathbb{Q}$ itself. A topological space can be partitioned in this way if and only if it is *idempotent*.

Definition 9. A topological space (X, τ) is *idempotent* if it is homeomorphic to the disjoint union $(X, \tau) \oplus (X, \tau)$.

And thus:

Theorem 10. *Any dense-in-itself, metrisable and idempotent space (such as $\mathbb{Q}$ or the Cantor space) is a generic model for the fragments $\mathcal{L}_K, \mathcal{L}_{KB}, \mathcal{L}_{\forall K}, \mathcal{L}_{\forall\Box}$ and $\mathcal{L}_{\forall\Box\Box_0}$.*

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