Characterization of metrizable Esakia spaces via some forbidden configurations*

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Priestley duality [3, 4] provides a dual equivalence between the category Dist of bounded distributive lattices and the category Pries of Priestley spaces; and Esakia duality [1] provides a dual equivalence between the category Heyt of Heyting algebras and the category Esa of Esakia spaces. A *Priestley space* is a compact space X with a partial order $\leq$ such that $x \not\leq y$ implies there is a clopen upset U with $x \in U$ and $y \notin U$. An *Esakia space* is a Priestley space in which ${\downarrow}U$ is clopen for each clopen U.

The three spaces Z_1, Z_2, and Z_3 depicted in Figure 1 are probably the simplest examples of Priestley spaces that are not Esakia spaces. Topologically each of the three spaces is homeomorphic to the one-point compactification of the countable discrete space $\{y\} \cup \{z_n \mid n \in \omega\}$, with x being the limit point of $\{z_n \mid n \in \omega\}$. For each of the three spaces, it is straightforward to check that with the partial order whose Hasse diagram is depicted in Figure 1, the space is a Pristley space. On the other hand, neither of the three spaces is an Esakia space because $\{y\}$ is clopen, but ${\downarrow}y = \{x, y\}$ is no longer open.

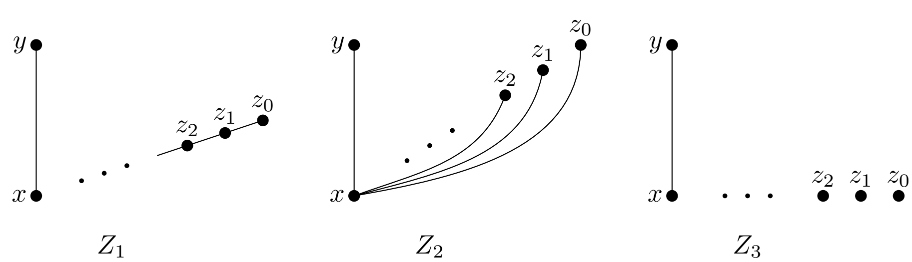

Figure 1: The three Priestley spaces Z_1, Z_2, and Z_3.

We show that a metrizable Priestley space is not an Esakia space exactly when one of these three spaces can be embedded in it. The embeddings we consider are special in that the point y plays a special role. We show that this condition on the embeddings, as well as the metrizability condition, cannot be dropped by presenting some counterexamples. An advantage of our characterization lies in the fact that when a metrizable Priestley space X is presented by a Hasse diagram, it is easy to verify whether or not X contains one of the three "forbidden configurations".

*An expanded version of this abstract, containing the proofs of all reported results, has been submitted for publication.

Definition 1. Let X be a Priestley space. We say that Z_i $(i = 1, 2, 3)$ is a *forbidden configuration* for X if there are a topological and order embedding $e : Z_i \to X$ and an open neighborhood U of $e(y)$ such that $e^{-1}(\downarrow U) = \{x, y\}$.

Theorem 2 (Main Theorem)**.** *A metrizable Priestley space X is not an Esakia space iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X.*

To give the dual statement of Theorem 2, let L_1, L_2, and L_3 be the dual lattices of Z_1, Z_2, and Z_3, respectively. They can be depicted as shown in Figure 2.

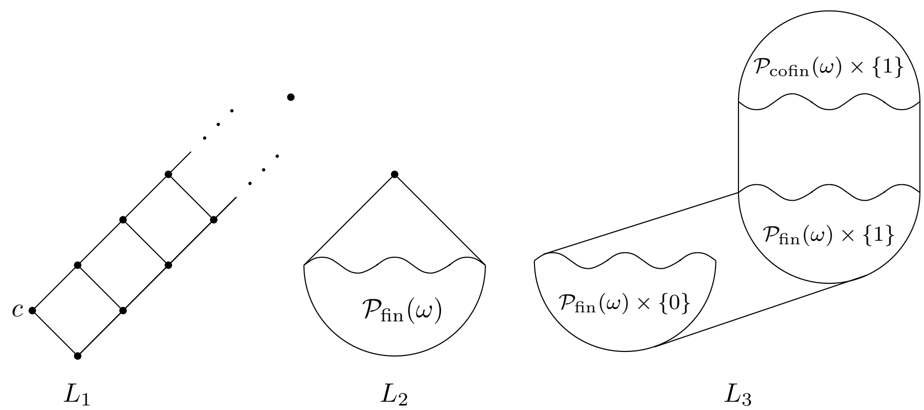

Figure 2: The lattices L_1, L_2 and L_3.

We have that L_2 is isomorphic to the lattice of finite subsets of ω together with a top element, and L_3 is isomorphic to the sublattice of $\mathbf{CF}(\omega) \times \mathbf{2}$ given by the elements of the form (A, n) where A is finite or $n = 1$. Here $\mathbf{CF}(\omega)$ is the Boolean algebra of finite and cofinite subsets of ω and $\mathbf{2}$ is the two-element Boolean algebra.

Neither of L_1, L_2, L_3 is a Heyting algebra: L_1 is not a Heyting algebra because $\neg c$ does not exist; L_2 is not a Heyting algebra because $\neg F$ does not exist for any finite subset F of ω; and L_3 is not a Heyting algebra because $\neg(F, 1)$ does not exist for any finite F.

Definition 3. Let $L \in \mathsf{Dist}$ and let $a, b \in L$. Define

$$I_{a\to b} := \{c \in L \mid c \wedge a \leq b\}$$

It is easy to check that $I_{a\to b}$ is an ideal, and that $I_{a\to b}$ is principal iff $a \to b$ exists in L, in which case $I_{a\to b} = \downarrow(a \to b)$.

Observe that if L is a bounded distributive lattice and X is the Priestley space of L, then X is metrizable iff L is countable. Thus, the following dual statement of Theorem 2 yields a characterization of countable Heyting algebras.

Theorem 4. *Let L be a countable bounded distributive lattice. Then L is not a Heyting algebra iff one of L_i $(i = 1, 2, 3)$ is a homomorphic image of L such that the homomorphism $h_i : L \to L_i$ satisfies the following property: There are $a, b \in L$ such that $h_i[I_{a\to b}] = I_{c_i\to 0}$, where $c_1 = c$, $c_2 = \{0\}$, or $c_3 = (\varnothing, 1)$.*

This characterization easily generalizes to countable p-algebras (=pseudocomplemented distributive lattices). Priestley duality for p-algebras was developed in [5]. We call a Priestley space X a *p-space* provided the downset of each clopen upset is clopen. Then a bounded distributive lattice L is a p-algebra iff its dual Priestley space X is a p-space.

Definition 5. Let X be a Priestley space. We say that Z_i $(i = 1, 2, 3)$ is a *p-configuration* for X if Z_i is a forbidden configuration for X and in addition the open neighborhood U of $e(y)$ is an upset.

We point out that neither of the bounded distributive lattices L_1, L_2, L_3 that are dual to Z_1, Z_2, Z_3 is a p-algebra. The next result is a direct generalization of Theorems 2 and 4:

Corollary 6. *Let L be a countable bounded distributive lattice, and let X be its Priestley space, which is then a metrizable space.*

1. *X is not a p-space iff one of Z_1, Z_2, Z_3 is a p-configuration for X.*
2. *L is not a p-algebra iff one of L_i $(i = 1, 2, 3)$ is a homomorphic image of L such that the homomorphism $h_i : L \to L_i$ satisfies the following property: There is $a \in L$ such that $h_i[I_{a\to 0}] = I_{c_i\to 0}$, where $c_1 = c$, $c_2 = \{0\}$, or $c_3 = (\varnothing, 1)$.*

We recall that *co-Heyting algebras* are order-duals of Heyting algebras. The Priestley spaces dual to co-Heyting algebras are the ones with the property that the upset of each clopen is clopen [2]. Let Z_1^*, Z_2^*, Z_3^* be the Priestley spaces obtained by reversing the order in Z_1, Z_2, Z_3, respectively. Then dualizing Theorem 2 yields:

Corollary 7. *A metrizable Priestley space X is not the dual of a co-Heyting algebra iff there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.*

We recall that *bi-Heyting algebras* are the lattices which are both Heyting algebras and co-Heyting algebras. Priestley spaces dual to bi-Heyting algebras are the ones in which the upset and downset of each clopen is clopen. Putting together the results for Heyting algebras and co-Heyting algebras yields:

Corollary 8. *A metrizable Priestley space X is not dual to a bi-Heyting algebra iff one of Z_1, Z_2, Z_3 is a forbidden configuration for X or there are a topological and order embedding e from one of Z_1^*, Z_2^*, Z_3^* into X and an open neighborhood U of $e(y)$ such that $e^{-1}(\uparrow U) = \{x, y\}$.*

We conclude by two examples showing that Theorem 2 is false without the metrizability assumption, and that in Definition 1 the condition on the open neighborhood U of $e(y)$ cannot be dropped.

Example 9. Let ω_1 be the first uncountable ordinal, and let X be the poset obtained by taking the dual order of $\omega_1 + 1$. Endow X with the interval topology. Consider the space Z given by the disjoint union of a singleton space $\{y\}$ and X with the partial order as depicted in Figure 3. Since $\downarrow\{y\}$ is not clopen, Z is not an Esakia space. On the other hand, there is no sequence in $X \setminus \{\omega_1\}$ converging to ω_1. Thus, Z does not contain the three forbidden configurations.

Example 10. Let X be the disjoint union of two copies of the one-point compactification of the discrete space ω, and let the order on X be defined as in Figure 4. It is straightforward to check that X is a metrizable Esakia space, and yet there is a topological and order embedding of Z_1 into X, described by the white dots in the figure.

Analogous examples can be found for all three forbidden configurations.

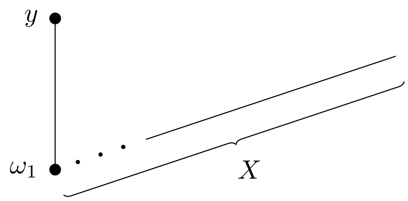

Figure 3: The space Z of Example 9.

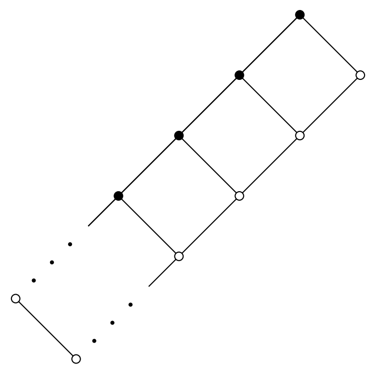

Figure 4: The space X of Example 10. The white dots represent the image of Z_1 under the embedding of Z_1 into X.

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