Trees and Topological Semantics of Modal Logic

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1 Introduction

Topological semantics of modal logic has a long history. It was shown by McKinsey and Tarski [11] that if we interpret $\Box$ as interior and hence $\Diamond$ as closure, then $\mathsf{S4}$ is the modal logic of all topological spaces. Many topological completeness results have been obtained since the inception of topological semantics. We list some relevant results: (1) $\mathsf{S4}$ is the logic of any crowded metric space [11, 13] (this result is often referred to as the *McKinsey-Tarski theorem*); (2) Grz is the logic of any ordinal space $\alpha \geq \omega^\omega$ [1, 8]; (3) Grz_n (for nonzero $n \in \omega$) is the logic of any ordinal space α satisfying $\omega^{n-1} + 1 \leq \alpha \leq \omega^n$ [1] (see also [7, Sec. 6]); (4) $\mathsf{S4.1}$ is the logic of the Pełczyński compactification of the discrete space ω (that is, the compactification of ω whose remainder is homeomorphic to the Cantor space) [6, Cor. 3.19]. If in (2) we restrict to a countable α, then all the above completeness results concern metric spaces. In fact, as was shown in [3], the above logics are the only logics arising from metric spaces.

The McKinsey-Tarski theorem yields that $\mathsf{S4}$ is the logic of the Cantor space. An alternative proof of this result was given in [12] (see also [2]), where the infinite binary tree was utilized. Kremer [10] used the infinite binary tree with limits to prove that $\mathsf{S4}$ is strongly complete for any crowded metric space. Further utility of trees with limits is demonstrated in [4].

Herein we summarize a general technique of topologizing trees which allows us to provide a uniform approach to topological completeness results for zero-dimensional Hausdorff spaces. It also allows us to obtain new topological completeness results with respect to non-metrizable spaces. Embedding these spaces into well-known extremally disconnected spaces (ED-spaces for short) then yields new completeness results for the logics above $\mathsf{S4.2}$ indicated in Figure 1.

It was proved in [5] that $\mathsf{S4.1.2}$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space ω, and this result was utilized in [6] to show that $\mathsf{S4.2}$ is the logic of the Gleason cover of the real unit interval $[0, 1]$. However, these results require a set-theoretic axiom beyond ZFC, and it remains an open problem whether these results are true in ZFC. In contrast, all our results are obtained within ZFC.

We briefly outline some of the techniques employed to obtain the indicated completeness results. A unified way of obtaining a zero-dimensional topology on an infinite tree with limits, say T, is by designating a particular Boolean algebra of subsets of T as a basis. If T has countable branching, then the topology ends up being metrizable. If the branching is 1, then the obtained space is homeomorphic to the ordinal space $\omega + 1$; if the branching is ≥ 2 but

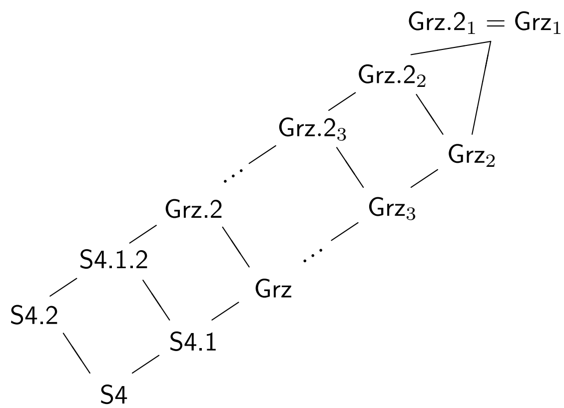

Figure 1: Some well-known extensions of S4.

finite, then it is homeomorphic to the Pełczyński compactification of ω; and if the branching is countably infinite, then there are subspaces homeomorphic to the space of rational numbers, the Baire space, as well as to the ordinal spaces $\omega^n + 1$.

For uncountable branching, it is required to designate a Boolean σ-algebra as a basis for the topology. This leads to topological completeness results for $\mathsf{S4}, \mathsf{S4.1}, \mathsf{Grz}$, and Grz_n with respect to non-metrizable zero-dimensional Hausdorff spaces.

To obtain topological completeness results for logics extending S4.2, we select a dense subspace of either the Čech-Stone compactification βD of a discrete space D with large cardinality or the Gleason cover E of a large enough power of $[0,1]$. This selection is realized by embedding a subspace of an uncountable branching tree with limits into either βD or E. The latter gives rise to S4.2, while the former yields the other logics of interest extending S4.2. We point out that these constructions can be done in ZFC.

2 Topologizing trees and topological completeness results

Let κ be a nonzero cardinal. The *κ-ary tree with limits* is $\mathcal{T}_\kappa = (T_\kappa, \leq)$ where T_κ is the set of all sequences, both finite and infinite, in κ and $\leq$ is the initial segment partial ordering of T_κ. For any $\sigma \in T_\kappa$, let $\uparrow\sigma = \{\varsigma \in T_\kappa \mid \sigma \leq \varsigma\}$. The following table presents some topologies on T_κ; τ is a spectral topology, π is the patch topology of τ, and we introduce the *σ-patch* topology Π of τ.

Topology	Generated by
τ	the set $\mathscr{S} := \{\uparrow\sigma \mid \sigma \in T_\kappa \text{ is a finite sequence}\}$
π	the least Boolean algebra $\mathscr{B}$ containing $\mathscr{S}$
Π	the least Boolean σ-algebra $\mathscr{A}$ containing $\mathscr{S}$

2.1 The patch topology π

Here we are concerned with the space $\mathfrak{T}_\kappa := (T_\kappa, \pi)$ and its subspaces $\mathfrak{T}_\kappa^\infty$, $\mathfrak{T}_\kappa^\omega$, and $\mathfrak{T}_\kappa^n$ ($n \in \omega$) whose underlying sets are $T_\kappa^\infty = \{\sigma \in T_\kappa \mid \sigma \text{ is an infinite sequence}\}$, $T_\kappa^\omega = \{\sigma \in T_\kappa \mid$

σ is a finite sequence$\}$, and $T_\kappa^n = \{\sigma \in T_\kappa \mid \sigma$ is a finite sequence of length $n\}$, respectively. It ends up that $\mathfrak{T}_\kappa$ is metrizable iff κ is countable. The following table for $1 \leq \kappa \leq \omega$ indicates a subspace X of $\mathfrak{T}_\kappa$ and a well known space Y that are homeomorphic.

X	Y
$\mathfrak{T}_1$	the ordinal space $\omega + 1$
$\mathfrak{T}_\kappa^\infty$ $(2 \leq \kappa < \omega)$	the Cantor discontinuum
$\mathfrak{T}_\kappa$ $(2 \leq \kappa < \omega)$	the Pełczyński compactification of the countable discrete space ω
$\mathfrak{T}_\omega^\infty$	the Baire space
$\mathfrak{T}_\omega^\infty$	the space of irrational numbers
$\mathfrak{T}_\omega^\omega$	the space of rational numbers
$\mathfrak{T}_\omega^n$ $(n \in \omega)$	the ordinal space $\omega^n + 1$

Assuming familiarity with S4, for $n \geq 1$, we recall the formulas $\mathsf{bd}_1 := \Diamond\Box p_1 \to p_1$ and $\mathsf{bd}_{n+1} := \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1}$ as well as the logics $\mathsf{S4.1} := \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p$, $\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$, and $\mathsf{Grz}_n := \mathsf{Grz} + \mathsf{bd}_n$. In the following table, the indicated subspace X of $\mathfrak{T}_\kappa$ satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X, giving the logic of X is L. In conjunction with the above table, this yields new proofs for many known topological completeness results.

L	X
S4	$\mathfrak{T}_\omega^\omega$, $\mathfrak{T}_\omega^\infty$, and $\mathfrak{T}_\kappa^\infty$ $(2 \leq \kappa < \omega)$
S4.1	$\mathfrak{T}_\kappa$ $(2 \leq \kappa < \omega)$
Grz	$\bigoplus_{n\in\omega} \mathfrak{T}_\omega^n$
Grz_{n+1}	$\mathfrak{T}_\omega^n$ $(n \in \omega)$

2.2 The σ-patch topology Π

We now fucus on the space $\mathbb{T}_\kappa := (T_\kappa, \Pi)$ and its subspaces $\mathbb{T}_\kappa^\infty$, $\mathbb{T}_\kappa^\omega$, and $\mathbb{T}_\kappa^n$ $(n \in \omega)$ whose underlying sets are T_κ^∞, T_κ^ω, and T_κ^n, respectively. It turns out that $\mathbb{T}_\kappa$ is a P-space; that is, a Tychonoff space such that every G_δ-set is open, and $\mathbb{T}_\kappa$ is discrete iff κ is countable. Thus, we consider only uncountable κ. In the following table, just as we had for the patch topology, the logic of the indicated subspace X of $\mathbb{T}_\kappa$ is L since X satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X. Hence, we obtain completeness for the same logics as in the previous section but for non-metrizable spaces.

L	X
S4	$\mathbb{T}_\kappa^\omega$
S4.1	$\mathbb{T}_\kappa$
Grz	$\bigoplus_{n\in\omega} \mathbb{T}_\kappa^n$
Grz_{n+1}	$\mathbb{T}_\kappa^n$ $(n \in \omega)$

2.3 Moving to the ED setting

Finally, we transfer these results into the setting of ED-spaces. By an unpublished result of van Douwen, see [9], the space $\mathbb{T}_\kappa^\omega$ embeds into the (remainder of the) Čech-Stone compactification

$\beta(2^\kappa)$ of the discrete space 2^κ. Consider $X^\omega_\kappa := \mathbb{T}^\omega_\kappa \cup 2^\kappa$ and $X^n_\kappa := \mathbb{T}^n_\kappa \cup 2^\kappa$ as subspaces of $\beta(2^\kappa)$ where we identify both $\mathbb{T}^\omega_\kappa$ and 2^κ with their image in $\beta(2^\kappa)$. Then X^ω_κ and X^n_κ are ED. Moreover, $\beta(2^\kappa)$, and hence $\mathbb{T}^\omega_\kappa$, can be embedded into a closed nowhere dense subspace F of the Gleason cover E of $[0,1]^{2^{2^\kappa}}$, where $[0,1]$ denotes the real unit interval. Identify $\mathbb{T}^\omega_\kappa$ with its image in E. Then the subspace $X_\kappa := \mathbb{T}^\omega_\kappa \cup (E \setminus F)$ is ED.

We recall that $\mathsf{S4.2}$, $\mathsf{S4.1.2}$, $\mathsf{Grz.2}$, and $\mathsf{Grz.2}_n$ are obtained respectively from $\mathsf{S4}$, $\mathsf{S4.1}$, Grz, and Grz_n by postulating the formula $\Diamond\Box p \to \Box\Diamond p$, which expresses that a space is ED. As previously, in the following table the space X satisfies the properties defined by the logic L and every finite rooted L-frame is an interior image of X, giving the logic of X is L.

L	X
S4.2	X_κ
S4.1.2	X^ω_κ
Grz.2	$\bigoplus_{n\in\omega} X^n_\kappa$
Grz.2_{n+2}	X^n_κ $(n \in \omega)$

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