Epimorphisms in varieties of Heyting algebras

T. Moraschini[1] and J.J. Wannenburg[2]

[1] Czech Academy of Sciences, Prague, Czech Republic
`moraschini@cs.cascz`
[2] University of Pretoria, Pretoria, South Africa
`jamie.wannenburg@up.ac.za`

A morphism $f\colon \mathbf{A} \to \mathbf{B}$ in a category is an *epimorphism* [5, 10, 11] provided it is right-cancellative, i.e. that for every pair of morphisms $g, h\colon \mathbf{B} \to \mathbf{C}$,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

We will focus on epimorphisms in varieties K of algebras, which we regard as categories whose objects are the members of K and whose morphisms are the algebraic homomorphisms. It is immediate that in such categories all surjective morphisms are indeed epimorphisms. However the converse need not be true in general: for instance, the embedding of three-element chain into the four-element diamond happens to be a non-surjective epimorphism in the variety of distributive lattices. Accordingly, a variety K of algebras is said to have the *epimorphism surjectivity property* (ES property for short), if its epimorphisms are surjective.

The failure of the ES property in distributive lattices can be explained in logical terms as the observation that complements are *implicitly*, but not *explicitly*, definable in distributive lattices, in the sense that when complements exist they are uniquely determined, even if there is no unary term witnessing their explicit definition. In general, the algebraic counterpart K of an algebraizable logic $\vdash$ [4] has the ES property if and only if $\vdash$ satisfies the so-called *(infinite deductive) Beth definability property*, i.e. the demand that all implicit definitions in $\vdash$ can be turned explicit [3, 9, 17]. This raises the question of determining which varieties of Heyting algebras have the ES property or, equivalently, which intermediate logics have the Beth definability property.

Classical results by Kreisel and Maksimova, respectively, state that *all* varieties of Heyting algebras have a weak form of the ES property [12], while only *finitely many* of them have a strong version of it [6, 14, 15, 16]. Nonethelss the standard ES property in varieties of Heyting algebras seems to defy simple characterizations, and very little is known about it. One of the few general results about the topic states that the ES property holds for all varieties with bounded depth [2, Thm. 5.3], yielding in particular a continuum of varieties with the ES property. Remarkably, this observation has been recently generalized [18, Thm. 13] beyond the setting of *integral* and *distributive* residuated lattices [7] as follows:

Theorem 1 (M., Raftery and Wannenburg)**.** *Let* K *be a variety of commutative square-increasing (involutive) residuated lattices. If the finitely subdirectly irreducible members of* K *have finite depth and are generated by their negative cones, then* K *has the ES property.*

On the other hand, the first (*ad hoc*) example of a variety of Heyting algebras lacking the ES property was discovered in [2, Cor. 6.2]. We enhance that observation by ruling out the ES property for a range of well-known varieties. To this end, let $\mathbf{RN}$ be the Rieger-Nishimura lattice. Relyng on Esakia duality we establish the following:

Theorem 2.

(i) *For every $2 \leqslant n \in \omega$, the variety of all Heyting algebras of width $\leqslant n$ lacks the ES property.*

(ii) *The variety $\mathbb{V}(\mathbf{RN})$ lacks the ES property, and has a continuum of locally finite subvarieties without the ES property.*

Recall that the Kuznetsov-Gerčiu variety KG is the variety generated by finite sums of cyclic (i.e. one-generated) Heyting algebras [8, 13, 1]. As a case study, we will present a full description of subvarieties of KG with the ES property. This yields an alternative proof of the well-known fact that varieties of Gödel algebras have the ES property.

References

[1] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The Kuznetsov-Gerciu and Rieger-Nishimura logics: the boundaries of the finite model property. *Logic and Logical Philosophy*, 17:73–110, 2008.

[2] G. Bezhanishvili, T. Moraschini, and J. G. Raftery. Epimorphisms in varieties of residuated structures. *Journal of Algebra*, 492:185–211, 2017.

[3] W. J. Blok and E. Hoogland. The Beth property in Algebraic Logic. *Studia Logica*, 83(1–3):49–90, 2006.

[4] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[5] M. Campercholi. Dominions and primitive positive functions. *J. Symbolic Logic*, 83(1):40–54, 2018.

[6] D. M. Gabbay and L. Maksimova. *Interpolation and definability*, volume 46 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, Oxford, 2005. Modal and intuitionistic logics.

[7] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, vol. 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[8] V. Ja. Gerčiu and A. V. Kuznetsov. The finitely axiomatizable superintuitionistic logics. *Soviet Mathematics Doklady*, 11:1654–1658, 1970.

[9] E. Hoogland. *Definability and interpolation: model-theoretic investigations*. Ph. D. Dissertation DS-2001-05, Institute for Logic, Language and Computation, University of Amsterdam, 2001.

[10] J. R. Isbell. *Epimorphisms and dominions*, chapter of Proceedings of the Conference on Categorical Algebra, La Jolla, California, 1965, pages 232–246. Springer, New York, 1966.

[11] E. W. Kiss, L. Márki, P. Pröhle, and W. Tholen. Categorical algebraic properties. a compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. *Studia Sci. Math. Hungar.*, 18:79–141, 1983.

[12] G. Kreisel. Explicit definability in intuitionistic logic. *J. Symbolic Logic*, 25:389–390, 1960.

[13] A. V. Kuznetsov and V. Ja. Gerčiu. Superintuitionistic logics and finite approximability. *Soviet Mathematics Doklady*, 11:1614–1619, 1970.

[14] L. Maksimova. Implicit Definability and Positive Logics. *Algebra and Logic*, 42(1):37–53, 2003.

[15] L. L. Maksimova. Projective Beth properties in modal and superintuitionistic logics. *Algebra i Logika*, 38:316–333, 1999.

[16] L. L. Maksimova. Intuitionistic logic and implicit definability. *Annals of Pure and Applied Logic*, 105(1-3):83–102, 2000.

[17] T. Moraschini, J. G. Raftery, and J. J. Wannenburg. Epimorphisms, definability and cardinalities. *To appear in Studia Logica*, 2018.

[18] T. Moraschini, J. G. Raftery, and J. J. Wannenburg. Epimorphisms in varieties of square-increasing residuated structures. *Manuscript, available online*, 2018.