The One-Variable Fragment of Corsi Logic

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It is well-known that the one-variable fragments of first-order classical logic and intuitionistic logic can be understood as notational variants of the modal logic $\mathsf{S5}$ and the intuitionistic modal logic MIPC, respectively. Similarly, the one-variable fragment of first-order Gödel logic may be viewed as a notational variant of the many-valued Gödel modal logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, axiomatized in [4] as an extension of MIPC with the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ and the constant domains axiom $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$. Further results and general methods for establishing correspondences between one-variable fragments of first-order intermediate logics and intermediate modal logics have been obtained in, e.g., [7, 1].

In this work, we establish such a correspondence for a weaker extension of propositional Gödel logic: the first-order logic of totally ordered intuitionistic Kripke models with increasing domains QLC, axiomatized by Corsi in [5] as an extension of first-order intuitionistic logic with the prelinearity axiom, and often referred to as "Corsi logic". We show that its one-variable fragment QLC_1 corresponds both to the Gödel modal logic $\mathsf{S5}(\mathbf{G})$, axiomatized in [4] as an extension of MIPC with the prelinearity axiom, and also to a one-variable fragment of a "Scott logic" studied in, e.g., [6]. Since $\mathsf{S5}(\mathbf{G})$ enjoys an algebraic finite model property (see [1]), validity in both this logic and QLC_1 are decidable, and indeed — as can be shown using methods from [3] — co-NP-complete.

Let us first recall the Kripke semantics for Corsi logic, restricted for convenience to its one-variable fragment. A QLC_1-*model* is a 4-tuple $\mathcal{M} = \langle W, \preceq, D, I\rangle$ such that

- W is a non-empty set;
- $\preceq$ is a total order on W;
- for all $w \in W$, D_w is a non-empty set called the *domain* of w, and $D_w \subseteq D_v$ whenever $w \preceq v$;
- for all $w \in W$, I_w maps each unary predicate P to some $I_w(P) \subseteq D_w$, and $I_w(P) \subseteq I_v(P)$ whenever $w \preceq v$.

We define inductively for $w \in W$ and $a \in D_w$:

$$\begin{array}{lll}
\mathcal{M}, w \models^a \bot & \Leftrightarrow & \text{never} \\
\mathcal{M}, w \models^a \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, w \models^a P(x) & \Leftrightarrow & a \in I_w(P) \\
\mathcal{M}, w \models^a \varphi \wedge \psi & \Leftrightarrow & \mathcal{M}, w \models^a \varphi \ \text{ and } \ \mathcal{M}, w \models^a \psi \\
\mathcal{M}, w \models^a \varphi \vee \psi & \Leftrightarrow & \mathcal{M}, w \models^a \varphi \ \text{ or } \ \mathcal{M}, w \models^a \psi \\
\mathcal{M}, w \models^a \varphi \to \psi & \Leftrightarrow & \mathcal{M}, v \models^a \varphi \ \text{ implies } \ \mathcal{M}, v \models^a \psi \ \text{ for all } v \succeq w \\
\mathcal{M}, w \models^a (\forall x)\varphi & \Leftrightarrow & \mathcal{M}, v \models^b \varphi \ \text{ for all } v \succeq w \text{ and } b \in D_v \\
\mathcal{M}, w \models^a (\exists x)\varphi & \Leftrightarrow & \mathcal{M}, w \models^b \varphi \ \text{ for some } b \in D_w.
\end{array}$$

We write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models^a \varphi$ for all $w \in W$, and $a \in D_w$. We say that a one-variable first-order formula φ is QLC_1-*valid* if $\mathcal{M} \models \varphi$ for all QLC_1-models $\mathcal{M}$. As mentioned above, it follows from results of Corsi [5] that φ is QLC_1-valid if and only if it is derivable in first-order intuitionistic logic extended with the prelinearity axiom.

The semantics for the modal logic $\mathsf{S5}(\mathbf{G})$ is defined for a set of formulas Fm built as usual over the language of intuitionistic logic extended with $\Box$ and $\Diamond$ and a countably infinite set of variables Var, where $\mathbf{G}$ denotes the standard Gödel algebra $\langle [0,1], \wedge, \vee, \to, 0, 1\rangle$. An $\mathsf{S5}(\mathbf{G})$-*model* $\mathfrak{M} = \langle W, R, V\rangle$ consists of a non-empty set of *worlds* W, a $[0,1]$-*accessibility relation* $R\colon W \times W \to [0,1]$ satisfying for all $u, v, w \in W$,

$$Rww = 1, \quad Rwv = Rvw, \quad \text{and} \quad Ruv \wedge Rvw \leq Ruw,$$

and a *valuation* map $V\colon \mathrm{Var} \times W \to [0,1]$. The valuation map is extended to $V\colon \mathrm{Fm} \times W \to [0,1]$ by $V(\bot, w) = 0$, $V(\top, w) = 1$, $V(\varphi_1 \star \varphi_2, w) = V(\varphi_1, w) \star V(\varphi_2, w)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$V(\Box\varphi, w) = \bigwedge\{Rwv \to V(\varphi, v) \mid v \in W\}$$
$$V(\Diamond\varphi, w) = \bigvee\{Rwv \wedge V(\varphi, v) \mid v \in W\}.$$

We say that $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-*valid* if $V(\varphi, w) = 1$ for all $\mathsf{S5}(\mathbf{G})$-models $\langle W, R, V\rangle$ and $w \in W$.

Let us make the correspondence between one-variable fragments and modal logics explicit, recalling the following standard translations $(-)^*$ and $(-)^\circ$ between the propositional language of $\mathsf{S5}(\mathbf{G})$ and the one-variable first-order language of QLC_1, assuming $\star \in \{\wedge, \vee, \to\}$:

$$\begin{aligned}
\bot^* &= \bot & \bot^\circ &= \bot\\
\top^* &= \top & \top^\circ &= \top\\
(P(x))^* &= p & p^\circ &= P(x)\\
(\varphi \star \psi)^* &= \varphi^* \star \psi^* & (\varphi \star \psi)^\circ &= \varphi^\circ \star \psi^\circ\\
((\forall x)\varphi)^* &= \Box\varphi^* & (\Box\varphi)^\circ &= (\forall x)\varphi^\circ\\
((\exists x)\varphi)^* &= \Diamond\varphi^* & (\Diamond\varphi)^\circ &= (\exists x)\varphi^\circ.
\end{aligned}$$

Note that the composition of $(-)^\circ$ and $(-)^*$ is the identity map. Therefore to show that $\mathsf{S5}(\mathbf{G})$ corresponds to the one-variable fragment of QLC, it suffices to show that $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if φ° is QLC_1-valid. It is easily shown that the translations under $(-)^\circ$ of the axioms and rules of the axiomatization of $\mathsf{S5}(\mathbf{G})$ given in [4] are QLC_1-valid and preserve QLC_1-validity, respectively. Hence if φ is $\mathsf{S5}(\mathbf{G})$-valid, then φ° is QLC_1-valid. To prove the converse, we proceed contrapositively and show that if $\varphi \in \mathrm{Fm}$ fails in some $\mathsf{S5}(\mathbf{G})$-model, then φ° fails in some QLC_1-model.

Let us say that an $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ is *irrational* if $V(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}$ and $w \in W$. We first prove the following useful lemma.

Lemma 1. *For any countable* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M} = \langle W, R, V\rangle$*, there exists an irrational* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M}' = \langle W, R', V'\rangle$ *such that* $V(\varphi, w) < V(\psi, w)$ *if and only if* $V'(\varphi, w) < V'(\psi, w)$ *for all* $\varphi, \psi \in \mathrm{Fm}$ *and* $w \in W$.

Next we consider any irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ and fix $w_0 \in W$. We let $(0,1)_\mathbb{Q}$ denote $(0,1) \cap \mathbb{Q}$ and define a corresponding one-variable Corsi model

$$\mathcal{M}_\circ = \langle (0,1)_\mathbb{Q}, \geq, D, I\rangle$$

such that for all $\alpha \in (0,1)_\mathbb{Q}$,

- $D_\alpha = \{v \in W \mid Rw_0v \geq \alpha\}$;
- $I_\alpha(P) = \{v \in W \mid V(p,v) \geq \alpha\} \cap D_\alpha$ for each unary predicate P.

We are then able to prove the following lemma by induction on the complexity of $\varphi \in \mathrm{Fm}$. The fact that $\mathcal{M}$ is irrational ensures that $V(\varphi,w) \geq \alpha$ if and only if $V(\varphi,w) > \alpha$ for all $\alpha \in (0,1)_{\mathbb{Q}}$, which is particularly important when considering the case for $\varphi = \Diamond\psi$.

Lemma 2. *For any $\varphi \in \mathrm{Fm}$, $\alpha \in (0,1)_{\mathbb{Q}}$, and $w \in D_\alpha$,*

$$\mathcal{M}_\circ, \alpha \models^w \varphi^\circ \iff V(\varphi,w) \geq \alpha.$$

Hence, if $\varphi \in \mathrm{Fm}$ is not $\mathsf{S5}(\mathbf{G})$-valid, there exists, by Lemma 1, an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$ such that $V(\varphi,w) < \alpha < 1$ for some $\alpha \in (0,1)$, and then, by Lemma 2, a QLC_1-model $\mathcal{M}_\circ = \langle (0,1)_{\mathbb{Q}}, \geq, D, I\rangle$ such that $\mathcal{M}_\circ, \alpha \not\models^w \varphi^\circ$. That is, φ° is not QLC_1-valid, and we obtain the following result.

Theorem 1. *A formula $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if φ° is QLC_1-valid.*

We have also established a correspondence between $\mathsf{S5}(\mathbf{G})$ and the one-variable fragment of a "Scott logic" studied in, e.g., [6], that is closely related to the semantics of a many-valued possibilistic logic defined in [2]. Let us call a SL_1-*model* a triple $\mathcal{M} = \langle D, \pi, I\rangle$ such that

- D is a non-empty set;
- $\pi \colon D \to [0,1]$ is a map satisfying $\pi(a) = 1$ for some $a \in D$;
- for each unary predicate P, $I(P)$ is a map assigning to any $a \in D$ some $I_a(P) \in [0,1]$.

The interpretation I_a is extended to formulas by the clauses $I_a(\bot) = 0$, $I_a(\top) = 1$, $I_a(\varphi \star \psi) = I_a(\varphi) \star I_a(\psi)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$I_a((\forall x)\varphi) = \bigwedge\{\pi(b) \to I_b(\varphi) \mid b \in D\}$$
$$I_a((\exists x)\varphi) = \bigvee\{\pi(b) \wedge I_b(\varphi) \mid b \in D\}.$$

We say that a one-variable first-order formula φ is SL_1-*valid* if $I_a(\varphi) = 1$ for all SL_1-models $\langle D, \pi, I\rangle$ and $a \in D$. Using Theorem 1 and a result from [6] relating Scott logics to first-order logics of totally ordered intuitionistic Kripke models, we obtain the following correspondence

Theorem 2. *A formula $\varphi \in \mathrm{Fm}$ is $\mathsf{S5}(\mathbf{G})$-valid if and only if $(\Box\varphi)^\circ$ is SL_1-valid.*

Let us mention finally that $\mathsf{S5}(\mathbf{G})$ enjoys an algebraic finite model property (see [1]), and hence validity in this logic and QLC_1 are decidable. Moreover, using a version of the non-standard semantics developed in [3] to obtain a polynomial bound on the size of the algebras to be checked, we are able to obtain the following sharpened result.

Theorem 3. *The validity problem for $\mathsf{S5}(\mathbf{G})$ is co-NP-complete.*

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