From Tableaux to Axiomatic Proofs
A Case of Relating Logic

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Abstract

The aim of our talk consists of three following elements: the first one is to semantically define relating logics, new kind of non-classical logics which enable to express that constituent propositions are related because of some, for instance causal or temporal, relationship; the second one is to present axiomatic and tableaux systems for the logics we study; the third one is connected with an idea of proof of soundness and completeness theorems based on a transition from a tableaux-like proof to an axiomatic one.

1 Introduction

The idea behind relating logic is simple one. Truth-values of compound propositions depend not only on logical values but also whether their component propositions are related. Consider the following examples:

1. Jan arrived in Amsterdam and took part in conference SYSMICS 2019.
2. If you turned off the light, in the room was pretty dark.

In these propositions we want to express more than extensional relations. Propostion 1 might be false even if both component propositions are true, because it is also required that Jan first arrived in Amsterdam and then took part in conference SYSMICS 2019. Similarly in case of proposition 2. This time we require that the fact of turning off the light to be a cause of dark in a room. It is easy to imagine further examples concerning analytic relationship (cf. [2, 115–120]) or content relationship (relevance) in a general sense (cf. [1], [2, 61–72], [5], [6]). Such examples lead to an idea of relating connectives which enable to express standard extensional dependences and non-extensional ones, some kind of intensional relations.

Relating logics are based on an interpretation of propositions which involves two factors: logical value and relation between propositions. In the talk we define them semantically and introduce syntactic approach by means of axiomatic and tableaux systems.

2 Language and Semantics of Relating Logic

Language $\mathcal{L}$ of relating logic consists of propositional variables, Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, relating connectives which are counterparts of Boolean two argument connectives $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$ and brackets.[1] A set of formulas in $\mathcal{L}$ is defined in the standard way:

$$\mathrm{For} \ni A ::= p_n \mid \neg A \mid (A \star A) \mid (A \star^{\mathrm{w}} A),$$

where $n \in \mathbb{N}$ and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

[1] We use letter w for notation of relating connectives because of Polish words wiązać, wiążący, which might be translated as relate, relating.

A model based on relating relation is an ordered pair $\langle v, \mathrm{R}\rangle$ such that $v\colon \mathrm{Prop} \longrightarrow \{0,1\}$ is a valuation of propositional variables and $\mathrm{R} \subseteq \mathrm{For} \times \mathrm{For}$ is a relating relation. Relation R as an element of semantic structure enables to express in metalanguage, in a quite natural way, that some propositions are somehow related. Such relation is supposed to simulate or model various kinds of relationships like temporal or causal one.

An interpretation of propositional variables and formulas build by means of Boolean connectives are defined in the standard way. According to intuitions presented in an introduction we assume the following interpretation in case of formulas build by relating connectives:

$$\langle v, \mathrm{R}\rangle \models A \star^{\mathrm{w}} B \text{ iff } [\langle v, \mathrm{R}\rangle \models A \star B \text{ and } \mathrm{R}(A, B)].$$

We might define notion of a truth with respect to a relating relation R, i.e $\mathrm{R} \models A$ iff $\forall_{v \in \{0,1\}^{\mathrm{Prop}}} \langle v, \mathrm{R}\rangle \models A$. A notion of tautology might be defined in two ways: as a true formula in all models and as a true formula with respect to all relating relations.

A logic is identified with a set of tautologies determined on the ground of some non-empty class of relating relations. The smallest relating logic is logic W (cf. [5]). Its extensions are defined by means of various classes of relations which are determined by some relational conditions. We distinguish three types of classes of relating relations:

- a horizontal class (for short: h-class) — a class of relations satisfying conditions in which we do not refer to complexity of formulas;
- a vertical class (for short: v-class) — a class of relations satisfying some conditions in which we refer to complexity of formulas
- a diagonal class (for short: d-class) — a class which is horizontal and vertical at the same time.

Examples of h-classes are: class of reflexive relations, class of symmetric relations or class of transitive relations (cf. [1], [2, 61–143], [5], [6]).

An example of v-class is a class of relations of eliminations and introductions of binary connectives (cf. [1], [2, 61–143], [5], [6]), i.e. the class of relations determined by the following conditions:

$$\begin{aligned}
&(\circ_1 \Rightarrow \mathrm{or})\ \ \mathrm{R}(A \circ B, C) \Longrightarrow [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)]\\
&(\mathrm{or} \Rightarrow \circ_1)\ \ [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)] \Longrightarrow \mathrm{R}(A \circ B, C)\\
&(\circ_2 \Rightarrow \mathrm{or})\ \ \mathrm{R}(A, B \circ C) \Longrightarrow [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)]\\
&(\mathrm{or} \Rightarrow \circ_2)\ \ [\mathrm{R}(A, C) \text{ or } \mathrm{R}(B, C)] \Longrightarrow \mathrm{R}(A, B \circ C).
\end{aligned}$$

Examples of d-classes are intersections of h-classes and v-classes, for instance a class of reflexive relations which are relations of eliminations and introductions of connectives (cf. [1], [2, 61–143], [5], [6]).

Because of three types of classes of relations we distinguish three types of extensions of W, i.e. h-logics (determined by h-classes), v-logics (determined by v-classes) and d-logics (determined by d-classes). In the talk we are going to focus on logics determined by the distinguished classes.

3 Axiomatic and Tableaux Systems of Relating Logics

In many cases relating logics are not difficult to axiomatize. The most important thing we need to know is that relating relation is expressible in language $\mathcal{L}$ by formula $(A \vee^{\mathrm{w}} B) \vee (A \rightarrow^{\mathrm{w}} B)$.

Let $A \looparrowright B := (A \vee^{\mathrm{w}} B) \vee (A \to^{\mathrm{w}} B)$. Axiomatic system of W consists of the following axiom schemata:

$$(A \star^{\mathrm{w}} B) \leftrightarrow ((A \star B) \wedge (A \looparrowright B)). \qquad (\mathrm{ax}_{\star^{\mathrm{w}}})$$

The only rules of inference is modus ponens:

$$\frac{A, A \to B}{B}$$

By means of $\looparrowright$ it is quite easy to present axiom schemata which express relational conditions that characterize a class of relation. For instance, we have that:

$A \looparrowright A$	expresses condition of reflexivity
$(A \looparrowright B) \to (B \looparrowright A)$	expresses condition of symmetry
$((A \looparrowright B) \wedge (B \looparrowright C)) \to (A \looparrowright C)$	expresses condition of reflexivity.

In turn, tableaux systems might be defined by some methods introduced in [3], [4] and [5]. A set of tableaux expressions is an union of For and the following sets $\{A\mathbf{r}B : A, B \in \mathrm{For}\}$, $\{A\bar{\mathbf{r}}B : A, B \in \mathrm{For}\}$. By expressions $A\mathbf{r}B$ and $A\bar{\mathbf{r}}B$ we say that formulas A, B are related and not related respectively. A tableaux inconsistency is either A and $\neg A$ or $A\mathbf{r}B$ and $A\bar{\mathbf{r}}B$.

In order to define a sound and complete tableaux system for a relating logic we use standard rules for formulas build by Boolean connectives (cf. [3], [4] and [5]). Then it is easy to determine tableaux rules concerning relating connectives:

$$(\mathrm{R}_{\neg\star^{\mathrm{w}}})\ \frac{\neg(A \star^{\mathrm{w}} B)}{\neg(A \star B) \mid A\bar{\mathbf{r}}B} \qquad (\mathrm{R}_{\star^{\mathrm{w}}})\ \frac{A \star^{\mathrm{w}} B}{\begin{array}{c} A \star B \\ A\mathbf{r}B \end{array}}$$

Specific rules which enable to express relational conditions are also not difficult to express. For instance, for reflexivity, transitivity and condition $(\mathrm{or} \Rightarrow \circ_1)$ we have:

$$(\mathrm{R_r})\ \frac{A}{A\mathbf{r}A} \qquad (\mathrm{R_t})\ \frac{A\mathbf{r}B, B\mathbf{r}C}{A\mathbf{r}C} \qquad (\mathrm{R}_{1:\mathrm{or}\Rightarrow\circ_1})\ \frac{A\mathbf{r}C}{(A \circ B)\mathbf{r}C} \qquad (\mathrm{R}_{2:\mathrm{or}\Rightarrow\circ_1})\ \frac{B\mathbf{r}C}{(A \circ B)\mathbf{r}C}$$

where B in rule $(\mathrm{R}_{1:\mathrm{or}\Rightarrow\circ_1})$ and A in rule $(\mathrm{R}_{2:\mathrm{or}\Rightarrow\circ_1})$ already appeared on a branch.

In the talk, we will focus on a problem of passing from tableaux-like proof to axiomatic proof. The main theorem we would like to present will say that if formula A is a branch consequence of set Σ (i.e A has proof from Σ in a tableaux system), then it is also an axiomatic consequence of Σ (i.e. A has proof from Σ in an axiomatic system).

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References

[1] Richard L. Epstein. Relatedness and implication. *Philosophical Studies*, 36:137–173, 1979.

[2] Richard L. Epstein. *The Semantic Foundations of Logic. Volume 1: Propositional Logics*. Springer Science+Business Media, Dordrecht, 1990.

[3] Tomasz Jarmużek. Tableau metatheorem for modal logics. In *Recent Trends in Philosophical Logic*. R. Ciuni, H. Wansing and C. Willkommen (eds.), Trends in Logic 41, Springer, 103–126, 2014.

[4] Tomasz Jarmużek. Construction of tableaux for classical logic: Tableaux as combinations of branches, branches as chains of sets. *Logic and logical Philosophy*, 1 (16):85–101, 2007.

[5] Tomasz Jarmużek and Bartosz Kaczkowski. On some logic with a relation imposed on formulae: Tableau system F. *Bulletin of the Section of Logic*, 43 (1/2):53–72, 2014.

[6] Mateusz Klonowski. A Post-style proof of completeness theorem for Symmetric Relatedness Logic S. *Bulletin of the Section of Logic*, forthcoming.