Representation theorems for Grzegorczyk contact algebras

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1 Introduction

In [5] Andrzej Grzegorczyk presented one of the first systems of point-free topology developed in the setting of the so-called *connection structures*. Even today, after a lot has been done and achieved in the area, Grzegorczyk's construction keeps being interesting, especially due to his definition of *point*, which embodies the geometrical intuition of point as diminishing system of regions of space (see Section 3). This talk is devoted to the following standard problems: what kind of topological spaces can be obtained from Grzegorczyk contact algebras and *vice versa*, which topological spaces give rise to Grzegorczyk algebras? The topic of the presentation is located well within the scope of the tradition of Boolean Contact Algebras (see e.g. [1, 6]).

2 Basic concepts

Consider a triple $\mathfrak{B} = \langle R, \leqslant, \mathsf{C}\rangle$, where $\langle R, \leqslant\rangle$ is a boolean lattice and $\mathsf{C} \subseteq R \times R$ satisfies:

$$\neg(\mathit{0} \mathrel{\mathsf{C}} x), \tag{C0}$$

$$x \leqslant y \Longrightarrow x \mathrel{\mathsf{C}} y, \tag{C1}$$

$$x \mathrel{\mathsf{C}} y \Longrightarrow y \mathrel{\mathsf{C}} x, \tag{C2}$$

$$x \leqslant y \Longrightarrow \forall_{z\in R}(z \mathrel{\mathsf{C}} x \Longrightarrow z \mathrel{\mathsf{C}} y). \tag{C3}$$

Elements of R are called regions and C is a *contact* (*connection*) relation. In $\mathfrak{B}$ we define *non-tangential* inclusion relation:

$$x \ll y \Longrightarrow \neg(x \mathrel{\mathsf{C}} -y)\,,$$

where $-y$ is the boolean complement of y (while $\neg$ is the standard negation operator).

We define $x \bigcirc y$ to mean that $x \cdot y \neq \mathit{0}$ (with $\cdot$ being the standard meet operation), and take $\perp \subseteq R \times R$ to be the set-theoretical complement of $\bigcirc$.

3 Grzegorczyk contact algebras

A *pre-point* of $\mathfrak{B}$ is a non-empty set X of regions such that:

$$0 \notin X\,, \tag{r0}$$

$$\forall_{u,v\in X}(u = v \vee u \ll v \vee v \ll u)\,, \tag{r1}$$

$$\forall_{u\in X}\exists_{v\in X}\ v \ll u\,, \tag{r2}$$

$$\forall_{x,y\in R}\big(\forall_{u\in X}(u \bigcirc x \wedge u \bigcirc y) \Longrightarrow x \mathrel{\mathsf{C}} y\big)\,. \tag{r3}$$

The purpose of this definition is to formally grasp the intuition of point as the system of diminishing regions determining the unique location in space (see the figures for geometrical intuitions on the Cartesian plane).

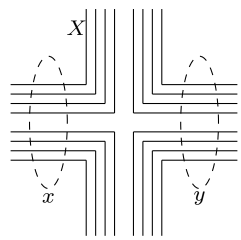

Figure 1: The set X of cross-like regions is not a pre-point, since regions x and y overlap all regions in X but are not in contact.

Let $\mathbf{Q}$ be the set of all pre-points of $\mathfrak{B}$. We extend the set of axioms for $\mathfrak{B}$ with the following postulate:

$$\forall_{x,y\in R}\Big(x \mathrel{\mathsf{C}} y \Longrightarrow \exists_{Q\in\mathbf{Q}_{\mathfrak{R}}}\big((x \perp y\ \vee\ \exists_{z\in Q}\ z \leqslant x \sqcap y)\ \wedge\ \forall_{z\in Q}(z \bigcirc x\ \wedge\ z \bigcirc y)\big)\Big), \tag{G}$$

called *Grzegorczyk axiom*, introduced in [5]. Any $\mathfrak{B}$ which satisfies all the aforementioned axioms is called *Grzegorczyk Contact Algebra* (GCA in short). Every such algebra is a Boolean Contact Algebra in the sense of [1].

A *point* of GCA is any filter generated by a pre-point:

$$\mathfrak{p} \text{ is a point iff } \exists_{Q\in\mathbf{Q}}\, \mathfrak{p} = \{x \in R \mid \exists_{q\in Q}\, q \leqslant x\}\,.$$

In every GCA we can introduce a topology in the set of all points, first by defining the set of all internal points of a region x:

$$\mathbf{Irl}(x) \coloneqq \{\mathfrak{p} \mid x \in \mathfrak{p}\}\,,$$

and second, taking all $\mathbf{Irl}(x)$ as a basis. The natural questions arise: what kind of topological spaces are determined by GCAs and what is the relation between GCAs and the boolean algebras of regular open sets of their topological spaces?

4 Representation theorems and topological duality

Let $\mathcal{T} = \langle S, \mathscr{O}\rangle$ be a topological space. The standard method for obtaining BCAs is via taking RO(X)—the complete algebra of all regular-open subsets of S— as regions, defining the connection relation C by:

$$U \mathrel{\mathsf{C}} V \Longrightarrow \mathrm{Cl}\, U \cap \mathrm{Cl}\, V \neq \emptyset\,.$$

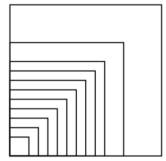

Figure 2: The set of rectangular regions is not a pre-point since the regions are not ordered by non-tangential inclusion.

This practice can be extended to GCAs as well, however one must narrow down the class of spaces, since not all topological spaces give rise to GCAs in the way just described.

In our talk we would like to focus on representation theorems for Grzegorczyk algebras. In particular we would like to show that there is a very strong kinship between GCAs and topological regular spaces. To this end we introduce the class of *concentric* spaces (a subclass of regular spaces), which are T_1-spaces such that in every point p there is a local basis B_p satisfying:

$$\forall_{U,V\in B_p}\big(U = V \vee \mathrm{Cl}\,U \subseteq V \vee \mathrm{Cl}\,V \subseteq U\big)\,. \tag{R1}$$

By an ω-*concentric* space we mean a space which at every point have a local countable basis which satisfies (R1). We prove, among others, the following theorems:

Theorem 1. *Every GCA is isomorphic to a dense subalgebra of a GCA for a concentric topological space.*

Every complete GCA is isomorphic to a GCA for a concentric topological space.

Theorem 2. *Every GCA with c.c.c.* (*countable chain condition*) *is isomorphic to a dense subalgebra of a GCA with c.c.c. for an ω-concentric topological space with c.c.c.*

Every complete GCA with c.c.c. is isomorphic to a GCA with c.c.c. for an ω-concentric topological space having c.c.c.

Theorem 3. *Every countable GCA is isomorphic to a dense subalgebra of a GCA for a second-countable regular space.*

Every complete countable GCA is isomorphic to a GCA for a second-countable regular space.

We also demonstrate the following topological duality theorem for a subclass of Grzegorczyk contact algebras:

Theorem 4 (Object duality theorem)**.** *Every complete GCA with c.c.c. is isomorphic to a GCA for a concentric space with c.c.c.; and every concentric c.c.c. space is homeomorphic to a concentric c.c.c. space for some complete GCA with c.c.c.*

The following two problems remain open:

1. full duality for algebras and topological spaces from Theorem 4,
2. duality for the full class of Grzegorczyk algebras (without c.c.c.).

5 Grzegorczyk and de Vries

Last, but not least, we would like to compare GCAs to de Vries constructions from [2]. Among others, we show that every Grzegorczyk point is a maximal concordant filter in the sense of [2] (*maximal round filter* or *end* in contemporary terminology), but not vice versa.

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