Infinitary connectives and Borel functions in Łukasiewicz logic

Antonio Di Nola[1], Serafina Lapenta[1], and Ioana Leuştean[2]

[1] Department of Mathematics, University of Salerno Fisciano, Salerno, Italy
adinola@unisa.it, slapenta@unisa.it
[2] Department of Computer Science, Faculty of Science University of Bucharest, Bucharest, Romania
ioana.leustean@unibuc.ro

Riesz Spaces are lattice-ordered linear spaces over the field of real numbers $\mathbb{R}$ [3]. They have had a predominant rôle in the development of functional analysis over ordered structures, due to the simple remark that most of the spaces of functions one can think of are indeed Riesz Spaces. Such spaces are also related to expansions of Łukasiewicz infinite-valued logic.

In particular, one can consider MV-algebras – the variety of algebras that model Łukasiewicz logic – and endow them with a scalar multiplication, where scalars are elements of the standard MV-algebra $[0, 1]$. Such MV-algebras with scalar multiplication form a variety and they are known in literature with the name of Riesz MV-algebras [1]. Moreover, Riesz MV-algebras are categorical equivalent with Riesz Spaces with a strong unit.

In this talk will exploit the connection between Riesz Spaces and MV-algebras as a bridge between algebras of Borel-measurable functions and Łukasiewicz logic. To do so we will define the infinitary logical systems $\mathcal{IRL}$, whose models are algebras of $[0, 1]$-valued continuous functions defined over some basically-disconnected compact Hausdorff space X. We will further discuss completeness of $\mathcal{IRL}$ with respect to σ-complete Riesz MV-algebras and characterize the Lindenbaum-Tarski algebra of it by means of Borel-measurable functions.

The logical system $\mathcal{IRL}$ is obtained in [2] starting from an expansion of Łukasiewicz logic introduced in [1], namely $\mathcal{RL}$, and by adding an infinitary operator that models a countable disjunction. In more details, we can consider a countable set of propositional variables and the connectives $\neg$, $\rightarrow$, $\{\nabla_\alpha\}_{\alpha\in[0,1]}$, $\bigvee$. The connectives $\neg$, $\rightarrow$, $\{\nabla_\alpha\}_{\alpha\in[0,1]}$ are inherited from the logic $\mathcal{RL}$, while the latter is a connective of arity less or equal to ω, i.e. it is defined for any set of formulas which is at most countable. Consider now the following set of axioms:

(L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(L3) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$
(L4) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
(R1) $\nabla_r(\varphi \rightarrow \psi) \leftrightarrow (\nabla_r\varphi \rightarrow \nabla_r\psi)$
(R2) $\nabla_{(r\odot q^*)}\varphi \leftrightarrow (\nabla_q\varphi \rightarrow \nabla_r\varphi)$
(R3) $\nabla_r(\nabla_q\varphi) \leftrightarrow \nabla_{r\cdot q}\varphi$
(R4) $\nabla_1\varphi \leftrightarrow \varphi$
(S1) $\varphi_k \rightarrow \bigvee_{n\in\mathbb{N}}\varphi_n$, for any $k \in \mathbb{N}$.

The logic $\mathcal{IRL}$ is obtained from these axioms, Modus Ponens and the following deduction rule.

$$\text{(SUP)} \quad \frac{(\varphi_1 \rightarrow \psi), \ldots, (\varphi_k \rightarrow \psi) \ldots}{\bigvee_{n\in\mathbb{N}}\varphi_n \rightarrow \psi}$$

Axioms (L1)-(L4) are the axioms of Łukasiewicz logic, axioms (R1)-(R4) make the connectives $\{\nabla_r\}_{r\in[0,1]}$ into the “de Morgan dual” of a scalar multiplication (that is, $\neg\nabla_r\neg$ behaves like a scalar multiplication in the sense of Riesz MV-algebras), axioms (S1) and (SUP) ensure that the new connective behaves as a least upper bound for a given sequence.

The new system has σ-complete Riesz MV-algebras as models and the following results hold.

Theorem 1. *[2] (1) IRL, the Lindenbaum-Tarski algebra of $\mathcal{IRL}$, is a σ-complete Riesz MV-algebra and it is the smallest σ-complete algebra that contains the Lindenbaum-Tarski algebra of $\mathcal{RL}$.*

(2) $\mathcal{IRL}$ is complete with respect to all algebras in $\mathbf{RMV}_\sigma$, the class of σ-complete Riesz MV-algebras;

A functional description of the Linbenbaum-Tarski algebra IRL is possible by recalling that any σ-complete Riesz MV-algebra is semisimple. Kakutani's duality, a result by Nakano and the duality between Riesz MV-algebras and vector lattices entail the following theorem.

Theorem 2. *Let A be a Dedekind σ-complete Riesz MV-algebra. There exists a basically disconnected compact Hausdorff space (i.e. it has a base of open F_σ sets) X such that $A \simeq C(X)$, where $C(X)$ is the algebra of $[0,1]$-valued and continuous functions defined over X.*

In particular, $IRL \simeq C(X)$ for some basically disconnected compact Hausdorff space X.

The above theorem, as strong as it is, does not allow for a more concrete description of the algebra IRL in the spirit of functional representation that holds for Łukasiewicz logic. Indeed, the Linbenbaum-Tarski algebras of Łukasiewicz logic and of the logic $\mathcal{RL}$ have both a clear-cut description: they are the algebras of all piecewise linear functions (in the first case, with integer coefficient) over some unit cube $[0,1]^\mu$. Having this in mind, to obtain a description of IRL as an appropriate subalgebra of $[0,1]^{[0,1]^\mu}$, we need to develop the algebraic theory of σ-complete Riesz MV-algebras.

To this end, one can consider the work of Słomiński on infinitary algebras. It turns out that σ-complete Riesz MV-algebras are a proper class of infinitary algebras in the sense of [4] and we can prove the following results.

Theorem 3. *[2] The following hold.*

(1) The ω-generated free algebra in the class of σ-complete Riesz MV-algebras exists and it is isomorphic with IRL.

(2) Consider the algebra of term functions of $\mathbf{RVM}_\sigma$ in n variables, denoted by $\mathcal{RT}_n$. If we consider the elements of $\mathcal{RT}_n$ as functions from $[0,1]^n$ to $[0,1]$, $\mathcal{RT}_n$ becomes a Riesz MV-algebras closed to pointwise defined countable suprema.

(3) The algebra $\mathcal{RT}_n$ is isomorphic with the Lindenbaum-Tarski algebra of $\mathcal{IRL}$ build upon n-propositional variables.

Finally, using the previous results, we can prove the following.

Theorem 4. *$\mathcal{RT}_n$ is isomorphic with the algebra of $[0,1]$-valued Borel-measurable functions defined over $[0,1]^n$. Whence, IRL is isomorphic to the algebra of all Borel measurable functions from $[0,1]^n$ to $[0,1]$.*

Moreover, the Loomis-Sikorski theorem holds for Riesz MV-algebras.

Theorem 5. *[2] Let $A \subseteq C(X)$ be a σ-complete Riesz MV-algebra, where $X = Max(A)$, and let $\mathcal{T} \subseteq [0,1]^X$ be the set of functions f that are essentially equal to some function of A. Then $\mathcal{T}$ is a Riesz tribe, each $f \in \mathcal{T}$ is essentially equal to a unique $f^* \in A$ and the map $f \mapsto f^*$ is a σ-homomorphism of $\mathcal{T}$ onto A.*

As a consequence, $\mathbf{RMV}_\sigma$ is an infinitary variety and the logic $\mathcal{IRL}$ is standard complete.

Theorem 6. *[2]* $\mathbf{RMV}_\sigma$ *is the infinitary variety generated by* $[0,1]$.

For the reader convenience, we summarize the main results of this abstract.

1. we introduce an infinitary logic starting from the logic of Riesz MV-algebras,
2. we discuss its models looking at them as infinitary algebras and via Kakutani's duality with compact Hausdorff spaces,
3. we use notions from infinitary universal algebra to obtain a characterization of the Linbenbaum-Tarski algebra of the infinitary logic as algebra of Borel-measurable functions,
4. via the Loomis-Sikorski theorem for Riesz MV-algebras, we prove that σ-complete Riesz MV-algebras are the infinitary variety generated by $[0,1]$ and we infer the standard completeness of $\mathcal{IRL}$.

References

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