# Preservation theorems in graded model theory*

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Graded model theory is the generalized study, in mathematical fuzzy logic (MFL), of the construction and classification of graded structures. The field was properly started in [8] and it has received renewed attention in recent years [1, 3–7]. Part of the programme of graded model theory is to find non-classical analogues of results from classical model theory (e.g., [2, 9, 10]). This will not only provide generalizations of classical theorems but will also provide insight into what avenues of research are particular to classical first-order logic and do not make sense in a broader setting.

On the other hand, classical model theory was developed together with the analysis of some very relevant mathematical structures. In consequence, its principal results provided a logical interpretation of such structures. Thus, if we want the model theory's idiosyncratic interaction with other disciplines to be preserved, the redefinition of the fundamental notions of graded model theory cannot be obtained from directly fuzzifying every classical concept. Quite the contrary, the experience acquired in the study of different structures, the results obtained using specific classes of structures, and the potential overlaps with other areas should determine the light the main concepts of graded model theory have to be defined in. It is in this way that several fundamental concepts of the model theory of mathematical fuzzy logic have already appeared in the literature.

The goal of this talk is to give syntactic characterizations of classes of graded structures; more precisely, we want to study which kind of formulas can be used to axiomatize certain classes of structures based on finite MTL-chains. Traditional examples of such sort of results are preservation theorems in classical model theory, which, in general, can be obtained as consequences of certain amalgamation properties (cf. [9]). We provide some amalgamation results using the technique of diagrams which will allow us to establish analogues of the Łoś–Tarski preservation theorem [9, Theorem 6.5.4] and the Chang–Łoś–Suszko theorem [9, Theorem 6.5.9].

The formalism of first-order fuzzy logics uses classical syntax with a signature $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ (predicate and functional symbols with their arities) and a many-valued semantics as in Mostowski–Rasiowa–Hájek tradition in which *models* are pairs $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where:

- $\boldsymbol{A}$ is an algebra of truth-values (for the propositional language)

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- $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where
  - $M$ is a set
  - $P_{\mathbf{M}}$ is a function $M^n \to A$, for each $n$-ary predicate symbol $P \in \mathbf{P}$
  - $F_{\mathbf{M}}$ is a function $M^n \to M$ for each $n$-ary function symbol $F \in \mathbf{F}$.
- An $\mathfrak{M}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to M$

$$
\begin{array}{rcl}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} & = & \mathrm{v}(x), \\
\|F(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|\circ(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}, \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}.
\end{array}
$$

In this talk, we will assume that the algebra $\boldsymbol{A}$ of truth values is an MTL-algebra. MTL-algebras are algebraic structures of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$
\begin{array}{lr}
a \,\&^{\boldsymbol{A}}\, b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, & \text{(residuation)} \\
(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} & \text{(prelinearity)}
\end{array}
$$

$\boldsymbol{A}$ is called an MTL-chain if its underlying lattice is linearly ordered.

Let us fix a finite non-trivial MTL-chain $\boldsymbol{A}$. Finiteness ensures that the infima and suprema used in the interpretation of quantifiers always exist. We will consider the expansion of a signature, denoted $\mathcal{P}^{\mathbf{A}}$, in which we add a propositional constant $\overline{a}$ for each element $a$ of $\boldsymbol{A}$. Also, we assume signatures to have crisp equality. We write $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \varphi[e]$ if $\varphi(x)$ has a free variable $x$ and $\|\varphi\|^{\langle \boldsymbol{A}, \mathbf{M} \rangle}_{\mathrm{v}} = \overline{1}^{\boldsymbol{A}}$ for any evaluation v that maps $x$ to $e$.

We will write $\langle \mathbf{A}, \mathbf{M}_2, \overrightarrow{d} \rangle \Rrightarrow_{\exists_n} \langle \mathbf{A}, \mathbf{M}_1, \overrightarrow{d} \rangle$ if for any $\exists_n$ formula $\varphi$, $\langle \mathbf{A}, \mathbf{M}_2 \rangle \models \varphi[\overrightarrow{d}]$ only if $\langle \mathbf{A}, \mathbf{M}_1 \rangle \models \varphi[\overrightarrow{d}]$. Also, we need to speak about embeddability of one model into another; see the usual definitions in e.g. [6].

In classical model theory amalgamation properties are often related in elegant ways to preservation theorems (see e.g. [9]). We will try an analogous approach to obtain our desired preservation result. The importance of this idea is that the problem of proving a preservation result reduces then to finding a suitable amalgamation counterpart. This provides us with proofs that have a neat common structure.

**Proposition 1.** (Existential amalgamation) *Let $\langle \boldsymbol{A}, \mathbf{M}_1 \rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2 \rangle$ be two structures for $\mathcal{P}^{\boldsymbol{A}}$ with a common part $\langle \boldsymbol{A}, \mathbf{M} \rangle$ with domain generated by a sequence of elements $\overrightarrow{d}$. Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d} \rangle \Rrightarrow_{\exists_1} \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d} \rangle.$$

*Then there is a structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ into which $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ can be strongly embedded by $\langle f, g\rangle$ while $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is $\mathcal{P}^{\boldsymbol{A}}$-elementarily strongly embedded (taking isomorphic copies, we may assume that $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is just a $\mathcal{P}^{\boldsymbol{A}}$-elementary substructure). The situation is described by the following picture:*

$$\begin{tikzcd}
& \langle \boldsymbol{A}, \mathbf{N}\rangle & \\
\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle \arrow[ur, dashed, "{\langle f,g\rangle}"] \arrow[rr, phantom, "\Rrightarrow_{\exists_1}"] & & \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle \arrow[ul, dashed, "\preccurlyeq"'] \\
& \langle \boldsymbol{A}, \mathbf{M}\rangle \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

*Moreover, the result is also true when $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ have no common part.*

**Proposition 2.** ($\exists_2$ amalgamation) *Let $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ be two structures for $\mathcal{P}^{\boldsymbol{A}}$ with a common part $\langle \boldsymbol{A}, \mathbf{M}\rangle$ with domain generated by a sequence of elements $\overrightarrow{d}$. Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle \Rrightarrow_{\exists_2} \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle.$$

*Then there is a structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ into which $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ can be strongly embedded by $\langle f, g\rangle$ preserving all $\forall_1$ formulas, while $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is $\mathcal{P}^{\boldsymbol{A}}$-elementarily strongly embedded (taking isomorphic copies, we may assume that $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is just a $\mathcal{P}^{\boldsymbol{A}}$-elementary substructure). The situation is described by the following picture:*

$$\begin{tikzcd}
& \langle \boldsymbol{A}, \mathbf{N}\rangle & \\
\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle \arrow[ur, dashed, "{\langle f,g\rangle}"] \arrow[rr, phantom, "\Rrightarrow_{\exists_2}"] & & \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle \arrow[ul, dashed, "\preccurlyeq"'] \\
& \langle \boldsymbol{A}, \mathbf{M}\rangle \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

*Moreover, the result is also true when $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ have no common part.*

These amalgamation properties let us prove analogues of the classical preservation theorems listed below. For their formulation we use the following notation: given a theory $T$ and two sets of formulas $\Phi$ and $\Psi$, we denote by $T \vdash \Phi \Rightarrow \Psi$ the fact that for every model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ of $T$, if $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a model of $\Phi$, then $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is also a model of $\Psi$.

**Theorem 3.** (Łoś–Tarski preservation theorem) *Let $T$ be a theory and $\Phi(\overrightarrow{x})$ a set of formulas in $\mathcal{P}^{\boldsymbol{A}}$. Then the following are equivalent:*

*(i) For any models of $T$, $\langle \boldsymbol{A}, \mathbf{M}\rangle \subseteq \langle \boldsymbol{A}, \mathbf{N}\rangle$, we have: if $\langle \boldsymbol{A}, \mathbf{N}\rangle \models \Phi$, then $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \Phi$.*

*(ii) There is a set of $\forall_1$-formulas $\Theta(\overrightarrow{x})$ such that: $T \vdash \Phi \Rightarrow \Theta$ and $T \vdash \Theta \Rightarrow \Phi$.*

**Theorem 4.** *Let $\mathbb{K}$ be a class of structures. Then the following are equivalent:*

*(i)* $\mathbb{K}$ *is closed under isomorphisms, substructures, and ultraproducts.*

*(ii)* $\mathbb{K}$ *is axiomatized by a set of universal* $\mathcal{P}^{\boldsymbol{A}}$*-sentences.*

By a $\forall_2^*$-formula we will mean any formula which is either $\forall_2$ or of the form

$$(\exists \overrightarrow{x})(\forall \overrightarrow{y})\varphi(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) \to \overline{a},$$

for $\varphi$ quantifier-free and $a$ the immediate predecessor of $\overline{1}^{\boldsymbol{A}}$.

**Theorem 5.** (Chang–Łoś–Suszko preservation theorem) *Let $T$ be a theory and $\Phi(\overrightarrow{x})$ a set of formulas in $\mathcal{P}^{\boldsymbol{A}}$. Then the following are equivalent:*

*(i)* $\Phi(\overrightarrow{x})$ *is preserved under unions of chains of models of $T$.*

*(ii) There is a set of* $\forall_2^*$*-formulas* $\Theta(\overrightarrow{x})$ *such that:* $T \vdash \Phi \Rightarrow \Theta$ *and* $T \vdash \Theta \Rightarrow \Phi$.

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