Positive Subreducts in Finitely Generated Varieties of MV-algebras

Leonardo M. Cabrer, Peter Jipsen[1], and Tomáš Kroupa[2]

[1] Chapman University, Orange, USA
jipsen@chapman.edu
[2] The Czech Academy of Sciences, Prague, Czech Republic
kroupa@utia.cas.cz

Abstract

Positive MV-algebras are negation-free and implication-free subreducts of MV-algebras. In this contribution we show that a finite axiomatic basis exists for the quasivariety of positive MV-algebras coming from any finitely generated variety of MV-algebras.

1 Positive subreducts of MV-algebras

Let $\mathcal{MV}$ be the variety of MV-algebras [1] in the language containing all the usual definable operations and constants. Using this signature we denote an MV-algebra $\mathbf{M} \in \mathcal{MV}$ as

$$\mathbf{M} = \langle M, \oplus, \odot, \vee, \wedge, \to, \neg, 0, 1\rangle.$$

An algebra

$$\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1\rangle$$

is a *positive subreduct* of $\mathbf{M}$ if $\mathbf{A}$ is a subreduct of $\mathbf{M}$.

Definition 1. Let $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$ be a set of function symbols, where $\oplus, \odot, \vee, \wedge$ are interpreted as binary operations and $0, 1$ as constants. An algebra $\mathbf{P}$ of type F is a *positive MV-algebra* if it is isomorphic to a positive subreduct of some MV-algebra.

Clearly, every MV-algebra gives rise to a positive MV-algebra and every bounded distributive lattice is a positive MV-algebra. In fact, positive MV-algebras are to MV-algebras as distributive lattices are to Boolean algebras.

Example 1 (Lower Chang algebra)**.** Let $\mathbf{C}$ be Chang algebra and

$$\operatorname{Rad}\mathbf{C} = \{0, \varepsilon, 2\varepsilon, \dots\}$$

be its radical, where the symbol ε denotes the least positive infinitesimal. Then the algebra $\mathbf{C}_l$ having the universe $\operatorname{Rad}\mathbf{C} \cup \{1\}$ is a positive subreduct of $\mathbf{C}$.

Example 2 (Non-decreasing McNaughton functions)**.** For each natural number n, the free n-generated MV-algebra is isomorphic to the algebra $\mathbf{F}_n$ of McNaughton functions $[0,1]^n \to [0,1]$. Then the algebra $\mathbf{F}_n^{\leq}$ of nondecreasing McNaughton functions is a positive subreduct of $\mathbf{F}_n$.

The class of all positive MV-algebras is denoted by $\mathcal{P}$. Since $\mathcal{P}$ is a class of algebras containing the trivial algebra and closed under isomorphisms, subalgebras, direct products and ultraproducts, it is a quasivariety. The following example shows that $\mathcal{P}$ is not a variety.

Example 3. Let θ be an equivalence relation on the algebra $\mathbf{C}_l$ from Example 1 with classes $\{0\}$, $\{\varepsilon, 2\varepsilon, \dots\}$, and $\{1\}$. Then θ is a $\mathcal{P}$-congruence on $\mathbf{C}_l$. The quotient $\mathbf{C}_l/\theta$ is isomorphic to the three-element algebra $\{\bar{0}, \bar{\varepsilon}, \bar{1}\}$ that satisfies the identities $\bar{\varepsilon} \oplus \bar{\varepsilon} = \bar{\varepsilon}$ and $\bar{\varepsilon} \odot \bar{\varepsilon} = \bar{0}$. However, the two equations cannot hold simultaneously in any MV-algebra. Hence, $\mathbf{C}_l/\theta$ is not a positive MV-algebra.

It can be shown that the quasivariety $\mathcal{P}$ is generated by the positive reduct of the standard MV-algebra $[0, 1]$. Moreover, the free n-generated positive MV-algebra is isomorphic to the positive subreduct $\mathbf{F}_n^{\leq}$ from Example 2.

2 Axiomatization

We define a class $\mathcal{Q}$ of algebras of type $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$. Specifically, an algebra $\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1\rangle$ belongs to $\mathcal{Q}$ if $\mathbf{A}$ satisfies the following identities and quasi-identities:

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice
2. $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids
3. $x \oplus 1 = 1$ and $x \odot 0 = 0$
4. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$ and $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$
5. $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$ and $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$
6. $x \oplus y = (x \oplus y) \oplus (x \odot y)$
7. $x \oplus y = (x \vee y) \oplus (x \wedge y)$
8. $x \odot y = (x \odot y) \odot (x \oplus y)$
9. If $x \oplus y = x$, then $z \oplus y \leq z \vee x$
10. If $x \oplus y = x \oplus z$ and $x \odot y = x \odot z$, then $y = z$

Every positive MV-algebra is a member of $\mathcal{Q}$ since 1.–10. are valid for any MV-algebra. The main open problem is to prove the opposite, that is, to show that any $\mathbf{A} \in \mathcal{Q}$ is a positive MV-algebra. We solve this problem for those $\mathbf{A} \in \mathcal{Q}$ satisfying additional identities of a special form. Namely let $\mathcal{V}$ be any finitely generated variety of MV-algebras. Di Nola and Lettieri proved in [3] that there exists a finite set S of identities axiomatizing the variety $\mathcal{V}$ within $\mathcal{MV}$, and every identity in S uses only terms of the language $\{\oplus, \odot, \vee, \wedge, 0, 1\}$.

Theorem 1. *The quasivariety of positive subreducts of $\mathcal{V}$ is axiomatized by the quasi-identities 1.–10. and the identities from S.*

The essential ingredient of the proof of Theorem 1 is a certain non-trivial generalization of the technique of good sequences, which was introduced by Mundici [2]. It remains an open problem to extend this result beyond finitely generated varieties of MV-algebras, possibly using an axiomatization different from 1.–10.

References

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