

Free algebras endomorphisms: Ruitenburg's Theorem and Beyond

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Ruitenburg's Theorem says that every endomorphism f of a finitely generated free Heyting algebra is ultimately periodic if f fixes all the generators but one. More precisely, there is $N \geq 0$ such that $f^{N+2} = f^N$, thus the period equals 2. We give a semantic proof of this theorem, using duality techniques and bounded bisimulations ranks. By the same techniques, we tackle investigation of arbitrary endomorphisms between free algebras. We show that they are not, in general, ultimately periodic. Yet, when they are (e.g. in the case of locally finite subvarieties), the period can be explicitly bounded as function of the cardinality of the set of generators.

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