Algebraic Functions

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Let $\mathbf{A}$ be an algebraic structure, and consider the system of equations

$$\alpha(\bar{x}, \bar{z}) := \begin{cases} t_1(x_1, \ldots, x_n, z_1, \ldots, z_m) = s_1(x_1, \ldots, x_n, z_1, \ldots, z_m) \\ \qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ t_k(x_1, \ldots, x_n, z_1, \ldots, z_m) = s_k(x_1, \ldots, x_n, z_1, \ldots, z_m) \end{cases}$$

where t_j and s_j are a terms for $j \in \{1, , \ldots, k\}$. Suppose that for all $\bar{a} \in A^n$ there is exactly one $\bar{a} \in A^m$ such that $\alpha(\bar{a}, \bar{b})$ holds. Then the system $\alpha(\bar{x}, \bar{z})$ defines m functions $f_1, \ldots, f_m : A^n \to A$ by

$$(f_1(\bar{a}), \ldots, f_m(\bar{a})) := \text{unique } \bar{b} \text{ such that } \alpha(\bar{a}, \bar{b}).$$

A function is called *algebraic* on $\mathbf{A}$ if it is one of the functions defined by a system of equations in the manner just described. For example, the complement function is algebraic on the two-element bounded lattice, as witnessed by the system

$$\alpha(x, z) := \begin{cases} x \wedge z = 0 \\ x \vee z = 1. \end{cases}$$

Given an algebraic structure $\mathbf{A}$, it is easy to see that every term-function of $\mathbf{A}$ is algebraic on $\mathbf{A}$, and that algebraic functions on $\mathbf{A}$ are closed under composition; that is, they a form a clone on A. Algebraic functions can be seen as a natural generalization of term-functions, and share some of their basic properties (e.g., they are preserved by endomorphisms and direct products).

Algebraic functions have been characterized for algebras in several well-known classes such as: Boolean Algebras, Distributive Lattices, Vector Spaces and Abelian Groups, among others. In our talk we will review these characterizations and discuss the main tools used to obtain them. We will also show how algebraic functions can be used in the study of epimorphisms and to describe intervals in the lattice of clones over a finite set.