Semi-analytic Rules and Craig Interpolation

Amir Akbar Tabatabai[1] and Raheleh Jalali[2]

[1] Academy of Sciences of the Czech Republic,
Prague, Czech Republic
tabatabai@math.cas.cz

[2] Academy of Sciences of the Czech Republic,
Prague, Czech Republic
jalali@math.cas.cz

Abstract

In [1], Iemhoff introduced the notion of a centered axiom and a centered rule as the building blocks for a certain form of sequent calculus which she calls a centered proof system. She then showed how the existence of a terminating centered system implies the uniform interpolation property for the logic that the calculus captures. In this paper we first generalize her centered rules to semi-analytic rules, a dramatically powerful generalization, and then we will show how the semi-analytic calculi consisting of these rules together with our generalization of her centered axioms, lead to the feasible Craig interpolation property. Using this relationship, we first present a uniform method to prove interpolation for different logics from sub-structural logics $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_{ew}}$ and **IPC** to their appropriate classical and modal extensions, including the intuitionistic and classical linear logics. Then we will use our theorem negatively, first to show that so many sub-structural logics including $Ł_n$, G_n, BL, R and RM^e and almost all super-intutionistic logics (except at most seven of them) do not have a semi-analytic calculus.

Let us begin with some preliminaries. First fix a propositional language extending the language of $\mathbf{FL_e}$. By the meta-language of this language we mean the language with which we define the sequent calculi. It extends our given language with the formula symbols (variables) such as ϕ and ψ. A meta-formula is defined as the following: Atomic formulas and formula symbols are meta-formulas and if $\bar{\phi}$ is a set of meta-formulas, then $C(\bar{\phi})$ is also a meta-formula, where $C \in \mathcal{L}$ is a logical connective of the language. Moreover, we have infinitely many variables for meta-multisets and we use capital Greek letters again for them, whenever it is clear from the context whether it is a multiset or a meta-multiset variable. A meta-multiset is a multiset of meta-formulas and meta-multiset variables. By a meta-sequent we mean a sequent where the antecedent and the succedent are both meta-multisets. We use meta-multiset variable and context, interchangeably.

For a meta-formula ϕ, by $V(\phi)$ we mean the meta-formula variables and atomic constants in ϕ. A meta-formula ϕ is called p-free, for an atomic formula or meta-formula variable p, when $p \notin V(\phi)$.

And finally note that by $\mathbf{FL_e}^-$ we mean the system $\mathbf{FL_e}$ minus the following axioms:

$$\frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

And $\mathbf{CFL_e}^-$ has the same rules as $\mathbf{FL_e}^-$, this time in their full multi-conclusion version, where $+$ is added to the language and also the usual left and right rules for $+$ are added to the system.

Now let us define some specific forms of the sequent-style rules:

Definition 1. A rule is called a *left semi-analytic rule* if it is of the form

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where Π_j, Γ_i and Δ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

and it is called a *right semi-analytic rule* if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

where Γ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,r} V(\bar{\psi}_{ir}) \subseteq V(\phi)$$

For the multi-conclusion case, we define a rule to be *left multi-conclusion semi-analytic* if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

with the same variable condition as above and the same condition that all Γ_i's and Δ_i's are meta-multiset variables. A rule is defined to be a *right multi-conclusion semi-analytic* rule if it is of the form

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

again with the similar variable condition and the same condition that all Γ_i's and Δ_i's are meta-multiset variables.
Moreover, the usual modal rules in the cut-free Gentzen calculus for the logics **K**, **K4**, **KD** and **S4** are considered as *semi-analytic modal rules.*

Definition 2. A sequent is called a *centered axiom* if it has the following form:

(1) Identity axiom: $(\phi \Rightarrow \phi)$

(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$

(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$

(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and the variables in any pair of elements in $\bar{\alpha}$ or in $\bar{\beta}$ or in $\bar{\phi}$ are equal.

The main theorem of the paper is the following:

Theorem 3. *(i) If* $\mathbf{FL_e} \subseteq L$, *(*$\mathbf{FL_e}^- \subseteq L$*) and* L *has a single-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then* L *has Craig interpolation.*

(*ii*) *If* $\mathbf{CFL_e} \subseteq L$*, (*$\mathbf{CFL_e}^- \subseteq L$*) and* L *has a multi-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then* L *has Craig interpolation.*

Proof. Call the centered sequent system G. Use the Maehara technique to prove that for any derivable sequent $S = (\Sigma, \Lambda \Rightarrow \Delta)$ in G there exists a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \Delta)$ are provable in G and $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, where $V(A)$ is the set of the atoms of A. □

As a positive result, our method provides a uniform way to prove the Craig interpolation property for substructural logics. For instance we have:

Corollary 4. *The logics* $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$, **ILL**, **CLL**, **IPC**, **CPC** *and their* **K**, **KD** *and* **S4** *versions have the Craig interpolation property. The same also goes for* **K4** *and* **K4D** *extensions of* **IPC** *and* **CPC**.

Proof. The usual cut-free sequent calculus for all of these logics consists of semi-analytic rules and centered axioms. Now, use Corollary 3. □

As a much more interesting negative result, which is also the main contribution of our investigation, we show that many different sub-structural logics do not have a complete sequent calculus consisting of semi-analytic rules and cenetered axioms. Our proof is based on the prior works (for instance [4] and [2]) that established some negative results on the Craig interpolation of some sub-structural logics. Considering the naturalness and the prevalence of these rules, our negative results expel so many logics from the elegant realm of natural sequent calculi.

Corollary 5. *None of the logics* R*,* BL *and* $Ł_\infty$*,* $Ł_n$ *for* $n \geqslant 3$ *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms.*

Corollary 6. *Except* G*,* $G3$ *and* **CPC***, none of the consistent* BL*-extensions have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free centered axioms.*

Corollary 7. *Except eight specific logics, none of the consistent extensions of* RM^e *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms.*

Corollary 8. *Except seven specific logics, none of the consistent super-intuitionistic logics have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules, context-sharing semi-analytic rules and centered axioms.*

A more detailed version of the Corollaries 7 and 8 can be found in [3].

References

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