An algebraic study of the First Order intuitionistic fragment of 3-valued Łukasiewicz logic

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Abstract

MV-algebras are semantic for Łukasiewicz logic and MV-algebras generated for finite chain are Heyting algebras where the Gödel implication can be written in terms of De Morgan and Moisil's modal operators. In our work, a fragment of Łukasiewicz logic is studied in the trivalent case. The propositional and first order logic is presented. The maximal consistent theories is studied as Monteiro's maximal deductive system of the Lindenbaum-Tarski algebra, in both cases. Consequently, the strong adequacy theorem with respect to the suitable algebraic structures is proven. Our algebraic strong completeness theorem does not need a negation in the language, in this sense Rasiowa's work is improved. The techniques presented in this work are adaptable to the other algebrizable logics where the variety of algebras from these logics is semisimple.

1 Trivalent modal Hilbert algebras with supremum

In this section, we shall introduce and study $\{\to, \vee, \triangle, 1\}$-reduct of 3-valued MV-algebra.

Definition 1 *An algebra $\langle A, \to, \vee, \triangle, 1\rangle$ is trivalent modal Hilbert algebra with supremum (for short, $H_3^{\vee,\triangle}$-algebra) if the following properties hold:*

(1) *the reduct $\langle A, \vee, 1\rangle$ is a join-semilattice with greatest element 1, and the conditions* (a) $x \to (x \vee y) = 1$ *and* (b) $(x \to y) \to ((x \vee y) \to y) = 1$ *hold.*

(2) *The reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra that verifies: $((x \to y) \to z) \to ((z \to x) \to z) \to z = 1$, and the operator $\triangle$ verify the following identities:* **(M1)** $\triangle x \to x = 1$, **(M2)** $((y \to \triangle y) \to (x \to \triangle\triangle x)) \to \triangle(x \to y) = \triangle x \to \triangle\triangle y$, *and* **(M3)** $(\triangle x \to \triangle y) \to \triangle x = \triangle x$.

Theorem 2 *The variety of $H_3^{\vee,\triangle}$-algebras is semisimple. The simple algebras are $\mathbb{C}_3^{\to,\vee}$ and $\mathbb{C}_2^{\to,\vee}$.*

Let $\mathfrak{Fm}_s$ be the absolutely free algebra over the language $\Sigma = \{\to, \vee, \triangle\}$ generated by a set Var of variables. Consider now the following logic:

Definition 3 *We denote by $\mathcal{H}^3_{\vee,\triangle}$ the Hilbert calculus determined by the followings axioms and inference rules, where $\alpha, \beta, \gamma, ... \in Fm$:*

Axiom schemas

(Ax1) $\alpha \to (\beta \to \alpha)$, **(Ax2)** $(\alpha \to (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$, **(Ax3)** $((\alpha \to (\beta \to \gamma)) \to (((\gamma \to \alpha) \to \gamma) \to \gamma)$, **(Ax4)** $\alpha \to (\alpha \vee \beta)$, **(Ax5)** $\beta \to (\alpha \vee \beta)$,

(Ax6) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$, **(Ax7)** $\triangle\alpha \to \alpha$, **(Ax8)** $\triangle(\triangle\alpha \to \beta) \to (\triangle\alpha \to \triangle\beta)$, **(Ax9)** $((\beta \to \triangle\beta) \to (\alpha \to \triangle(\alpha \to \beta))) \to \triangle(\alpha \to \beta)$, **(Ax10)** $((\triangle\alpha \to \beta) \to \gamma) \to ((\triangle\alpha \to \gamma) \to \gamma)$.

Inference rules

(MP), **(NEC-S)** $\dfrac{\Gamma \vdash_\vee \alpha}{\Gamma \vdash_\vee \triangle\alpha}$.

Let $\Gamma \cup \{\alpha\}$ be a set formulas of $\mathcal{H}^3_{\vee,\triangle}$, we define the derivation of α from Γ in usual way and denote by $\Gamma \vdash_\vee \alpha$.

Theorem 4 (Lindenbaum-Łos) *Let $\mathcal{L}$ be a Tarskian and finitary logic (see [2, pag. 48]) over the language $\mathbb{L}$. Let $\Gamma \cup \{\varphi\} \subseteq \mathbb{L}$ be such that $\Gamma \not\vdash \varphi$. Then exists a set Ω such that $\Gamma \subseteq \Omega \subseteq \mathbb{L}$ with Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

Theorem 5 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. Let $\Gamma/\equiv_\vee = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum $\mathfrak{Fm}/\equiv_\vee$, then:* **1.** *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$ then $\beta \in \Gamma$,* **2.** *$\Gamma/\equiv_\vee$ is a modal deductive system of $\mathfrak{Fm}/\equiv_\vee$. Also, if $\varphi \notin \Gamma/\equiv_\vee$ and for any modal deductive system $\overline{D}$ which contains properly to $\Gamma/\equiv_\vee$, then $\overline{\varphi} \in \overline{D}$.*

The notion deductive systems considered in the last Theorem, part 2, was named *Systèmes deductifs liés à "a"* by A. Monteiro, where a is an element of some given algebra such that the congruences are determined by deductive systems [3, pag. 19]. This was studied by Monteiro himself and other authors for diferent algebraic system where it is possible to define an implication in terms of the operations of lenguage form this systems.

Lemma 6 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. If $\alpha \notin \Gamma$ then $\triangle\alpha \to \beta \in \Gamma$ for any $\beta \in \mathfrak{Fm}_s$.*

Theorem 7 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, with Γ non-trivial maximal respect to φ in $\mathcal{H}^3_{\vee,\triangle}$. The map $v : \mathfrak{Fm}_s \to \mathbb{C}_3$, defined by:*

$$v(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Gamma_0 \\ 1/2 & \text{if } \alpha \in \Gamma_{1/2} \\ 1 & \text{if } \alpha \in \Gamma \end{cases}$$

for all $\alpha \in \mathfrak{Fm}_s$ it is a valuation for $\mathcal{H}^3_{\vee,\triangle}$, where $\Gamma_{1/2} = \{\alpha \in \mathfrak{Fm}_s : \alpha \notin \Gamma, \nabla\alpha \in \Gamma\}$ and $\Gamma_0 = \{\alpha \in \mathfrak{Fm}_s : \alpha, \nabla\alpha \notin \Gamma\}$.

Theorem 8 *(Soundness and completeness of $\mathcal{H}^3_{\vee,\triangle}$ w.r.t. $H_3^{\vee,\triangle}$-algebras) Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_s$, $\Gamma \vdash_\vee \varphi$ if and only if $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$.*

2 Model Theory and first order logics of $\mathcal{H}_3^{\vee,\triangle}$ without identities

Let Λ be the propositional signature of $\mathcal{H}_3^{\vee,\triangle}$, the simbols $\forall$ (universal quantifier) and $\exists$ (existential quantifier), with the punctuation marks (commas and parenthesis). Let $Var = \{v_1, v_2, ...\}$ a numerable set of individual variables. A first order signature $\Sigma = \langle \mathcal{P}, \mathcal{F}, \mathcal{C} \rangle$ consists of: a set $\mathcal{C}$ of individual constants; for each $n \geq 1$, $\mathcal{F}$ a set of functions with n-ary, for each $n \geq 1$, $\mathcal{P}$ a set of predicates with n-ary. It will be denoted by T_Σ and $\mathfrak{Fm}_\Sigma$ the sets of all terms and formulas, respectively.

Let Σ be a first order signature. The logic $\mathcal{QH}_3^{\vee,\triangle}$ over Σ is obtained from the axioms and rules of $\mathcal{H}_3^{\vee,\triangle}$ by substituting variables by formulas of $\mathfrak{Fm}_\Sigma$, by extending the following axioms and rules:

Axioms Schemas **(Ax11)** $\varphi_x^t \to \exists x\varphi$, if t is a term free for x in φ, **(Ax12)** $\forall x\varphi \to \varphi_x^t$, if t is a term free for x in φ, **(Ax13)** $\triangle\exists x\varphi \leftrightarrow \exists x\triangle\varphi$, **(Ax14)** $\triangle\forall x\varphi \leftrightarrow \forall x\triangle\varphi$, **(Ax15)** $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$ if α does not contain free occurrences of x. **Inferences Rules** **(R3)** $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$, **(R4)** $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$ where x does not occur free in φ.

Let Σ be a first-order signature. A first-order structure over Σ is pair $\mathfrak{U} = \langle A, \cdot^{\mathfrak{U}}\rangle$ where A is a non-empty set and $\cdot^{\mathfrak{U}}$ is a interpretation mapping defined on Σ as follows: for each individual constant symbol c of Σ, $c^{\mathfrak{U}} \in A$; for each function symbol f n-ary of Σ, $f^{\mathfrak{U}} : A^n \to A$; for each predicate symbol P n-ary of Σ, $P^{\mathfrak{U}} : A^n \to B$, where B is a complete $H_3^{\vee,\triangle}$-algebra.

For a given Σ-structure $\langle A, \cdot^{\mathfrak{U}}\rangle$, let us consider the signature $\Sigma' = \Sigma \cup \{c_a\}_{a\in A}$ which is the signature Σ extended by a set with new constants. Let us denote the extended language by $\mathfrak{Fm}(\Sigma')$. We want to define the truth value a closed formula. For this task, we consider the structure $\mathfrak{U}$ and the map $m : CT_{\Sigma'} \to A$, where $CT_{\Sigma'}$ is the set of closed terms (without free variables) of the language $\mathfrak{Fm}_{\Sigma'}$, is defined as follows: if τ is c_a, then $m(\tau) = m(c_a) = a$; if τ is $f(\tau_1, ..., \tau_n)$ and $\tau_i \in CT_{\Sigma'}$, then $m(\tau) = f^{\mathfrak{U}}(m(\tau_1), ..., m(\tau_n))$.

Let φ be a closed formula (sentence) from Σ', then we define $m : \mathfrak{Fm}_{\Sigma'} \to B$ inductively over the complexity of φ as follows: if φ is $P(\tau_1, \cdots, \tau_n)$ with P a n-ary predicate and $\tau_i \in CT_{\Sigma'}$, then $m(\varphi) = P^{\mathfrak{U}}(m(\tau_1), ..., m(\tau_n))$; if φ is $\gamma \vee \psi$ then $m(\varphi) = m(\gamma) \vee m(\psi)$; if φ is $\gamma \to \psi$ then $m(\varphi) = m(\gamma) \to m(\psi)$; if φ is $\triangle\psi$ then $m(\varphi) = \triangle m(\psi)$; let $\psi = \psi(x)$ a formula with x is a unique free variable, we denote $\psi_x^{c_a}$ the formula obtained by replacing x for c_a. Then: if φ is $\exists x\psi$ then $m(\varphi) = \bigvee\limits_{c_a\in\Sigma'} m(\psi_x^{c_a})$; if φ is $\forall x\psi$ then $m(\varphi) = \bigwedge\limits_{c_a\in\Sigma'} m(\psi_x^{c_a})$.

We say that $m : \mathfrak{Fm}_{\Sigma'} \to B$ is $\mathcal{QH}_3^{\vee,\triangle}$-valuation or simply a valuation.

As usual, we can define $\Gamma \vDash \alpha$, that is, for any structure $\mathfrak{U}$, if $\mathfrak{U} \vDash \psi$ for every $\psi \in \Gamma$, then $\mathfrak{U} \vDash \alpha$.

Lemma 9 *Let α be a formula of $\mathcal{QH}_3^{\vee,\triangle}$ and β an instance of α, then there exits $\mathfrak{U}$ such that $\mathfrak{U} \vDash \alpha$ implies $\mathfrak{U} \vDash \beta$.*

Theorem 10 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, if $\Gamma \vdash_\vee \varphi$ then $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$.*

It is important to note that from Theorem 10 and Lemma 9, it is easy to see that every instance of a theorem is valid.

It is clear that $\mathcal{QH}_3^{\vee,\triangle}$ is a tarskian logic. So, we can consider the notion of maximal theories with respect to some formula and the notion of closed theories is defined in the same way. Therefore, we have that Lindenbaum- Los' Theorem for $\mathcal{QH}_3^{\vee,\triangle}$. Then, we have the following

Now, let us consider the relation $\equiv$ defined by $x \equiv y$ iff $\vdash x \to y$ and $\vdash y \to x$, then we have the algebra $\mathfrak{Fm}_{\Sigma'}/\equiv$ is a $H_3^{\vee,\triangle}$-algebra and the proof is exactly the same as in the propositional case.

Theorem 11 *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, with Γ non-trivial maximal respect to φ in $\mathcal{QH}_3^{\vee,\triangle}$. Let $\Gamma/\equiv_\vee = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum*

$\mathfrak{Fm}_\Sigma/\equiv_\vee$, *then:* **1.** *If* $\alpha \in \Gamma$ *and* $\overline{\alpha} = \overline{\beta}$, *then* $\beta \in \Gamma$. *If* $\overline{\alpha} \in \Gamma/\equiv_\vee$, *then* $\overline{\forall x \alpha} \in \Gamma/\equiv_\vee$; *in this case we say that* $\Gamma/\equiv_\vee$ *is monadic.* **2.** $\Gamma/\equiv_\vee$ *is a modal deductive system of* $\mathfrak{Fm}_\Sigma/\equiv_\vee$. *Also, if* $\varphi \notin \Gamma/\equiv_\vee$ *and for any modal deductive system* $\overline{D}$ *being monadic in the sense of 1 and containing properly to* $\Gamma/\equiv_\vee$, *then* $\overline{\varphi} \in \overline{D}$.

Theorem 12 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, *with* Γ *non-trivial maximal respect to* φ *in* $\mathcal{QH}^3_{\vee,\triangle}$. *The map* $v : \mathfrak{Fm}_\Sigma \to \mathbb{C}_3$, *defined by:*

$$v(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Gamma_0 \\ 1/2 & \text{if } \alpha \in \Gamma_{1/2} \\ 1 & \text{if } \alpha \in \Gamma \end{cases}$$

for all $\alpha \in \mathfrak{Fm}_\Sigma$ *it is a valuation for* $\mathcal{H}^3_{\vee,\triangle}$, *where* $\Gamma_{1/2} = \{\alpha \in \mathfrak{Fm}_\Sigma : \alpha \notin \Gamma, \nabla\alpha \in \Gamma\}$ *and* $\Gamma_0 = \{\alpha \in \mathfrak{Fm}_\Sigma : \alpha, \nabla\alpha \notin \Gamma\}$.

Theorem 13 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, *if* $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$ *then* $\Gamma \vdash_\vee \varphi$.

Proof: Let us suppose $\Gamma \vDash_{\mathcal{H}^3_{\vee,\triangle}} \varphi$ and $\Gamma \nvdash_\vee \varphi$. Then, there exists Δ maximal theory such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_\vee \varphi$. From the latter and Theorem 12, there exists a structure $\mathfrak{U}$ such that $\Delta \nvDash_{\mathfrak{U}} \varphi$ but $\Delta \vDash_{\mathcal{A}} \gamma$ for every $\gamma \in \Delta$, which is a contradiction.

□

It is possible to adapt our proof of strong Completeness Theorem in the propositional and first order cases to logics from the certain semisimple varieties of algebras. This is so because the maximal congruences play the same role as the maximal consistent theories in the Lindenbaum-Tarski algebra. From the latter and results of universal algebra, we have the algebra quotient by maximal congruences are isomorphic to semisimple algebras. Therefore, we always have a homomorphism from the Lindenbaum-Tarski algebra to the semisimple algebras. This homomorphism is the same one constructed by Carnielli and Coniglio to prove strong completeness theorems for different logics ([2]). On the other hand, we can observe that A. V. Figallo constructed this homomorphism to study different semisimple varieties. The general presentation of these ideas will be part of a future work.

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