HYPERREAL EXPECTED UTILITIES AND PASCAL'S WAGER

ABSTRACT. This paper re-examines two major concerns about the validity of
Pascal's Wager: (1) The classical von Neumann-Morgenstern Theorem seems
to contradict the rationality of maximising expected utility when the utility
function's range contains infinite numbers (McClennen 1994). (2) Apparently,
the utility of salvation cannot be reflexive under addition by real numbers
(which Pascal's *Pensée* 233 demands) and strictly irreflexe under multiplication
by scalars < 1 at the same time (Hájek 2003).

Robinsonian nonstandard analysis is used to establish a hyperreal version of
the von Neumann-Morgenstern Theorem: an affine utility representation the-
orem for internal, complete, transitive, independent and infinitesimally contin-
uous preference orderings on lotteries with hyperreal probabilities. (Herein, a
preference relation $\preceq$ on lotteries is called *infinitesimally continuous* if and only
if for all $x \prec y \prec z$, there exist hyperreal, possibly infinitesimal, numbers p, q
such that the "perturbed preference ordering" $px + (1-p)z \prec y \prec qx + (1-q)z$
holds. Infinitesimal Continuity is hence a much weaker condition than conti-
nuity.) This Hyperreal von Neumann-Morgenstern Theorem yields a hyperreal
version of the Expected Utility Theorem — affirming a conjecture by Sobel
(1996). This responds to objection (1).

To address objection (2), a convex linearly ordered superset S of the reals
whose maximum is both reflexive under addition by finite numbers and strictly
irreflexive under multiplication by scalars < 1 is constructed.

If the Wagerer is indifferent among the pure outcomes except salvation
(a common soteriological position) and some technical conditions hold, then
the Hyperreal Expected Utility Theorem allows to represent the Wagerer's
preference ordering through an S-valued (not just hyperreal-valued) utility
function, answering objections (1) and (2) simultaneously.

Behold, I set before you the way of life and the way of death.	*But your happiness? Let us weigh the gain and the loss in wagering that God is. Let us estimate these two chances. If you gain, you gain all; if you lose, you lose nothing. Wager, then, without hesitation that He is.*
Jeremiah 21,8 (King James Version)	Blaise Pascal, *Pensées*, §233 (Trotter translation)

1. INTRODUCTION

1.1. **The context of Pascal's Wager.** Pascal's Wager [16, *Pensée* 233] is a Chris-
tian apologetic argument.[1] It is meant to address individuals who already hold cer-
tain beliefs about the supernatural (cf. Rescher [17]), which explains the strength
of some of the argument's premises (see Subsection 1.2):

First, the subjective probability for the existence of the Christian God is as-
sumed to be positive and non-infinitesimal. (Zero probabilities would make the

---

[1]The context of *Pensée* 233, in particular Pensée 181, implies that the Wager is argument is
Christian apologetics, not specific Jansenist or Catholic apologetics. (Jansenism was the Roman
Catholic sect Pascal belonged to.) It is not addressing non-Roman-Catholic or non-Jansenist
Christian believers, but rather individuals who lack interest in a personal faith (hence Section III
of the *Pensées* is entitled "Of the necessity of the Wager") or lean towards atheism.

argument invalid, infinitesimal probabilities would require a sufficiently high utility of salvation, cf. Oppy 1990 [15] and Hájek 2003 [5].)

Secondly, the audience is assumed to consider the Christian faith the only viable alternative to atheism for themselves. (Otherwise they might as well become attracted to any religion that promises paradise to its followers. Several variants of this so-called many-gods objection have been studied systematically by Bartha [1].)

From a theological perspective, it is important to note that Pascal did not expect that anyone who is convinced of the conclusion of the argument could earn their salvation themselves — let alone by merely accepting the rationality of some gambling strategy.[2] To the contrary, Pascal's (Jansenist) theology places a great emphasis on grace and predestination.

The purpose of this apologetic argument is, therefore, simply to "incite to the search after God" [*Pensée* 181].

1.2. **The structure of the Wager. Mixed strategies.** Pascal's argument — directed at someone who is choosing between either Christianity or atheism and, in addition, assigns positive, non-infinitesimal probability to the existence of the Christian God — can be formalised as follows:

(1) Premise: One has to wager for or against God, and the payoff of the wager is as follows:

	Christian God exists (with some probability $p \gg 0$)	Christian God does not exist
Wager for God	I	f_2
Wager against God	f_3	f_4

Herein, I denotes an infinitely large number[3], f_2, f_3, f_4 are finite[4], and $p \gg 0$ means that p is non-infinitesimal[5].

(2) Premise: Reason demands to maximise expected utility.

(3) Conclusion: Reason demands to wager for God.

Formalisations of the Wager have to identify I mathematically in some superset S of the field of the reals.

Now, in order to clarify Premise 2, one needs to define what kind of choices the Wagerer is allowed to make. In this paper, we allow the Wagerer to base his/her decision to wager for or against God on a random event of some probability q. (For instance, by tossing a coin to determine what to wager for.) Such a strategy is called a *mixed* strategy.

If one were to exclude the possibility of mixed strategies, the decision of the Pascalian Wagerer amounts to the choice of one of two continuum-size sets of lotteries. For, there are then two possible lotteries (wagering for or against God) for each value for the probability p of God's existence.

As we do allow for mixed strategies, the Wagerer has to choose one continuum-size set of lotteries among a continuum of continuum-size set of lotteries. For, there is a continuum of possible lotteries — one for each value for the chance that he wagers for God — for each value for the probability p of God's existence.

Thus, Premise 2 can now be phrased as follows: Let $\langle \bar{p}, \bar{q} \rangle$ denote the lottery where the probability that the Christian God exists is $\bar{p}$ and the probability that the Wagerer actually wagers for Him is $\bar{q}$. Then, for every $\bar{p} \in (0, 1]$ (excluding

---

[2]Cf. Pascal in *Pensée* 240.

[3]I.e., $I > n$ holds for every $n \in \mathbb{N}$.

[4]I.e. $|f_2|, |f_3|, |f_4| \le n$ for some $n \in \mathbb{N}$.

[5]I.e., there exists some $n \in \mathbb{N}$ such that $p > \frac{1}{n}$.

infinitesimal probabilities $\bar{p}$, see Premise 1), a rational Wagerer must strictly prefer $\langle \bar{p}, 1 \rangle$ over $\langle \bar{p}, \bar{q} \rangle$ for any $\bar{q} < 1$.

This statement is exactly what the Pascalian must prove (in some formal setting) in order to justify Premise 2.

1.3. **Two concerns about Pascal's Wager.** Pascal's Wager faces at least two major challenges: (1) McClennen's decision-theoretic objection, and (2) Hájek's dilemma.

(1) McClennen (1994) [13] points out that Premise 2, the rationality of maximising expected utility, lacks a decision-theoretic justification (such as the von Neumann-Morgenstern Theorem) since the Wagerer's utility function is allowed to take infinite values: For, the classical von Neumann-Morgenstern Theorem only says that a preference ordering on lotteries can be represented by a real-valued expected utility function if and only if the preference ordering has certain properties, among them continuity. Now, on the one hand, continuous preference-orderings are inconsistent with infinite utilities (which Pascal's Wager entails), and on the other hand, the Pascalian wants to allow for infinite (not just real-valued) expected utility functions. Hence, the Pascalian cannot justify Premise 2 through classical utility theory. Instead, a new expected utility theorem is needed in order to defend Premise 2.

(2) Hájek (2003) [5] contends that there is a dilemma for any conceivable mathematical (re)formulation of the Wager: On the one hand, a historically faithful reading of Pascal's *Pensée* 233 demands that the utility of salvation be reflexive under addition by real numbers.[6] On the other hand, the utility of salvation must be irreflexive under multiplication (by probabilities > 0), in order to ensure that one can distinguish between the expected utility of outright wagering for God and mixed strategies (where the Wagerer only ends up wagering for God with some probability $p > 0$, cf. Duff 1986 [3]). Hence, one must find a convex linearly ordered set which contains the reals and has a maximum that is both reflexive under addition by reals and strictly irreflexive under multiplication by positive scalars < 1. However, Hájek thought that this is impossible: "[There are] no prospects for characterizing a notion of the utility of salvation that is reflexive under addition without being reflexive under multiplication by positive, finite probabilities" (Hájek 2003 [5, p. 49]).

1.4. **Outline of the argument.** In a recent paper, Bartha (2007) [1] proposed a new formalisation of Pascal's Wager, based on generalised utility ratios, which addresses both McClennen's objection and Hájek's dilemma. The aim of this article is to demonstrate how McClennen's objection and Hájek's dilemma can also be addressed by means of *one-place* hyperreal[7]-valued utility functions; if one drops the requirement of reflexivity under addition, this approach can be simplified even further.

In particular, we shall prove:

(1) There is an expected-utility representation theorem for hyperreal utility functions: Every standard-definable, complete, transitive, independent and infinitesimally continuous preference relation can be represented by a

---

[6] In an appendix, we shall reexamine this claim and see that, in fact, irreflexivity under addition may be more in accordance with Pascal's theology.

[7] The field of hyperreals — in the sense of Robinson's (1966) nonstandard analysis [18] — is an ordered field, containing all real numbers, as well as infinitesimals and infinite numbers. It is a nonstandard model of the reals.

hyperreal-valued affine utility function. (See Section 2, in particular the Hyperreal Expected Utility Theorem 3.)

(2) There are two candidates for a mathematical model of the Wagerer's utility function where the maximal utility is both reflexive under addition and irreflexive under multiplication by positive probabilities. In particular, there exists a convex subset of a two-dimensional vector space which allows for a linear ordering which is consistent with addition and multiplication and whose maximum is reflexive under addition without being reflexive under multiplication by positive probabilities. (See Subsection 4.3 and Appendix B.)

Hence, each of the challenges by McClennen and Hájek can be addressed separately. The combination of Hájek's dilemma and McClennen's critique is potentially troublesome for the Pascalian. However, we shall prove that under additional hypotheses on the Wagerer's metaphysical stance, the range of this utility function can even be chosen as a linearly-ordered convex set whose maximum is both reflexive under addition and strictly irreflexive under multiplication, thus answering at the same time McClennen's objection and Hájek's dilemma. Besides, in an appendix to this paper, we shall argue that despite its philosophical merits and faithfulness to Pascal's statement in *Pensée* 233, the reflexivity under addition of the utility of salvation is in tension with other aspects of Pascalian theology.

Now, it has been argued that there is some "arbitrariness" in modelling subjective utility of salvation of a given human individual by some particular infinite hyperreal (cf. Hájek 2003 [5] and Bartha 2007 [1]). The Hyperreal Expected Utility Theorem clarifies that this degree of freedom simply reflects an ubiquitious phenomenon in decision theory with cardinal preferences: Von Neumann-Morgenstern utility functions are only unique up to a positive factor and a shift by an additive scalar.

Finally, in order to apply the Hyperreal Expected Utility Theorem to Pascal's Wager, we must assume that the Pascalian Wagerer has a completely defined preference relation over lotteries with arbitrary hyperreal chances. This does, of course, by no means entail that the Wagerer is assumed to assign infinitesimal probability to the existence of God—which would be inconsistent with Premise 1. It only means that the Wagerer is able to compare those lotteries where the probability for the event that he wagers for God while God does not exist is hyperreal (e.g. infinitesimal) with other lotteries.

2. The Hyperreal Expected Utility Theorem

In order to justify Premise 2, i.e. the rationality of maximising expected utility whilst permitting infinite values in the utility function's range, a decision-theoretic argument is required. Typically, the rationality of maximising expected utility is justified in decision theory via the classical Expected Utility Theorem of von Neumann and Morgenstern, but this rules out infinite values of the utility function.

However, as explained by Bartha (2007) [1, p. 10], reflexivity under multiplication (i.e. the axiom that $x \cdot I = I$ for all $x > 0$) is responsible for the inconsistency of infinite utilities with the hypotheses of the von Neumann-Morgenstern Theorem. Thus, since infinite hyperreals and surreals are (strictly) irreflexive under multiplication, there may be some hope to counter the first objection if I is a hyperreal or a surreal number.

To the present author, it is unclear how to develop a von Neumann-Morgenstern Theorem for *surreal* utilities and thus to respond to McClennen's objection through a surreal formalisation of the Wager.

By means of Robinson's (1966) [18] nonstandard analysis, we shall prove a von Neumann-Morgenstern Theorem for *hyperreal*-valued utility functions (*Hyperreal von Neumann-Morgenstern Theorem*, Theorem 1) and derive from there a rigorous decision-theoretic justification for the rationality of maximising hyperreal expected utility (*Hyperreal Expected Utility Theorem*, Theorem 3).

Hence, if the utility of salvation in Pascal's Wager is modeled by positive infinite hyperreals, then nonstandard analysis provides a response to McClennen's objection.

The Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) follows easily from the classical Expected Utility Theorem of von Neumann and Morgenstern. Indeed, if $*$ denotes a nonstandard embedding[8] from the standard universe into the nonstandard universe, then one can prove the following equivalence theorem, which is an easy consequence of applying the Transfer Principle[9] to the standard Expected Utility Theorem of von Neumann and Morgenstern in Jensen's (1967) formulation [8].

In the statement of this theorem, we employ the notion of internality in the sense of nonstandard analysis. *Internal* means to be an element of the $*$-image of a standard set—and this is also equivalent to being definable by a formula of set theory which treats the reals as atoms and has internal, e.g. standard, parameters.

A $*$-*linear space* is an internal linear space over the field $*\mathbb{R}$. Furthermore, an internal subset X of a $*$-linear space is $*$-*convex* if and only if $px + (1-p)y \in X$ for all $x, y \in X$ and $p \in {}^*[0,1]$. Finally, an internal function $U : X \to {}^*\mathbb{R}$, defined on some $*$-convex set X is called $*$-*affine* if and only if $U(px + (1-p)y) = pU(x) + (1-p)U(y)$ for all $x, y \in X$ and $p \in {}^*[0,1]$. Note that these definitions are consistent with the terminology regarding $*$-images of formulae in Footnote 8, when applied to the formal definitions of being a linear space or a convex set or an affine function; this consistency is crucial for the proof of the Theorem which relies on the use of the Transfer Principle.

Also, $*(0,1]$ and $*(0,1)$ denote the sets of hyperreals x satisfying $0 < x \leq 1$ and $0 < x < 1$, respectively. This definition, again, is consistent with the Transfer Principle outlined in Footnote 8.

Theorem 1 (Hyperreal von Neumann-Morgenstern Theorem). *Let X be an internal[10] $*$-convex subset of a $*$-linear space, and let $\preceq$ be a binary relation $\subseteq X \times X$.*

---

[8]This is an embedding of the superstructure over the reals into the superstructure of a non-Archimedean model of the ordered field of the reals—usually obtained via an ultrafilter construction—which satisfies the *Transfer Principle*, the *Countable Saturation Principle* and the *Internal Definition Principle*. (The superstructure $V(M)$ over some set M is defined via $V_0 = M$, $V_{n+1}(M) = V_n(M)$ for all $n \in \mathbb{N}_0$ and $V(M) = \bigcup_{n=0}^{\infty} V_n(M)$.) The Transfer Principle states the following: Any first-order proposition $\phi[a_1, \ldots, a_n]$ of set theory that treats the reals as atoms and has only bounded quantifiers (and parameters $a_1, \ldots, a_n$ from the superstructure over the reals), holds if and only if the proposition $\phi[{}^*a_1, \ldots, {}^*a_n]$, sometimes also referred to as $*\phi[a_1, \ldots, a_n]$ (the $*$-image of the formula $\phi[a_1, \ldots, a_n]$), holds in the nonstandard universe. The Countable Saturation Principle states that any decreasing countable chain of nonempty *internal* sets, i.e. sets that are elements of $*$-images of (standard) sets, must have a nonempty intersection. The Internal Definition Principle says that any subset of an internal set that is defined via a set-theoretic formula with internal parameters is itself internal. There are even definable (over Zermelo-Fraenkel set theory with the Axiom of Choice) nonstandard extensions of the superstructure over the reals, cf. Herzberg [7].

[9]See Footnote 8.

[10]Since all models in applications of nonstandard analyis are (standard parts) of internal objects (otherwise, it is impossible to obtain any information on these objects via the Transfer Principle), the requirement of internality is even in general not a relevant restriction. (Cf. also Herzberg [6].) But, as we shall see later on, we can even circumvent the notion of internality for the purposes of this article.

*There exists a *-affine function $U : X \to {}^*\mathbb{R}$ such that*

$$U(x) \le U(y) \iff x \preceq y$$

holds for all $x, y \in X$ if and only if $\preceq$ possesses all of the following properties:

(1) **Completeness.** *For all $x, y \in X$, either $x \preceq y$ or $y \preceq x$.*
(2) **Transitivity.** *For all $x, y, z \in X$ with $x \preceq y$ and $y \preceq z$, one has $x \preceq z$.*
(3) **Infinitesimal Continuity.** *For all $x, y, z \in X$ with $x \prec y \prec z$ [11], there exist hyperreals $p, q \in {}^*(0,1)$ such that*

$$px + (1-p)z \prec y \prec qx + (1-q)z.$$

(4) **Independence.** *For all $x, y, z \in X$ and every $p \in {}^*(0,1]$, the relation $x \preceq y$ is equivalent to $px + (1-p)z \preceq py + (1-p)z$.*

Herein, the interpretation of $x \preceq y$ should be read as 'x is not preferred over y' or 'either y is preferred over x or they are equivalent'.

The first two properties are just the weak order axioms[12].

When we compare Infinitesimal Continuity with ordinary continuity[13] of binary relations on convex spaces, we find that $(0,1)$ has been replaced by ${}^*(0,1)$. In particular, the hyperreals p and q in the definition of Infinitesimal Continuity may be infinitely close to 1, which corresponds to an *infinitesimal perturbation* $x' = px + (1-p)z$ of x and an infinitesimal perturbation $z' = qx + (1-q)z$ of z. In other words, Infinitesimal Continuity asserts the existence of a hyperreal (possibly infinitesimal) perturbation, while ordinary continuity asserts the existence of a real, non-infinitesimal perturbation. Hence, Infinitesimal Continuity is a much weaker condition than the ordinary continuity axiom in the sense of Jensen (1967) [8].

Our Independence axiom says that a preference relation $x \preceq y$ is preserved if x and y are both mixed with another lottery and the same, possibly hyperreal, probability p. Whilst this is a stronger axiom than ordinary independence in the sense of Jensen [8] (for, it replaces $(0,1]$ by the larger set ${}^*(0,1]$ in the definition of independence for binary relations on convex spaces), it is clearly the natural extension of the ordinary independence axiom to lotteries with hyperreal chances.

Proof of the Hyperreal von Neumann-Morgenstern Theorem. The (standard) Expected Utility Theorem of von Neumann and Morgenstern about preference relations on convex sets says the following: A binary relation on any convex subset X of a linear space satisfies the axioms of completeness, transitivity, continuity and independence if and only if there exists some affine function $U : X \to \mathbb{R}$ such that the estimate $U(x) \le U(y)$ is equivalent to $x \preceq y$ (for all $x, y \in X$).

We shall apply the Transfer Principle to this Expected Utility Theorem. Note, for this purpose, that the Transfer Principle also yields that the *-image of the set of binary relations on convex subsets of linear spaces is just the set of internal binary relations on *-convex subsets of *-linear spaces.

Hence, applying the Transfer Principle to the standard Expected Utility Theorem leads to the following result: Any internal binary relation on a *-convex subset of a *-linear space is *-complete, *-transitive, *-continuous and *-independent if and only if there exists a *-affine function $U : X \to {}^*\mathbb{R}$ satisfying

$$\forall x, y \in X \qquad U(x) \le U(y) \Leftrightarrow x \preceq y.$$

But, *-transitivity is the same as transitivity, and *-completeness is the same as completeness. Furthermore, if we apply the Tranfer Principle to the definitions of continuity and independence (for standard binary relations on convex sets), we

---

[11]For any $x, y \in X$, we define $x \prec y$ to be the negation of $y \preceq x$.

[12]In the terminology of Jensen [8], the first two axioms characterise *complete preorderings*.

[13]The Continuity axiom is also known as the Archimedean property (cf. e.g. Jensen 1967 [8]).

obtain that *-continuity is the same as Infinitesimal Continuity, and *-independence is the same as Independence. This completes the proof of the Theorem.

$\square$

Theorem 2 (Internal Expected Utility Theorem). *Let W be an internal finite-dimensional linear space over the field $^*\mathbb{R}$ of the hyperreals, let $x_1, \ldots, x_m \in W$, and consider $Y = \{\sum_{i=1}^{m} p_i x_i \; : \; p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^{m} p_i = 1\}$. Let $\preceq$ be an internal binary relation $\subseteq Y \times Y$. The relation $\preceq$ satisfies all the axioms of (1) Completeness, (2) Transitivity, (3) Infinitesimal Continuity and (4) Independence if and only if there exist hyperreals $u_1, \ldots, u_m$ such that*

$$\sum_{i=1}^{m} p_i x_i \preceq \sum_{i=1}^{m} q_i x_i \Longleftrightarrow \sum_{i=1}^{m} p_i u_i \leq \sum_{i=1}^{m} q_i u_i$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0,1]$ with $\sum_{i=1}^{m} p_i = 1$ and $\sum_{i=1}^{m} q_i = 1$.

The hypothesis of internality of the relation $\preceq$ may be replaced by a stronger, but conceptually more accessible assumption: standard-definability under a basis choice. $\preceq$, a relation on some subset Y of an n-dimensional linear space W over $^*\mathbb{R}$ ($n \in \mathbb{N}$) is said to be *standard-definable under a basis choice* if there exist

- an isomorphism $\psi : W \simeq {}^*\mathbb{R}^n$ (a bijective map that commutes both with addition and with multiplication by *hyperreals*) and
- a first-order formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in which the canonical extensions (*-images) of maps from $\mathbb{R}^M$ to $\mathbb{R}^N$ (for any $M, N \in \mathbb{N}$), as well as equality '=' and the order relation '<' may occur, with free variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ and constants from $^*\mathbb{R}$

such that

$$\forall v, w \in Y \qquad (x \preceq y \Longleftrightarrow \varphi[\psi(v), \psi(w)])$$

(in other words: $\preceq = \{\langle v, w \rangle \in Y^2 \; : \; \varphi[\psi(v), \psi(w)]\}$).

In particular, $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ may be any formula from the language of ordered rings[14]. Since, however, the theory of real-ordered fields admits quantifier elimination (which can, for instance, be proven via the so-called Tarski-Seidenberg Principle, cf. e.g. Bochnak, Coste, Roy [2, Proposition 5.5.2] or Marker [12, Theorem 3.3.15]), this would simply mean that there are polynomials $f_{i,j}$ ($i \leq M$, $j \leq N$) in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with coefficients from $^*\mathbb{R}$ such that

$$\preceq = \bigcup_{i=1}^{M} \bigcap_{j=1}^{N} \left\{ \langle v, w \rangle \in Y^2 \; : \; f_{i,j}(\psi(v), \psi(w)) \geq 0 \right\}.$$

Note that whenever $\chi : {}^*\mathbb{R}^n \simeq {}^*\mathbb{R}^n$ is an automorphism of the linear space $^*\mathbb{R}^n$ over $^*\mathbb{R}$ and $f : {}^*\mathbb{R}^{2n} \to {}^*\mathbb{R}$ is defined via canonical extensions of maps from $\mathbb{R}^M$ to $\mathbb{R}^N$ as well as constants from $^*\mathbb{R}$, then $f \circ \chi$ can also be defined that way. (The reason is that χ itself is definable, since it is a linear map from a finite-dimensional linear space onto itself.) Hence, the choice of ψ is irrelevant: It can be replaced by $\chi \circ \psi$ and thus by an arbitrary other isomosorphism between W and $^*\mathbb{R}^n$ (as linear spaces over $^*\mathbb{R}$).

With this new concept of standard-definability under a basis choice in mind, let us state the following Theorem as an immediate corollary to Theorem 2. This theorem affirms a conjecture by Sobel (1996) [20].

Theorem 3 (Hyperreal Expected Utility Theorem). *Let W be a finite-dimensional linear space over the field $^*\mathbb{R}$ of the hyperreals, let $x_1, \ldots, x_m \in W$ and suppose $Y = \{\sum_{i=1}^{m} p_i x_i \; : \; p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^{m} p_i = 1\}$ (the convex hull of*

---

[14]The operations in the language of ordered rings are addition, subtraction and multiplication; the relations in this language are equality '=' and the order relation '<'. Cf. e.g. Marker [12].

$x_1, \ldots, x_m$ *over* $^*\mathbb{R}$). *Let* $\preceq$ *be a binary relation* $\subseteq Y \times Y$ *and assume* $\preceq$ *to be standard-definable under a basis choice. The relation* $\preceq$ *on* Y *satisfies all the axioms of (1) Completeness, (2) Transitivity, (3) Infinitesimal Continuity and (4) Independence if and only if there exist hyperreals* $u_1, \ldots, u_m$ *such that*

$$\sum_{i=1}^m p_i x_i \preceq \sum_{i=1}^m q_i x_i \Longleftrightarrow \sum_{i=1}^m p_i u_i \leq \sum_{i=1}^m q_i u_i$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0,1]$ *with* $\sum_{i=1}^m p_i = \sum_{i=1}^m q_i = 1$.

Note that the statement of the Hyperreal Expected Utility Theorem (Theorem 3) does not involve the notion of an internal set any longer—in contrast to, e.g. the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1).

Proof of the Internal Expected Utility Theorem. First, suppose $Y = \{\sum_{i=1}^m p_i x_i \ : \ p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^m p_i = 1\}$ and $\preceq$ is an internal binary relation $\subseteq Y \times Y$. We have to show that the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) may be applied in the setting of the Hyperreal Expected Utility Theorem (Theorem 3).

For this sake, note that if $n = \dim_{^*\mathbb{R}} W$ is the dimension of W as a linear space over the field $^*\mathbb{R}$, then W is isomorphic (over $^*\mathbb{R}$) to $^*\mathbb{R}^n$. Let us denote this isomorphism by $\psi : W \simeq {}^*\mathbb{R}^n$.

If $y_1, \ldots, y_m$ are elements of an arbitrary *-linear space Z, then the set

$$C(y_1, \ldots, y_m) := \left\{ z \in Z \ : \ \exists p_1, \ldots, p_m \in {}^*[0,1] \left(\sum_{i=1}^m p_i = 1, \quad z = \sum_{i=1}^m p_i y_i \right) \right\}$$

is internally defined and therefore—according to the Internal Definition Principle[15]—itself an internal set. Moreover, it is closed under convex combinations with weights from $^*\mathbb{R}$. Hence it is a *-convex set (the *-convex hull of $y_1, \ldots, y_m$).

Since ψ is an isomorphism, we find that
(1)
$$\psi(Y) = \left\{ z \in {}^*\mathbb{R}^n \ : \ \exists p_1, \ldots, p_m \in {}^*[0,1] \left(\sum_{i=1}^m p_i = 1, \quad z = \sum_{i=1}^m p_i \psi(x_i) \right) \right\}.$$

Therefore, the observation of the previous paragraph may be applied to $C(\psi(x_1), \ldots, \psi(x_m)) = \psi(Y)$. Hence $X := \psi(Y)$ is a *-convex subset of the *-linear space $^*\mathbb{R}^n$.

Now, the internality of $\preceq$ on Y ensures that the relation $\preceq_X$, defined by

$$\xi_1 \preceq_X \xi_2 :\Leftrightarrow \psi^{-1}(\xi_1) \preceq \psi^{-1}(\xi_2),$$

is also internal (and if $\preceq$ is standard-definable under a basis choice, $\preceq_X$ will have that property, too: $\preceq_X = \{\langle x, y \rangle \in X^2 \ : \ \varphi(x,y)\}$). Since the formula φ only involves the canonical extensions (*-images) of standard maps as well as equality and the order relation, $\preceq_X$ is internally defined and thus, according to the Internal Definition Principle, internal.

Thus, we may apply the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) to the set $X = \psi(Y)$ and the relation $\preceq_X$ on X. Observe that $\preceq_X$ satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence if and only if the relation $\preceq$ on Y satisfies them (because ψ is an isomorphism and thus commutes with *-convex combinations, i.e. convex combinations with weights from $^*\mathbb{R}$). Furthermore, the equivalence assertion

$$\forall \xi_1, \xi_1 \in X \qquad \xi_1 \preceq_X \xi_2 \Leftrightarrow U(\xi_1) \leq U(\xi_2)$$

---

[15]See Footnote 8

is true if and only if

$$\forall y_1, y_2 \in Y \qquad y_1 \preceq y_2 \Leftrightarrow U\left(\psi\left(y_1\right)\right) \leq U\left(\psi\left(y_2\right)\right).$$

Hence, after we have applied the Hyperreal von Neumann-Morgenstern Theorem to X and $\preceq_X$, we actually obtain the following statement: The relation $\preceq$ on Y satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence if and only if there is some *-affine function $U : \psi\left(Y\right) \rightarrow {}^*\mathbb{R}$ with

$$\forall y_1, y_2 \in Y \qquad y_1 \preceq y_2 \Leftrightarrow U\left(\psi\left(y_1\right)\right) \leq U\left(\psi\left(y_2\right)\right).$$

Finally, we shall demonstrate that the existence of a *-affine function $U : \psi\left(Y\right) \rightarrow {}^*\mathbb{R}$ with $y_1 \preceq y_2 \Leftrightarrow U\left(\psi\left(y_1\right)\right) \leq U\left(\psi\left(y_2\right)\right)$ for all $y_1, y_2 \in Y$ is equivalent to the existence of the hyperreals $u_1, \ldots, u_m$ as in the statement of the Hyperreal Expected Utility Theorem. This is straightforward: Given $u_1, \ldots, u_m$, define

$$\forall i \in \{1, \ldots, m\} \qquad U\left(\psi\left(x_i\right)\right) := u_i.$$

This function allows for a unique *-affine extension $U : \psi(Y) \rightarrow \mathbb{R}$, given by

$$U\left(\sum_{i=1}^m p_i \psi\left(x_i\right)\right) = \sum_{i=1}^m p_i U\left(\psi\left(x_i\right)\right) = \sum_{i=1}^m p_i u_i$$

for all $p_1, \ldots, p_m \in {}^*[0,1]$ with $\sum_{i=1}^m p_i = 1$.

Conversely, given $U : \psi(Y) \rightarrow {}^*\mathbb{R}$, simply set

$$\forall i \in \{1, \ldots, m\} \qquad u_i := U\left(\psi\left(x_i\right)\right).$$

Since U is *-affine, this already entails that

$$U\left(\sum_{i=1}^m p_i \psi\left(x_i\right)\right) = \sum_{i=1}^m p_i u_i$$

for all $p_1, \ldots, p_m \in {}^*[0,1]$ with $\sum_{i=1}^m p_i = 1$.

$\square$

The Hyperreal Expected Utility Theorem shows that expected hyperreal-valued utility functions represent preference orderings among lotteries based on a finite set of pure outcomes and nonstandard probabilities—provided that we impose certain natural conditions which are, apart from definability or internality, the direct analogues (the *-images) of the original von Neumann-Morgenstern conditions.

Hyperreal-valued utility functions have also been studied by Skala (1974) [19], Kannai (1992) [9] and Lehmann (2001) [10], in chronological order. Lehmann's (2001) [10] article is also concerned with nonstandard von Neumann-Morgenstern utility functions, but only allows for standard probabilities, which leads to a different representation theorem. Kannai (1992) [9] shows that every convex preference ordering admits a concave utility function, provided one chooses an appropriate nonstandard extension of the reals as the range of the utility function. Skala's (1974) [19] results are, to the author's knowledge, the most relevant in the literature to the subject of this article. Skala, in refuting Fishburn's (1971) [4] impression that game theory with non-Archimedean utilities is "rather barren", constructs utility functions that represent *mean groupoids*. A mean groupoid is a generalisations of a convex set on which a complete transitive order is defined. This more general approach leads, however, when applied to our setting, to a significantly weaker result than our Hyperreal von Neumann-Morgenstern Theorem. In particular, Skala's (1974) representation theorem [19, Theorem 9] only works in one direction. More importantly, general weighted sums of pure outcomes as considered in the Hyperreal Expected Utility Theorem, are even undefined in the mean groupoid setting.

3. Application to Pascal's Wager

If $x_1, \ldots, x_m \in {}^*\mathbb{R}^n$, then the *convex hull of $x_1, \ldots, x_m$ over ${}^*\mathbb{R}$* is the set

$$\left\{ \sum_{i=1}^m p_i x_i \; : \; p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^m p_i = 1 \right\},$$

and the convex hull of $\{x_1, \ldots, x_m\}$ over $\mathbb{R}$ is defined as

$$\left\{ \sum_{i=1}^m p_i x_i \; : \; \sum_{i=1}^m p_i = 1, \quad p_1, \ldots, p_m \in [0,1] \right\}.$$

Corollary 1. *Let $x_1, \ldots, x_m \in {}^*\mathbb{R}^n$, let V and Y be the convex hulls of $x_1, \ldots, x_m$ over $\mathbb{R}$ and ${}^*\mathbb{R}$, respectively, and let $\preceq$ be an internal binary relation $\subseteq Y \times Y$ that satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence (cf. Theorem 1). Then, the restriction of $\preceq$ to V is transitive, complete and independent. Furthermore, there are $f_1, \ldots, f_m \in {}^*\mathbb{R}$ such that*

$$(2) \qquad \sum_{i=1}^m p_i x_i \preceq \sum_{i=1}^m q_i x_i \iff \sum_{i=1}^m p_i f_i \le \sum_{i=1}^m q_i f_i$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0,1]$ with $\sum_{i=1}^m p_i = 1$ and $\sum_{i=1}^m q_i = 1$.

If, moreover, $x_1 \succ x_2 \sim \cdots \sim x_m$ [16], then $f_2 = \cdots = f_m$ can be any hyperreal and f_1 can be any hyperreal $> f_2$ (e.g. f_1 positive infinite, $f_2 = \cdots = f_m = 1$).

Proof of Corollary 1. By Theorem 3 there are $f_1 = u_1, \ldots, f_m = u_m \in {}^*\mathbb{R}$ such that the equivalence statement (2) holds. If $x_1 \succ x_2 \sim \cdots \sim x_m$, then $f_1 > f_2 = \cdots = f_m$. Since the numbers $\langle u_1, \ldots, u_m \rangle$ in Theorem 3 is only unique up to affine transformations, $f_2 = \cdots = f_m$ can indeed be any given hyperreal and f_1 can be any hyperreal $> f_2$. $\qquad \square$

Let us now apply the Corollary 1 to Pascal's Wager. $n = 2$ and $m = 4$. Let $x_1 = \langle 1, 1 \rangle$ represent the pure outcome where the Christian God exists and the Wagerer opts for wagering for God, let $x_2 = \langle 0, 1 \rangle$ represent the pure outcome where the Wagerer chooses to wager for the Christian God, although He does not exist, let $x_3 = \langle 1, 0 \rangle$ represent the pure outcome where the Wagerer wagers against the Christian God, whilst He does exist, and let $x_4 = \langle 0, 0 \rangle$ be the pure outcome where the Christian God does not exist and the Wagerer also wagers against His existence.

Let us now rephrase Corollary 1 in non-technical terms: Whenever the Wagerer's preference relation is

- transitive
- complete (on the space of lotteries with hyperreal chances),
- unaffected by infinitesimal perturbations (Infinitesimal Continuity),
- unaffected by mixing with other lotteries (Independence), and
- internal, e.g. definable through standard functions with hyperreal parameters,

there are cardinal utilities $\{f_1, \ldots, f_4\}$ associated with the four pure outcomes $x_1, \ldots, x_4$, and for any two lotteries $\sum_{i=1}^4 p_i x_i$ and $\sum_{i=1}^4 q_i x_i$, the first lottery is not preferred over the first if and only if the expected utility from the first lottery ($\sum_{i=1}^4 p_i f_i$) is less than or equal to the expected utility from the second lottery ($\sum_{i=1}^4 q_i f_i$). If we interpret the assumptions on the Wagerer's preference relation as rationality axioms, then we obtain indeed that reason demands the maximisation of expected utility.

---

[16]We write $x \sim y$ if both $x \preceq y$ and $y \preceq x$.

In particular, the vector

$$\langle \bar{p}, \bar{q} \rangle = \langle \bar{p}, \bar{p}\bar{q} \rangle + \langle 0, \bar{q} - \bar{p}\bar{q} \rangle = \bar{p}\left(\bar{q}\langle 1,1 \rangle + (1-\bar{q})\langle 1,0 \rangle\right) + (1-\bar{p})\left(\bar{q}\langle 0,1 \rangle + (1-\bar{q})\langle 0,0 \rangle\right)$$

is the lottery described in Subsection 1.2, where $\bar{p}$ is the subjective probability for the existence of the Christian God and $\bar{q}$ is the probability that the Wagerer chooses to wager for Him. Hence, if

- $\bar{p} > 0$ is non-infinitesimal,
- $1 - \bar{q} > 0$ is non-infinitesimal,
- f_1 is a positive infinite hyperreal, and
- f_2, f_3, f_4 are finite,

then for all $\bar{q} \in {}^*[0,1)$,

(3) $$\underbrace{\bar{p}\left(\bar{q}f_1 + (1-\bar{q})f_3\right) + (1-\bar{p})\left(\bar{q}f_2 + (1-\bar{q})f_4\right)}_{=\bar{p}\bar{q}f_1 + \bar{p}(1-\bar{q})f_3 + (1-\bar{p})(\bar{q}f_2 + (1-\bar{q})f_4)} < \bar{p}f_1 + (1-\bar{p})f_2,$$

which is equivalent to $\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle$ by equivalence statement (2) of Corollary 1, and hence wagering for God with probability 1 is strictly preferable compared to wagering for God with probability $\bar{q} < 1$.

Before we consider the special case where $f_2 = f_3 = f_4$, let us note the following points about the use of hyperreals in formalising Pascal's Wager:

- The axiom of Completeness requires the preference relation to be *defined* between lotteries with hyperreal (including infinitesimal) chances for each of the pure outcomes (e.g. the event that the Wagerer wagers for God and the existence of God). This has no consequences whatsoever for the Wagerer's subjective probability for the existence of God; it does by no means imply that the Wagerer assigns a non-real or even infinitesimal probability to the existence of the Christian God (which would contradict Premise 1). In applying the Hyperreal Expected Utility Theorem to the Pascalian Wagerer, we merely require him to have a preference realation that is defined over lotteries with hyperreal — including infinitesimal — subjective probabilities for the existence of God and for the event that the Wagerer actually wagers for Him.
- Corollary 1 is a consequence of the Hyperreal von Neumann-Morgenstern Theorem and hence ultimately of the classical Expected Utility Theorem of von Neumann and Morgenstern. Therefore, the "arbitrariness" of modelling $f_1 = I$, the subjective utility of salvation of a given human individual, by some particular infinite hyperreal (an objection of Hájek 2003 [5] and Bartha 2007 [1] against the use of hyperreals as values for I) is due to the fact that von Neumann-Morgenstern utility functions are only unique up to a positive factor (and a shift by an additive scalar). Hence, this "arbitrariness" merely reflects a typical property of decision-theoretic cardinal utility functions.

If, as in the second part of the Corollary, $f_2 = f_3 = f_4 = 1$ and f_1 is a positive infinite hyperreal I, then the right-hand side of inequality (3), which is the utility of $\langle \bar{p}, \bar{q} \rangle$, can be simplified to

$$\bar{p}\left(\bar{q}I + 1 - \bar{q}\right) + (1-\bar{p})\left(\bar{q} + 1 - \bar{q}\right) = \bar{p}\bar{q}I + \bar{p}(1-\bar{q}) + 1 - \bar{p} = \bar{p}\bar{q}I + 1 - \bar{p}\bar{q} = \bar{p}\bar{q}(I-1) + 1.$$

Hence, for all $\bar{p}, \bar{p}', \bar{q}, \bar{q}' \in [0,1]$, one has

$$\langle \bar{p}, \bar{q} \rangle \preceq \langle \bar{p}', \bar{q}' \rangle \Leftrightarrow \bar{p}\bar{q} \leq \bar{p}'\bar{q}'.$$

Interestingly, this equivalence is even true if the convex combinations of utilities are computed not in ${}^*\mathbb{R}$, but in $S_{\text{RA-IM}} := \mathbb{R} \cup \{rI \ : \ r \in (0,1]\}$ (see Equation (6) below and the discussion surrounding it). For, in $S_{\text{RA-IM}}$, the the right-hand side

of inequality (3) (i.e. the utility of $\langle \bar{p}, \bar{q} \rangle$) can be simplified, for all $\bar{p}, \bar{p}', \bar{q}, \bar{q}' \in [0, 1]$ to

$$\bar{p} \left(\bar{q} I + 1 - \bar{q} \right) + (1 - \bar{p}) \left(\bar{q} + 1 - \bar{q} \right) = \bar{p} \bar{q} I + 1 - \bar{p} = \bar{p} \bar{q} I.$$

Therefore, if $f_2 = f_3 = f_4 = 1$ and f_1 is a positive infinite hyperreal I and the Wagerer's preference ordering $\preceq$ satisfies the hypotheses of Corollary 1, then $\preceq$ can be represented through an $S_{\text{RA-IM}}$-valued utility function.

Note, however, that the maximum of $S_{\text{RA-IM}}$ is both reflexive under addition and strictly irreflexive under multiplication. Hence, if $f_2 = f_3 = f_4$, one can respond, based on the Corollary, to both McClennen's objection and Hájek's dilemma by using a single-valued utility function, viz. a utility function with values in $S_{\text{RA-IM}}$.

The philosophical interpretation of the equality $f_2 = f_3 = f_4$ is, of course, that the Wagerer is indifferent among all pure outcomes except salvation. It can be decomposed into the following two theological propositions:

(1) *Separation from God as judgement for non-believers:* $f_3 = f_4$ holds if the Wagerer believes that the Christian God would not punish those who choose not to have fellowship with Him, but simply "leaves them alone", i.e. as well-off (viz. f_3) as they were if there was no God (f_4).

(2) *Utility-neutral sanctification:* $f_2 = f_4$ holds if the Wagerer assumes that he does not have to make any sacrifices for his faith on earth that would reduce his overall utility. Any sacrifices that he makes will, at least in the long run, result in an offsetting increase in utility, even without special divine intervention. If God does not exist and he wagers for Him nevertheless, he is just as well-off (viz. f_2) as he would be if he decides otherwise (f_4).

4. Resolving Hájek's dilemma

4.1. **Reflexivity under Addition and Pascal's soteriology.** In Subsection 1.3, we mentioned that Hájek (2003) [5] reads Pascal as assuming that the reward of salvation, I, is reflexive under addition, i.e. it does not change when a positive utility is added onto it:

$$\forall x \in \mathbb{R} \qquad x + I = I$$

(which may also be read as a definition of addition of reals onto I).

As an example, consider the most simple contemporary formalisation of Pascal's Wager — where the Wagerer's utility function takes values in the set of the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$ with their natural ordering. The utility of salvation is $I = +\infty$. Recalling the convention that $x + \infty = +\infty$ for every $x \in \mathbb{R}$, the condition of Reflexivity under Addition is clearly satisfied.

It should be noted at this point that there are good theological reasons not to interpret Pascal's *Pensée* 233 as stating that I should be reflexive under addition.[17] This would then particularly favour a model of the wager where I is a hyperreal or surreal number.

---

[17] For, reflexivity under addition directly contradicts a major and widely-held thesis in Biblical soteriology (in particular in Roman Catholicism, but by far not limited to it), viz. the belief that there is some hierarchy in Heaven: Not all of those who are saved will *a priori* receive the same reward at Judgement Day. In particular, it might be possible that Pascal himself shared that opinion: In Appendix A.2 we shall argue that Pascal seems to accept the soteriological claims of the New Testament in their literal meaning. These speak plainly about a hierarchy in Heaven, and hence of a non-trivial ordering of the utility associated with salvation. (Moreover, a distinctive of Jansenist doctrine of justification is salvation by grace alone and, at the same time, a hidden judgment.) Prior to these deliberations, in Appendix A.1, we shall reconsider Hájek's argument that Pascal viewed the utility of the saved as reflexive under addition and discuss some of the questions that it raises.

That said, Hájek's interpretation of *Pensée* 233 is sufficiently convincing, and it seems best to simply accept the tension between that passage and other aspects of Pascalian theology.

4.2. **Irreflexivity under Multiplication.** In our presentation of Pascal's Wager (see Subsection 1.2), the Wagerer is allowed to adopt mixed strategies: The Wagerer may base his decision to wager for or against God on some random event of non-infinitesimal probability $q > 0$ (e.g. through dicing, tossing coins etc.). Such a strategy will be called *mixed strategy of chance q.*

In order to apply Premise 2 in that setting, the Pascalian must prove that the expected utility of any mixed strategy of chance $q < 1$ is less than the the expected utility of outright wagering for God (the "mixed strategy" of chance 1). As we will see, this is only possible if one has

$$\forall q \in (0,1) \qquad qI < I,$$

an axiom called (Strict) Irreflexivity under Multiplication.

For, suppose there existed some $q < 1$ such that $qI = I$, and consider a mixed strategy of chance q. Then the conditional expected utility, conditioned with respect to God's existence, of the mixed strategy would be $qI + (1-q)f_3 = I + (1-q)f_3$. If f_3 is non-negative, this would be at least as much as I, the conditional expected utility associated with outright wagering for God. Hence, if there was some $q < 1$ such that $qI = I$, then the expected utility of the mixed strategy of chance q would always be at least as much as expected utility of outright wagering for God. (This may be termed a co-optimal mixed strategy. If such a strategy exists, flipping some biased coin about whether to wager for or against God maximises utility equally well as faith proper.)

This reasoning in favour of (Strict) Irreflexivity under Multiplication is due to Duff (1986) [3] and has been reiterated by Hájek [5] as well as Bartha (2007) [1]; it must be taken into account by every formalisation of the Wager which allows the Wagerer to adopt a mixed strategy.

As Hájek (2003) [5] and Bartha (2007) [1] noted, any formalisation of the Wager where I is a hyperreal or surreal number automatically satisfies Strict Irreflexivity under Multiplication (and hence is not susceptible to the reasoning above). We shall not replicate Hájek's argument here, since it appears to tacitly assume that $f_2 > f_4$. Instead, we give a new proof. Recall, for this sake, that whenever J is a surreal or hyperreal number the implication

(4) J infinite $\Rightarrow \forall r \gg 0 \quad rJ$ infinite

(wherein, as before, $a \ll b$ means that $b - a$ is positive and non-infinitesimal) holds for all J. From here, we can readily deduce that regardless of the exact values for f_2, f_3, f_4 (provided they are finite) mixed strategies always carry an infinitely lesser reward than outright wagering for God. Indeed, observe that choosing to wager for God yields expected utility $pI + (1-p)f_2$, whilst choosing to Wager for God with some probability q yields expected utility $p(qI + (1-q)f_3) + (1-p)(qf_2 + (1-q)f_4)$. The difference between the former and the latter value is

(5) $p(1-q)(I - f_3) + (1-p)(1-q)(f_2 - f_4).$

Now, whenever f_2, f_3, f_4 are finite and $p \gg 0$ as well as $q \ll 1$, the first addend is always positive infinite (due to implication (4) applied to $J = I - f_3$ and $r = p(1-q)$) whilst the second addend is finite. Hence the difference in expected utility between outright wagering for God and a mixed strategy is always positive, even infinite. Therefore, mixed strategies where the probability of wagering against God is non-infinitesimal always carry a lesser reward if I is some positive infinite hyperreal or surreal utility. Hence, Strict Irreflexivity under Multiplication holds whenever I is

an infinite hyperreal or surreal number. (Hájek's original argument tacitly assumed that $f_2 > f_4$ and proved that the difference in expected utility between outright wagering for God and a mixed strategy of chance q is strictly decreasing in q and zero for $q = 1$, which proves that mixed strategies are suboptimal.)

However, neither hyperreals nor surreals are reflexive under addition in $^*\mathbb{R}$ or FIELD (the field of surreal numbers, i.e. the Dedekind completion of the field generated by the ordinals), respectively. Thus, we have yet to show that there exists a set S of utilities of the Wagerer in which the utility of salvation satisfies both Reflexivity under Addition and Strict Irreflexivity under Multiplication.

This will be accomplished in Subsection 4.3: We will construct a convex linearly ordered set $S := S_{\text{RA-IM}}$ containing the reals which does satisfy both Reflexivity under Addition and Strict Irreflexivity under Multiplication.[18] Moreover, even Premise 2 can be defended for $S_{\text{RA-IM}}$-valued utility functions (as we saw in the discussion of Corollary 1) under additional hypotheses on the Wagerer's soteriological presuppositions. Hence, formalising Pascal's Wager through an $S_{\text{RA-IM}}$-valued utility function allows to respond to McClennen's (1994) [13] decision-theoretic objection and Hájek's dilemma at the same time.

4.3. **A model for S with Strict Irreflexivity under Multiplication and Reflexivity under Addition for all infinite utilities.** We shall construct a linearly-ordered set $S \supsetneq \mathbb{R}$ such that the maximum $I \in S$, the utility of salvation, has the property of Irreflexivity under Multiplication and Reflexivity under Addition. Furthermore, taking convex combinations of elements of S will be defined in a way that is consistent with the linear order on S. Taking convex combinations with 0 will implicitly define an operation of multiplication by elements of $[0, 1]$, furthermore it will define an operation of addition for some pairs of elements of S. These operations are, as we will see, associative as well as commutative and satisfy the law of distributivity. Hence the set S defined in this Subsection is a rather well-behaved model for the set of possible utilities of a Pascalian Wagerer.

Irreflexivity under Multiplication and Reflexivity under Addition for I imply, via the law of distributivity (in the form $q(x + y) = qx + qy$ for $q \in (0, 1]$ and $x, y \in S$), Reflexivity under Addition for qI. Indeed,

$$\forall x \in \mathbb{R} \quad \forall q \in (0, 1] \qquad qx + qI = q(x + I) = qI,$$

hence (inserting $x = y/q$): $y + qI = qI$ for all $y \in \mathbb{R}$. Thus, $qI \in S$ must be reflexive under addition for all $q \in (0, 1]$.

Hence a natural candidate for S is

(6) $$S := S_{\text{RA-IM}} := \mathbb{R} \cup \{qI \ : \ q \in (0, 1]\}.$$

So, in addition to the maximal utility $I := 1I$, there is a continuum of other infinite utilities, each denoted by qI for some $q \in (0, 1)$. (In all of this, $\mathbb{R}$ and its subintervals may be replaced by $^*\mathbb{R}$. This would allow to consider nonstandard probabilities as well.)

In order to develop decision theory under risk with this set of utilities, we need to be able to form convex combinations of elements of S. For $x, y \in \mathbb{R} \subset S$, convex combinations shall be defined in the ordinary way. For $x \in \mathbb{R}$, $q \in (0, 1]$ and $r \in [0, 1]$, we define

$$(1 - r)x + r(qI) = (rq)I$$

---

[18]In Appendix B, we construct another linearly ordered superset of the reals which satisfies Reflexivity under Addition and Strict Irreflexivity under Multiplication, but the ordering is inconsistent with mixing the utilities of mixed strategies. Also, it is not clear how to defend Premise 2 when S is the set constructed in Appendix B.

(with the convention—typical for probability theory—that $0I = 0$), in line with associativity of multiplication and Reflexivity under Addition for $(rq)I$. Finally, for $q, q' \in (0, 1]$ and $r \in [0, 1]$ we set

$$(1 - r)qI + r(q'I) = ((1 - r)q + rq') I.$$

This implicitly defines addition for some pairs of elements in S (viz. for those $\langle x, y \rangle \in S^2$ where $x \in \mathbb{R}$ or $y \in \mathbb{R}$ or $\langle x, y \rangle = \langle qI, q'I \rangle$ wherein $q + q' \in [0, 1]$ with $q + q' = 1$) is not closed under addition, e.g. $I + I$ is undefined), and it also defines multiplication by elements of $[0, 1]$ (simply take $y = 0$ as the second element of a convex combination). It is an easy exercise to check that the law of distributivity holds, and that both multiplication by elements of $[0, 1]$ and addition are associative as well as commutative.

Moreover, one can extend the linear order $<$ on the reals to S by setting

$$\forall q \in (0, 1] \quad \forall x \in \mathbb{R} \qquad qI > x$$

(thus making each qI infinite and hence S non-Archimedean) and

$$qI < rI \Leftrightarrow q < r$$

for all $q, r \in (0, 1]$.

The strict ordering $<$ is preserved by multiplication by elements of $(0, 1]$, i.e.

$$\forall x, y \in S \quad \forall r \in (0, 1] \qquad x < y \Rightarrow rx < ry$$

and the weak ordering $\leq$ is preserved by addition:

$$\forall x, y, z \in S \qquad x < y \Rightarrow x + z \leq y + z.$$

(Note that if $x < y$ are reals, then $x + qI = qI = y + qI$, hence addition by qI, for any $q \in (0, 1]$ does not preserve the strict ordering.) The strict ordering $<$ is preserved, however, by adding a real.

These observations yield that forming convex combinations is consistent with the strict ordering $<$:

$$\forall x, y \in S \quad \forall r \in (0, 1) \qquad x < y \Rightarrow x < rx + (1 - r)y < y.$$

(One can easily prove this directly as well: The right-hand side obviously holds whenever $x, y \in \mathbb{R}$ or both $x = qI$ and $y = q'I$ for some $q, q' \in (0, 1]$. It also holds whenever $x \in \mathbb{R}$ and $y = qI$ for some $q, q' \in (0, 1]$, since then $rx + (1 - r)y = (1 - r)qI$ is infinite, but dominated by $y = qI$.)

Finally, mixed strategies do not yield optimal utility in this setting: The expected utility of wagering for God with probability $q \in (0, 1)$ equals $p\left(qI + (1 - q)f_3\right) + (1 - p)\left(qf_2 + (1 - q)f_4\right)$ whilst the expected utility of outright wagering for God is $pI + (1 - p)f_2$, and, as we see from expression (5), the difference between the two expected utilities is

$$(1 - q)\left(p\left(I - f_3\right) + (1 - p)\left(f_2 - f_4\right)\right).$$

Recalling that f_2, f_3, f_4 are finite and in light of the Reflexivity under Addition for I and pI, we obtain that the expected utility of wagering for God with probability $q \in (0, 1)$ is $(1 - q)pI$, which is strictly less than I.

5. Conclusion

We have shown that the concept of hyperreal expected utility has a sound decision-theoretic basis: Under some natural conditions on the preference orderings, expected hyperreal-valued utility functions on convex sets represent preference orderings among lotteries based on a finite set of pure outcomes and hyperreal probabilities.

This is good news for the Pascalian since Pascal's Wager — and most of its generalisations, such as the many-gods wagers studied by Bartha (2007) [1] —, only allow a finite number of pure outcomes. In the original Wager, there are just four pure outcomes: the Wagerer believes in God and God exists, or he does not wager for Him, although He exists, or he does wager for Him, whilst He does not exist, or he does not wager for Him, nor does He exist.

Therefore, a formalisation of Pascal's argument by means of hyperreals is consistent with a decision theory that incorporates nonstandard probabilities, whilst every internal (nonstandard) probability measure canonically induces a standard real-valued probability measure. (In particular, if one composes a nonstandard probability measure on a finite set with the standard part map[19], one obtains a standard probability measure on that finite set[20]).

The Hyperreal Expected Utility Theorem (Theorem 3) provides a general decision-theoretic justification of hyperreal-valued expected utility functions. Thus, one may now consider, in the spririt of Bartha's conclusion ("Beyond Pascal's Wager" [1, p. 39-41]), the use of hyperreal stochastic utilities in other situations where either an infinite good is at stake, or where intolerable outcomes should be avoided, or where both Kantian and utilitarian deliberations seem to have their point.

In addition, we have constructed a convex linearly ordered superset S of the reals which has a maximum that is both reflexive under addition by finite numbers and strictly irreflexive under multiplication by scalars < 1, thereby proposing a way out of Hájek's dilemma

Moreover, if one assumes that the Wagerer is indifferent among all pure outcomes except salvation ($f_2 = f_3 = f_4$ in the notation of the Wagerer's payoff matrix), which e.g. follows from all those theological systems (*nota bene*, on the Wagerer's part) where judgment just means separation from God and where sanctification is utility-neutral, a corollary of the Hyperreal Expected Utility Theorem (Theorem 3) shows that the preference ordering can be represented by a utility function whose range is contained in the aforementioned convex linearly ordered set S.

Summing up, we have determined under which hypotheses one can simultaneously refute two major arguments against Pascal's Wager, viz. McClennen's decision-theoretic objection and Hájek's dilemma, through a formalisation with a single-valued utility function whose range is a certain subset S of the hyperreals: Aside from technicalities, one has to impose the assuption of $f_2 = f_3 = f_4$ (in the notation of the Wagerer's payoff matrix above), which can be upheld if the Wagerer views sanctification as utility-neutral and believes that the Christian God, if He exists, will judge non-believers by "mere" separation from Him.

---

[19]Let r be a hyperreal number such that r is S-bounded, i.e. there exists some standard natural number $N \in \mathbb{N}$ with $-N \le r \le N$. Then, due to the Hausdorff property of the order topology on $\mathbb{R}$, there exists a unique real number s, which minimises $|r - s|$ among all $s \in \mathbb{R}$. This s is then denoted $°r$ and referred to as the *standard part* of r. The function st $: r \mapsto °r$ is called *the standard part map*.

[20]This is just a special case of a general construction: Any internal probability function can be extended to a (σ-additive) probability measure with standard values, as was shown by Loeb [11]. Almost all contributions to probability theory using nonstandard methods rely on this basic result of Loeb.

Appendix A. Reflexivity under Addition
vs. Soterical Differentiation

A.1. **Pascal on Reflexivity under Addition.** As was mentioned in the Introduction, Hájek suggests that Pascal would have required I, the utility associated with salvation, to be *reflexive under addition* (and we have already indicated our disagreement with that statement):

$$(7) \qquad \forall x \in \mathbb{R}_{>0} \qquad x + I = I$$

(and thereby, by adding $y = -x$ to both sides of the equation, even $y + I = I$ for all $y \in \mathbb{R}_{<0}$, hence $y + I = I$ for every real y).

At first glance, Hájek's interpretation of Pascal as assuming the Reflexivity under Addition is convincing. First, Reflexivity under Addition seems to express that "Nothing could be better for you than your salvation" (Bartha). Moreover, Pascal writes in the preface to the Wager [*Pensée* 233]:

> Unity joined to infinity adds nothing to it, no more than one foot to an infinite measure. The finite is annihilated in the presence of the infinite, and becomes a pure nothing. [...]

The first sentence of this passage seems to support the axiom of Reflexitivity under Addition (7). The second sentence, however, explicates that a finite number is nothing *in the presence of*—in other words: *compared to*—an infinite value (italicisation of the author).

If we view this second sentence as an explanation of the first one, then we are not forced to adopt the assumption of Reflexitivity under Addition.

Rather, we could postulate that S, the set of possible utilities of the Pascalian Wagerer, is a convex subset of an ordered field[21] and satisfies for all I in some nonempty proper subset $\Sigma \subsetneq S$ the following estimate:

$$(8) \qquad \forall x \in \mathbb{R}_{>0} \quad \forall n \in \mathbb{N} \qquad 0 < \frac{x}{I} < \frac{1}{n}.$$

In words, this means that for all $x \in \mathbb{R}_{>0}$ and $I \in \Sigma$, $\frac{x}{I}$ is a positive infinitesimal. We may assume that the set Σ has been chosen as the maximal subset of S with the property that all elements $I \in \Sigma$ satisfy estimate (8).

Note that in this formalisation, Hájek's axiom of I being an *Overriding Utility* [5], holds—provided we model the utilities of the saved by elements of Σ and the values f_2, f_3, f_4 (as defined in the Wagerer's payoff matrix in Premise (1)) by $S \setminus \Sigma$. For, in order for someone to have an infinite level of utility in this model, she must wager for God.

In particular, the axiom (8) is satisfied if

- S is the ordered field of hyperreal numbers and Σ the subset of positive infinite hyperreals, or
- S is the ordered field of surreal numbers and Σ the subset of positive infinite surreal numbers.

In both of these ordered fields, 'finite' means being bounded by some $n \in \mathbb{N}$.

---

[21]An *ordered field* Q is a field on which a linear order is defined, in such a way that addition of any element of the field preserves the order relation between two elements, and so does multiplication by positive elements. A subset $A \subseteq Q$ is called *convex* if and only if for all $x, y \in A$ and every $p \in Q$ with $0_Q <_Q p <_Q 1_Q$, one has $p \times_Q x +_Q \ 1_Q -_Q p \ \times_Q y \in A$. One is inclined to demand the convexity of the set of possible utilities of the Wagerer in order to allow for mixed strategies.

A.2. **Pascal and Soterical Differentiation.** Up to now, we have only argued that Pascal need not be read as supporting the axiom of Reflexivity under Addition. But can he be interpreted as subscribing to the principle that the Wagerer's utility should have multiple levels of infinite utility in its range (*Soterical Differentiation*)?

It is difficult to argue directly in favour of this. Given the apologetic nature of most of his writings, Pascal has written next to nothing on eschatology (not even in the Prophecies section, Section XI, of the *Pensées*), and his soteriological comments are mainly concerned with matters of justification and salvation, in particular the doctrine of grace and predestination. Also, neither Cornelius Jansen, the founder of the Roman Catholic sect Pascal belonged to—nor Augustine, the church father whose soteriology greatly influenced Pascal and Jansen, seem to have published anything that would either favour or contradict Soterical Differentiation.

It is important to note at this point that Pascal's and Jansenism's emphasis on a justification by grace through faith (quite as the protestant *sola gratia*), as opposed to works, cannot be seen as an argument against Soterical Differentiation. With the same right that God proves Himself gracious to some and not to others, He may as well reward some of the saved more and some less.

Strictly speaking, we can therefore only speculate what Pascal's views regarding Soterical Differentiation might have been like. However, Pascal had a very high appreciation of Scripture, even of what he terms "obscure passages". For instance, in *Pensée* 575, he writes the following (which itself consists half of indirect Scripture quotations)

> All things work together for good to the elect [*cf. Romans 8,28*], even the obscurities of Scripture; for they honour them because of what is divinely clear. And all things work together for evil to the rest of the world, even what is clear; for they revile such, because of the obscurities which they do not understand [*cf. 2nd Peter 3,16*].

(Comments of the author in squared parentheses.) Similarly, in *Pensées* 568, 579 as well as 889 ("the true guardians of the Divine Word have preserved it unchangeably") he defends the divine inspiration of the whole of Scripture; he goes even further to claim, quoting Augustine that "He who will give the meaning of Scripture, and does not take it from Scripture, is an enemy of Scripture" [*Pensée* 900].

However, the New Testament plainly and multiply mentions a hierarchy in Heaven. For intance, in the first paragraphs of the Sermon on the Mount, Jesus says:

> Whosoever therefore shall break one of these least commandments, and shall teach men so, he shall be called the least in the kingdom of heaven: but whosoever shall do and teach them, the same shall be called great in the kingdom of heaven. [Matthew 5,19 (King James Version)]

In the same Gospel, Jesus teaches as follows:

> Verily I say unto you, Among them that are born of women there hath not risen a greater than John the Baptist: notwithstanding he that is least in the kingdom of heaven is greater than he. [Matthew 11,11 (King James Version)]

What is translated as "least" in the New International Version or the King James Version, would be ἐλάξιστος (the elative or superlative of μιχρός) and μιχρότερος (comparative of μιχρός), respectively in the original Greek New Testament sources, clearly suggesting that there is a hierarchy in Heaven. Outside the Gospel of Matthew, Soterical Differentiation can be found in the book of Revelation:

> And I saw thrones, and they sat upon them, and judgment was
> given unto them: and I saw the souls of them that were beheaded
> for the witness of Jesus, and for the word of God, and which had
> not worshipped the beast, neither his image, neither had received
> his mark upon their foreheads, or in their hands; and they lived
> and reigned with Christ a thousand years. [Revelation 20,4 (King
> James Version)]

This means that the Millennium — i.e. Christ's first reign on earth — will see a
proper subset of the faithful sharing power with Him. Note that there is agreement
on these verses among all major textual witnesses and early translations (cf. the
critical apparatus of Nestle-Aland's *Novum Testamentum Graece*, 27th rev. ed.).

Given that central passages of the New Testament mention a hierarchy in Heaven,
it is reasonable to assume that Pascal would have approved of the idea of Soterical
Differentiation.

It should be noted that under the Jansenist or Augustinian idea of predestination,
which Pascal subscribed to, "not only are the reasons for the judgment hidden
(which the Calvinists admit), but the judgment itself is also" (cf. Miel [14, p.
105-106]) and therefore, the believers should "work out [their] salvation with fear
and trembling" (Miel [14, p. 105] citing Philippians 2,12). Consequently (and
this is even consistent with Calvinism), humans may not know how much exactly,
compared to other saved ones, they will be rewarded in Heaven—all one can say is,
that, in case of salvation, it has to be an infinite value compared with any earthly
utility.

APPENDIX B. AN ALTERNATIVE WAY OF RESOLVING HÁJEK'S DILEMMA

In this Appendix, we construct a model for S where only the maximum satisfies Reflexivity under Addition. However, this model for S is not covered by the Corollary 1, whence it is susceptible to McClennen's objection.

Let S, the set of possible utilities of a Pascalian Wagerer, be defined as

$$S = \{\langle 1, 0 \rangle\} \cup [0, 1) \times \mathbb{R},$$

i.e. the union of the singleton $\{\langle 1, 0 \rangle\}$ with the set of all pairs of real numbers where the first entry is ≥ 0 and < 1. (Here and in the following, $\mathbb{R}$ and its various subintervals could again be replaced by any real-ordered field, thus allowing, for instance, for nonstandard probabilities.)

The first coordinate should be seen as representing 'heavenly utility' and the second coordinate 'earthly utility" . In the payoff matrix of the Wager, $f_2, f_3, f_4 \in \{0\} \times \mathbb{R}$ and $I = \langle 1, 0 \rangle$. The utility $I = \langle 1, 0 \rangle$ might be interpreted as the utility of someone who heeded Pascal's advice and "wager[ed for God] without hesitation", thus not considering mixed strategies.

Let the total order $<$ on S be just the lexicographic ordering:

$$\forall \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in S \qquad \langle x_1, y_1 \rangle < \langle x_2, y_2 \rangle \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2).$$

Hence, $\langle 1, 0 \rangle$ is the strict maximum of S:

$$\forall \langle x, y \rangle \in [0, 1) \times \mathbb{R} \qquad \langle x, y \rangle < \langle 1, 0 \rangle.$$

Let the operation of addition in S be defined as follows:

$$\forall \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in [0, 1) \times \mathbb{R} \qquad \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle \max\{x_1, x_2\}, y_1 + y_2 \rangle$$

and

$$\forall \langle x, y \rangle \in S \qquad \langle x, y \rangle + \langle 1, 0 \rangle = \langle 1, 0 \rangle + \langle x, y \rangle = \langle 1, 0 \rangle.$$

In particular, $I = \langle 1, 0 \rangle$ is reflexive under addition. Multiplication by elements of $[0, 1]$ will be defined as the ordinary multiplication on $\mathbb{R}^2$:

$$\forall \langle x, y \rangle \in S \qquad p \langle x, y \rangle = \langle px, py \rangle.$$

Hence, $I = \langle 1, 0 \rangle$ meets the requirement of Irreflexivity under Multiplication

$$\forall p < 1 \qquad p \langle 1, 0 \rangle = \langle 0, p \rangle < \langle 1, 0 \rangle.$$

This entails that both S and the proper subset $[0, 1) \times \mathbb{R}$ of S are closed under addition as well as under multiplication by elements of $[0, 1]$.

Having constructed S and observed that in this setting, the utility of salvation, I, is both reflexive under addition and irreflexive under multiplication, we close with the following remarks:

(1) Albeit exhibiting some resemblances to the set of vector-valued utilities considered by Hájek [5, Subsection 4.2, pp. 39-41], it is different in that S contains just one value with maximal 'heavenly utility', viz. I, the maximum of S itself.

(2) Adding up the overall utility of one individual with the utility of another one, does not make much sense if salvation is at stake. This is reflected by the ordering on S being inconsistent with addition: For, if $x_3 \geq x_2 > x_1$ but $y_2 < y_1$, then both $\langle x_1, y_1 \rangle < \langle x_2, y_2 \rangle$ and

$$\langle x_1, y_1 \rangle + \langle x_3, y_3 \rangle > \langle x_2, y_2 \rangle + \langle x_3, y_3 \rangle.$$

For, the left hand side in the inequality then equals $\langle x_3, y_1 + y_3 \rangle$ and the right hand side equals $\langle x_3, y_2 + y_3 \rangle$. Also, the ordering is obviously inconsistent with mixing the utilities of mixed strategies, i.e. elements of $(0, 1) \times \mathbb{R}$:

If $p \in (0, 1)$ and $\langle x_1, y_1 \rangle < \langle x_2, y_2 \rangle$ are elements of $(0, 1) \times \mathbb{R}$, then we will have

$$\langle x_1, y_1 \rangle > p \langle x_1, y_1 \rangle + (1 - p) \langle x_2, y_2 \rangle$$

whenever $(1 - p)x_2 < x_1$. Hence, in order to estimate the utility of a strategy that uses multiple random experiments, one must first compute the overall probability that the Wagerer will, at the very end, Wager for God.

(3) This, however, will lead to no further inconsistencies: The ordering is consistent with mixing pure strategies: If $p \in (0, 1)$ and $\langle x_1, y_1 \rangle < \langle x_2, y_2 \rangle$ are elements of $\{0\} \times \mathbb{R} \cup \{\langle 1, 0 \rangle\}$ (since $\{0\} \times \mathbb{R}$ is the set of utilities associated with wagering against God), then

$$\langle x_1, y_1 \rangle < p \langle x_1, y_1 \rangle + (1 - p) \langle x_2, y_2 \rangle .$$

(4) Mixed strategies no longer yield maximal utility: The utility of wagering for God with probability q is

$$p \left(q \langle 1, 0 \rangle + (1 - q) \langle 0, y_{f_3} \rangle \right) + (1 - p) \left(q \langle 0, y_{f_2} \rangle + (1 - q) \langle 0, y_{f_4} \rangle \right),$$

wherein $\langle 0, y_{f_i} \rangle = f_i$ for $i \in \{2, 3, 4\}$. This can be reduced to

$$\langle pq, p(1 - q)y_{f_3} \rangle + \langle 0, (1 - p)qy_{f_2} + (1 - p)(1 - q)y_{f_4} \rangle$$

which equals

$$\langle pq, p(1 - q) \left(y_{f_3} \right) + (1 - p)qy_{f_2} + (1 - p)(1 - q)y_{f_4} \rangle < \langle 1, 0 \rangle = I.$$

So, I is reflexive under addition, and mixed strategies are suboptimal.

REFERENCES

[1] P. Bartha. Taking stock of infinite value: Pascal's Wager and relative utilities. *Synthese*, 154(1):5–52, 2007.

[2] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry. Transl. from the French. Rev. and updated ed.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 36. Berlin: Springer, 1998.

[3] A. Duff. Pascal's Wager and infinite utilities. *Analysis*, 46(2):107–109, 1986.

[4] P.C. Fishburn. On the foundations of game theory: the case of non-Archimedean utilites. *Internat. J. Game Theory*, 1(2):65–71, 1971.

[5] A. Hajek. Waging war on Pascal's Wager. *Philosophical Review*, 112(1):27–56, 2003.

[6] F.S. Herzberg. Internal laws of probability, generalized likelihoods and Lewis' infinitesimal chances—a response to Adam Elga. *British J. Philos. Sci.*, 58(1):25–43, 2007.

[7] F.S. Herzberg. A definable nonstandard enlargement. *Math. Log. Q.*, 54(2):167–175, 2008.

[8] N.E. Jensen. An introduction to Bernoullian utility theory. *Swedish J. Econ.*, 69(3):163–183, 1967.

[9] Y. Kannai. Non-standard concave utility functions. *J. Math. Econ.*, 21(1):51–58, 1992.

[10] D. Lehmann. Expected qualitative utility maximization. *Games Econ. Behav.*, 35(1-2):54–79, 2001.

[11] P.A. Loeb. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Trans. Amer. Math. Soc.*, 211:113–122, 1975.

[12] D. Marker. *Model theory: An introduction*. Graduate Texts in Mathematics. 217. New York, NY: Springer, 2002.

[13] E.F. McClennen. Pascal's Wager and finite decision theory. In J. Jordan, editor, *Gambling on God: Essays on Pascal's Wager*, pages 115–138. Lanham, MD: Rowman & Littlefield, 1994.

[14] J. Miel. *Pascal and Theology*. Baltimore, MD: The Johns Hopkins Press, 1969.

[15] G. Oppy. On Rescher on Pascal's Wager. *Internat. J. Philos. Religion*, 30(3):159–168, 1991.

[16] B. Pascal. *Pascal's Pensées. Translated by W.F. Trotter*. New York: Harvard Classics, 1910.

[17] N. Rescher. *Pascal's Wager: a study of practical reasoning in philosophical theology*. Notre Dame, IN: University of Notre Dame Press, 1985.

[18] A. Robinson. *Non-standard analysis*. (Studies in Logic and the Foundations of Mathematics.) Amsterdam: North-Holland Publishing Company., 1966.

[19] H.J. Skala. Nonstandard utilities and the foundation of game theory. *Internat. J. Game Theory*, 3(2):67–81, 1974.

[20] J.H. Sobel. Pascalian wagers. *Synthese*, 108(1):11–61, 1996.