Lebesgue's Density Theorem for Ideals
Part I

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joint with Sandra Müller[1], Philipp Schlicht[2], and Thilo Weinert[1]

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KNAW Academy Colloquium
Generalized Baire Spaces
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The density points of a set

Let λ denote Lebesgue measure on $\mathbb{R}$.

Definition

Let $A \subseteq \mathbb{R}$ be measurable and let $x \in \mathbb{R}$. Define the *density of x in A as*

$$\mathcal{D}_x(A) = \lim_{\varepsilon \to 0} \frac{\lambda(A \cap U_\varepsilon(x))}{\lambda(U_\varepsilon)}$$

and the set of *points of density 1 in A* or *density points of A* as

$$\Phi(A) = \{x \in \mathbb{R} \mid \mathcal{D}_x(A) = 1\},$$

- This definition makes sense for any metric space equipped with a Radon measure μ, such as $\mathbb{R}^n$ with Lebesgue measure.
- ... In particular, for ${}^\omega\omega$ and ${}^\omega 2$, each with the usual metric and the usual product measure.

Lebesgue's Density Theorem

For simplicity, let $(\mathfrak{X}, \mu)$ be one of

- $\mathbb{R}^n$ with the standard metric and Lebesgue measure,
- Cantor space ${}^\omega 2$ with the usual metric and product measure (the coin-tossing measure).

Theorem

For any μ-measurable sets $A \subseteq \mathfrak{X}$,

$$\Phi(A) =_\mu A.$$

Note the theorem holds in many more Polish metric measure spaces—cf. two recent works by Andretta-Camerlo and Andretta-Camerlo-Constantini.

Φ as a selector on $\textsc{Malg}$

Let $\mathfrak{X}$ be a standard Borel space equipped a Borel probability measure μ.

Let $\textsc{Meas}(\mathfrak{X}, \mu)$ denote the measurable, $\textsc{Null}(\mathfrak{X}, \mu)$ the null, and $\textsc{Borel}(\mathfrak{X})$ the Borel subsets of $\mathfrak{X}$.

Recall that the *measure algebra* is defined as

$$\textsc{Malg} = \textsc{Meas}(\mathfrak{X}, \mu) / \textsc{Null}(\mathfrak{X}, \mu) = \textsc{Borel}(\mathfrak{X}) / \textsc{Null}(\mathfrak{X}, \mu)$$

In fact $\textsc{Malg}$ is always the same, regardless of $\mathfrak{X}$ and μ.

Lebesgue's Density Theorem for $(\mathfrak{X}, \mu)$ implies that Φ gives rise to a *selector*

$$\hat{\Phi} \colon \textsc{Malg} \to \textsc{Meas}(\mathfrak{X}, \mu),$$

i.e., $\hat{\Phi}([A]_\mu) \in [A]_\mu$.

Some properties of Φ

The map Φ is ‘natural’ in that it’s definition is not too complicated:

- For any measurable set A, $\Phi(A)$ is $\mathbf{\Pi}^0_3$.
- The following map is the restriction of a Σ^1_1 relation to a Π^1_1 set

  $$\Phi \restriction \textsc{Borel}(\mathfrak{X})\colon\ \textsc{Borel}(\mathfrak{X}) \to \mathbf{\Pi}^0_3$$

  when viewed as a map sending codes to codes.

The map Φ also has nice algebraic properties, for example:

- $A =_\mu B \Rightarrow \Phi(A) = \Phi(B)$ (well-defined on $[A]_\mu$)
- $A \subseteq_\mu B \Rightarrow \Phi(A) \subseteq \Phi(B)$ (monotonic)
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ (preserves $\cap$)
- $\Phi(A) \cap \Phi(A^{\complement}) = \emptyset$ (disjointness property, follows from previous)

A Lebesgue Density Theorem for ideals?

From forcing, we know many more ideals, each with their own notion of measurability…

Question

For which of those ideals is there an analogue of Lebesgue's Density Theorem?

The short answer:

- We can define a notion of density point with nice properties which works for a large class of ccc forcings,
- As a counterexample, we show no even remotely 'nice' notion of density point works for Sacks forcing.

Ideals from forcing

Let $\mathfrak{X}$ denote ${}^\omega\omega$ or ${}^\omega 2$, and suppose $\mathbf{P}$ is a set of perfect trees on ω or 2, respectively. As usual $[T]$ denotes the set of branches through T.

(Of course $\langle \mathbf{P}, \supseteq \rangle$ is a forcing, but we will not force in this talk.)

Definition

For $X \subseteq \mathfrak{X}$,

- $X \in N_{\mathbf{P}} \iff \forall T \in \mathbf{P} \quad \exists T' \in \mathbf{P}$ s.t. $T' \subseteq T$ and $[T'] \cap X = \emptyset$.
- Let $I_{\mathbf{P}}$ denote the σ-ideal generated by $I_{\mathbf{P}}$.
- $X \in I^*_{\mathbf{P}} \iff \forall T \in \mathbf{P} \quad \exists T' \in \mathbf{P}$ s.t. $T' \subseteq T$ and $[T'] \cap X \in I_{\mathbf{P}}$.
- $X \in \textsc{Meas}_{\mathbf{P}}(\mathfrak{X}) \iff \forall T \in \mathbf{P} \quad \exists T' \in \mathbf{P}$ s.t. $T' \subseteq T$ and $([T'] \subseteq_{I^*_{\mathbf{P}}} X$ or $[T'] \subseteq_{I^*_{\mathbf{P}}} X^{\complement})$.

In all cases currently of interest one can show $I_{\mathbf{P}} = I^*_{\mathbf{P}}$.

Some assumptions on $\mathbb{P}$

We make the following assumptions from now on:

- If $T \in \mathbf{P}$ and $s \in T$, $T_s = \{ t \in T \mid t \subseteq s \vee s \subseteq t\} \in \mathbf{P}$.
- $I^*_{\mathbf{P}} = I_{\mathbf{P}}$.
- $\textsc{Borel}(\mathfrak{X}) \subseteq \textsc{Meas}_{\mathbf{P}}(\mathfrak{X})$.

These assumptions hold for a very large class of forcings—e.g., the strongly arboreal forcings that satisfy the ccc or fusion.

The Main Definition

Recall $\mathbf{P}$ is a set of perfect trees on ω or 2. For $T \in \mathbf{P}$, recall that the *stem of* T is defined as follows:

$$\mathrm{stem}_T = \max\{t \in T \mid (\forall s \in T)\ (s \subseteq t \vee t \subseteq s)\}.$$

Definition

Given t in ${}^{<\omega}\omega$ or ${}^{<\omega}2$ let

$$L_t = \{T \in \mathbf{P} \mid \mathrm{stem}_T = t\ \}$$

and for $A \in \textsc{Meas}_{\mathbf{P}}(\mathfrak{X})$ let

$$\Phi_{\mathbf{P}}(A) = \{x \in \mathfrak{X} \mid (\forall^\infty n)(\forall T \in L_{x \restriction n})\ [T] \cap A \notin I_{\mathbf{P}}\}$$

In the relevant case, each set L_t will consist of pairwise compatible conditions…

Example: Random forcing

Let $\mathfrak{X} = {}^{\omega}2$ with the usual product measure μ (the coin-tossing measure).

We can regard Random forcing as the set of conditions

$$\mathbf{P} = \{T \mid T \text{ is a perfect tree on } 2 \text{ and } \frac{\mu([T])}{2^{\mathrm{lh}(\mathrm{stem}_T)}} > \frac{1}{2}\}$$

(ordered by $\supseteq$).

Then for all $A \in \textsc{Meas}(\mathfrak{X}, \mu)$,

$$\Phi_{\mathbf{P}}(A) =_{\mu} \Phi(A)$$

(but the two notions of density points don't coincide).

The Main Definition, Second Take

Equivalently, define

$$L = L_\emptyset = \{T \in \mathbf{P} \mid \mathrm{stem}_T = \emptyset\}.$$

Definition

- For $A \subseteq \mathfrak{X}$, say A is *L-positive*, or $A \in L^+$, **iff**

$$(\forall T \in L)\ [T] \cap A \notin I_{\mathbf{P}}.$$

- Given t in ${}^{<\omega}\omega$ or ${}^{<\omega}2$, define the *shift map* $\sigma_t \colon \mathfrak{X} \to \mathfrak{X}$ by

$$\sigma_t(x) = t \frown x$$

- Then

$$\Phi_{\mathbf{P}}(A) = \{x \in \mathfrak{X} \mid (\forall^\infty n)\ (\sigma_{x \restriction n})^{-1}[A] \in L^+\}$$

Properties of $\Phi_{\mathbf{P}}$

Theorem

Suppose $\mathbf{P}$ is strongly linked (cf. the following talk by Sandra Müller), and $\mathbf{P}$ and as well as $\perp_{\mathbf{P}}$ are Σ^1_1.
Then $\Phi_{\mathbf{P}} \restriction \textsc{Borel}(\mathfrak{X})$ is absolutely Δ^1_2 as a map from Borel codes to Borel codes, and for $A, B \in \textsc{Borel}(\mathfrak{X})$

- $\Phi_{\mathbf{P}}(A) \in [A]_{I_{\mathbf{P}}}$,
- $A =_{I_{\mathbf{P}}} B \Rightarrow \Phi_{\mathbf{P}}(A) = \Phi_{\mathbf{P}}(B)$,
- $\Phi_{\mathbf{P}}(A) \in \mathbf{\Sigma}^0_2$,
- $A \subseteq_{I_{\mathbf{P}}} B \Rightarrow \Phi_{\mathbf{P}}(A) \subseteq_{I_{\mathbf{P}}} \Phi_{\mathbf{P}}(B)$ *(almost preserves $\subseteq$)*,
- $\Phi_{\mathbf{P}}(A \cap B) =_{I_{\mathbf{P}}} \Phi_{\mathbf{P}}(A) \cap \Phi_{\mathbf{P}}(B)$ *(almost preserves $\cap$)*.

Sandra Müller will also present a theorem that rules out any such map for Sacks forcing satisfying even the first two requirements.

Thank you for your attention!

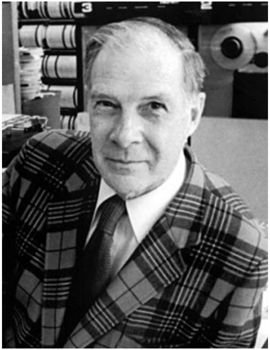

Foto source: `en.wikipedia.org/w/index.php?curid=40177109`.

“Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane.”

Richard W. Hamming, in: N. Rose, Mathematical Maxims and Minims, Raleigh NC: Rome Press Inc., 1988.

Next up: Part II by Sandra Müller

Lebesgue's Density Theorem for Ideals
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Density property

Let $\mathcal{I}$ be an ideal and $\Phi\colon \textsc{Borel} \to \textsc{Borel}$ be a function such that for all Borel sets A and B, $A =_{\mathcal{I}} B \Rightarrow \Phi(A) = \Phi(B)$.

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Definition

1. Say that Φ is $\mathcal{I}$-*compatible* iff

$$A \subseteq_{\mathcal{I}} B \Rightarrow \Phi(A) \subseteq_{\mathcal{I}} \Phi(B)$$

and
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for all Borel sets A and B.

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2. Say that Φ is $\mathcal{I}$-*positive* iff $\Phi(A) \cap A \notin \mathcal{I}$ for all Borel sets $A \notin \mathcal{I}$.

Proposition

The following statements are equivalent.

1. Φ *is* $\mathcal{I}$*-compatible and* $\mathcal{I}$*-positive.*
2. Φ *has the* $\mathcal{I}$*-density property, i.e.* $\Phi(A) =_{\mathcal{I}} A$ *for all Borel sets* A*.*

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We first show that $B_0 = A \setminus \Phi(A) \in \mathcal{I}$. Towards a contradiction, assume that $B_0 \notin \mathcal{I}$. Then $\Phi(B_0) \setminus \Phi(A) \notin \mathcal{I}$, since it contains $\Phi(B_0) \cap B_0$ as a subset, and the latter is not in $\mathcal{I}$ since Φ is $\mathcal{I}$-positive. On the other hand, we have $\Phi(B_0) \setminus \Phi(A) \in \mathcal{I}$ since $B_0 \subseteq_{\mathcal{I}} A$ and Φ is $\mathcal{I}$-compatible.

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It remains to show that $B_1 = \Phi(A) \setminus A \in \mathcal{I}$.

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It remains to show that $B_1 = \Phi(A) \setminus A \in \mathcal{I}$. Assume that $B_1 \notin \mathcal{I}$, so in particular $\Phi(A) \notin \mathcal{I}$. The set $C = \Phi(B_1) \cap B_1 \notin \mathcal{I}$, since Φ is $\mathcal{I}$-positive.

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For which ideals $\mathcal{I}$ is our density function Φ_s $\mathcal{I}$-compatible?

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Definition

A tree forcing $\mathbf{P}$ has the *stem property* if for all $T \in \mathbf{P}$ and $\mathcal{I}$-almost all $x \in [T]$, there are infinitely many $n \in \omega$ such that there is some $T' \leq T$ with $x \in [T']$ and $\mathrm{stem}_{T'} = x \restriction n$.

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Lemma

Let $\mathbf{P}$ *be a ccc tree forcing with the stem property and* $\mathcal{I} = \mathcal{I}_{\mathbf{P}}$*. Then* Φ_s *is* $\mathcal{I}$*-compatible.*

$\mathcal{I}$-positivity for strongly linked forcings

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A tree forcing $\mathbf{P}$ is *strongly linked* if any $S, T \in \mathbf{P}$ with $\mathrm{stem}_S \subseteq \mathrm{stem}_T$ and $\mathrm{stem}_T \in S$ are compatible in $\mathbf{P}$.

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Note that strongly linked implies σ-linked and hence ccc.

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Note that strongly linked implies σ-linked and hence ccc.

Lemma

Let $\mathbf{P}$ be a strongly linked tree forcing with the stem property and $\mathcal{I} = \mathcal{I}_{\mathbf{P}}$. Let $T \in \mathbf{P}$. Then $\mathcal{I}$-almost all $x \in [T]$ are $\mathcal{I}$-density points of $[T]$.

This implies that Φ_{s} is $\mathcal{I}_{\mathbf{P}}$-positive.

Examples: Ideals with the density property

Corollary

Suppose $\mathbf{P}$ is a strongly linked tree forcing with the stem property and let $\mathcal{I} = \mathcal{I}_{\mathbf{P}}$. Then Φ_s has the density property.

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Corollary

Suppose $\mathbf{P}$ is a strongly linked tree forcing with the stem property and let $\mathcal{I} = \mathcal{I}_{\mathbf{P}}$. Then Φ_s has the density property.

In particular, Φ_s has the density property for

- Cohen forcing $\mathbf{C}$,
- Hechler forcing $\mathbf{H}$,
- eventually different reals forcing $\mathbf{E}$,
- Laver forcing with a filter $\mathbf{L}_F$, and
- Mathias forcing with a translation invariant filter $\mathbf{R}_F$.

Ideals without the density property

How about non-ccc ideals?

Ideals without the density property

How about non-ccc ideals?

Proposition

Φ_s *does not have the density property for*

- *Mathias forcing* $\mathbf{R}$*,*
- *Silver forcing* $\mathbf{V}$*,*
- *Sacks forcing* $\mathbf{S}$*,*
- *Laver forcing* $\mathbf{L}$*, and*
- *Miller forcing* $\mathbf{M}$*.*

A strong failure of the density property for Sacks forcing

There is no Baire measurable function Φ with the density property yielding a notion of density points for the ideal $\mathcal{I}$ of countable sets.

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Definition

A *selector* for an equivalence relation is a function that picks an element from each equivalence class. Here we will have equivalence relations $E \subseteq F$ on a set Y and a selector for the equivalence relation induced by F on Y/E.

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Definition

A *selector* for an equivalence relation is a function that picks an element from each equivalence class. Here we will have equivalence relations $E \subseteq F$ on a set Y and a selector for the equivalence relation induced by F on Y/E.

Let Λ denote the set of Borel codes and let B_x denote the set with code $x \in \Lambda$. Moreover, consider the following equivalence relations on Λ:

$$(x, y) \in E_= \iff B_x = B_y$$

and

$$(x, y) \in E_{\mathcal{I}} \iff B_x \triangle B_y \in \mathcal{I}.$$

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Definition

A *selector for* $\mathcal{I}$ *with Borel values* is a selector for $E_{\mathcal{I}}/E_{=}$ on Λ.

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Definition

A *selector for $\mathcal{I}$ with Borel values* is a selector for $E_{\mathcal{I}}/E_{=}$ on Λ.

Theorem

There is no Baire measurable selector for $\mathcal{I}$ with Borel values.

A strong failure of the density property for Sacks forcing

Definition

A *selector for* $\mathcal{I}$ *with Borel values* is a selector for $E_{\mathcal{I}}/E_{=}$ on Λ.

Theorem

There is no Baire measurable selector for $\mathcal{I}$ *with Borel values.*

Almost the same proof also shows:

Theorem

1. *There is no Baire measurable selector for* $\mathcal{I}$ *with* $\mathbf{\Sigma}^1_2$ *values.*
2. *Assuming* PD*, there is no Baire measurable selector for* $\mathcal{I}$ *with projective values.*

Open questions

Question

Is the existence of a simply definable selector equivalent to the ccc for all homogeneous σ-ideals?

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Is the existence of a simply definable selector equivalent to the ccc for all homogeneous σ-ideals?

Question

Is there a Baire measurable selector with Borel values for other non-ccc ideals?

"This new integral of Lebesgue is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which were impregnable."

Edward Burr Van Vleck, Current Tendencies of Mathematical Research, Bulletin of the American Mathematical Society (Oct 1916)

Thank you for your attention!