# The Edinburgh Topology

Peter Holy / Wolfgang Wohofsky presenting joint work with Marlene Koelbing and Philipp Schlicht

University of Bonn / University of Kiel

pholy@math.uni-bonn.de / wolfgang.wohofsky@gmx.at

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## The Edinburgh topology

Let  $\kappa$  be a regular and uncountable cardinal. Let  $\mathrm{NS}_\kappa$  denote the non-stationary ideal on  $\kappa$ . In a nutshell, the Edinburgh topology is obtained from the bounded topology by working with non-stationary rather than bounded subsets of  $\kappa$ . One could also generalize this further by using arbitrary  $<\kappa$ -closed ideals on  $\kappa$ .

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#### Definition

Let  $\heartsuit = \{f : A \to 2 \mid A \in \mathrm{NS}_{\kappa}\}$ . The Edinburgh topology on  $2^{\kappa}$  is provided by the basis  $\{[f] \mid f \in \heartsuit\}$  of (Edinburgh) clopen subsets of  $2^{\kappa}$ , where  $[f] = \{g \in 2^{\kappa} \mid f \subseteq g\}$ .

Many, but not all of our results also apply to the generalized Baire space  $\kappa^{\kappa}$  rather than the generalized Cantor space  $2^{\kappa}$ .

### Some basic observations

- The Edinburgh topology refines the bounded topology.
- The (above/smallest) basis of the Edinburgh topology has size  $2^{\kappa}$ .
- There are  $2^{\kappa}$  many disjoint sets from the above basis.
- Hence there are  $2^{2^{\kappa}}$  many Edinburgh open sets.

In the following, we will abbreviate Edinburgh by the letter E, for example, we will say E-open, E-closed, etc.

I want to provide some arguments showing that the Edinburgh topology leads to an interesting structure theory of Edinburgh Borel sets.

### A normal form for E-closed sets

For  $x \in 2^{\kappa}$ , let  $F_x = \{x \upharpoonright A \mid A \in \mathrm{NS}_{\kappa}\}$  be the filter on  $\heartsuit$  generated by x.

## Proposition

If  $P\subseteq \heartsuit$ , then  $[P]=\{x\in 2^\kappa\mid F_x\subseteq P\}$  is an E-closed subset of  $2^\kappa$ . Conversely, every E-closed subset of  $2^\kappa$  is of the form [P] for some  $P\subseteq \heartsuit$  that is closed under restrictions.

*Proof:* If  $X \subseteq 2^{\kappa}$  is E-closed, let  $P = \{x \upharpoonright A \mid x \in X \land A \in \mathrm{NS}_{\kappa}\}$ . Now if  $x \in X$ , then clearly  $F_x \subseteq P$ . If  $x \notin X$ , then since X is E-closed, there is  $A \in \mathrm{NS}_{\kappa}$  with  $X \cap [x \upharpoonright A] = \emptyset$ . But then,  $x \upharpoonright A \notin P$ , hence also  $F_x \not\subseteq P$ . The first statement of the proposition is verified similarly.  $\square$ 

We will see in the following that this normal form is actually useful.

# An Edinburgh open set that is not Edinburgh $F_{\sigma}$

Let  $U = \{x \in 2^{\kappa} \mid x \subseteq \kappa \text{ is unbounded}\}$ . Note: U is Edinburgh open!

### Proposition

*U* is not Edinburgh  $F_{\sigma}$ , i.e. no  $\kappa$ -union of *E*-closed sets.

*Proof:* Assume for a contradiction that it is, i.e.  $U=\bigcup_{\alpha<\kappa}[P_\alpha]$ , with each  $P_\alpha\subseteq \heartsuit$ . We inductively construct an unbounded subset of  $\kappa$  which is not in U. We say that  $f\in \heartsuit$  is bounded in  $\kappa$  if  $\{\gamma<\kappa\mid f(\gamma)=1\}$  is. Starting with  $f_0=\emptyset$ , we construct a continuous and increasing  $\kappa$ -sequence of bounded  $f_\alpha$ 's in  $\heartsuit$  so that  $f_{\alpha+1}(\gamma)=1$  for some  $\gamma\geq\alpha$ , and so that  $f_{\alpha+1}\not\in P_\alpha$  for all  $\alpha<\kappa$ : If some  $P_\alpha$  contained all bounded extensions of some extension of  $f_\alpha$ , then  $[P_\alpha]$  would have to contain a bounded set. In the end,  $f=\bigcup_{\alpha<\kappa}f_\alpha$  is an unbounded subset of  $\kappa$  which is not in  $\bigcup_{\alpha<\kappa}[P_\alpha]$ , yielding our desired contradiction.  $\square$ 

# An Edinburgh $G_{\delta}$ set that is neither E-open nor E- $F_{\sigma}$

Let  $\mathcal{C}=\{x\in 2^\kappa\mid x\subseteq \kappa \text{ is club}\}$ . Since  $\mathcal{C}$  is  $G_\delta$ , it is also E- $G_\delta$ . Since every E-open set contains a nonstationary set,  $\mathcal{C}$  is not E-open.

### **Proposition**

 $\mathcal{C}$  is not Edinburgh  $F_{\sigma}$ .

*Proof Sketch:* Assume for a contradiction that  $\mathcal{C} = \bigcup_{\alpha < \kappa} [P_{\alpha}]$  with each  $P_{\alpha} \subseteq \heartsuit$ . A slightly more careful diagonalization argument as for the set U above, this time using closed and bounded subsets of  $\kappa$ , adding only a single function value 1 in each step, and making sure this happens exactly on a club subset C of  $\kappa$ , yields that  $C \not\in \bigcup_{\alpha < \kappa} [P_{\alpha}]$ , our desired contradiction.  $\square$ 

# The club filter is not Edinburgh Borel

Note first that the club filter is both Edinburgh dense and co-dense. Moreover, as usual, one can verify the Baire category theorem for the Edinburgh topology. That is, every  $\kappa$ -intersection of Edinburgh open dense sets is Edinburgh dense. The same proof shows also that every  $\kappa$ -intersection of Edinburgh open dense sets contains both an element of the club filter, and of the nonstationary ideal. It follows, as usual, that the club filter cannot have the Edinburgh Baire property. However, again as usual, every Edinburgh Borel set does have the Edinburgh Baire property.

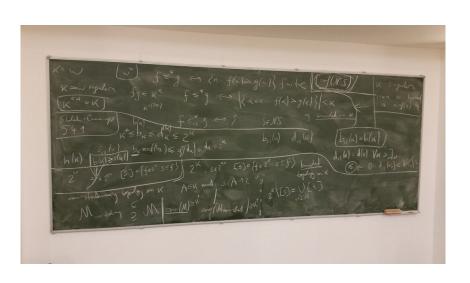
### Corollary

The club filter is not Edinburgh Borel. In particular, not every subset of  $2^{\kappa}$  is Edinburgh Borel.









# Edinburgh meager sets

# Edinburgh meager sets



**Recall:**  $\heartsuit$  denotes the set of partial functions f from  $\kappa$  to 2 with dom(f) non-stationary.

## Edinburgh topology

... is generated by  $\{[f]: f \in \heartsuit\}$ 

... where 
$$[f] = \{x \in 2^{\kappa} : f \subseteq x\}$$

- $X \subseteq 2^{\kappa}$  is Edinburgh open if  $X = \bigcup_{i \in I} [f_i]$  (with  $f_i \in \heartsuit$ )
- $X \subseteq 2^{\kappa}$  is Edinburgh nowhere dense if for each  $f \in \emptyset$  there is  $g \in \emptyset$  such that  $g \supseteq f$  and  $X \cap [g] = \emptyset$
- $X \subseteq 2^{\kappa}$  is Edinburgh meager if  $X \subseteq \bigcup_{i < \kappa} A_i$  with each  $A_i$  Edinburgh (closed) nowhere dense
- $\bullet$  Baire Category holds: intersection of  $\kappa$  many Edinburgh open dense sets is Edinburgh dense
  - 2<sup>κ</sup> is not Edinburgh meager
  - ▶ [f] is not Edinburgh meager (for any  $f \in \heartsuit$ )

Basic properties of Edinburgh nowhere dense sets:

- ullet every set of size  $< 2^{\kappa}$  is Edinburgh nowhere dense
- there is an Edinburgh nowhere dense set of size  $2^{\kappa}$ 
  - there is even a closed such set:

$$\{x \in 2^{\kappa} : x(\alpha) = x(\alpha + 1) \text{ for each even } \alpha < \kappa\}$$

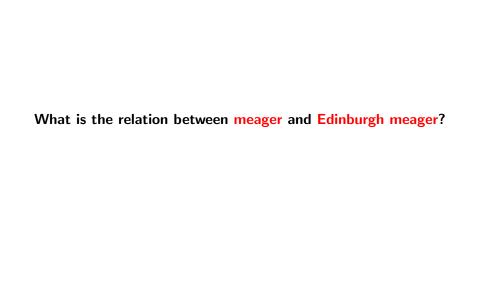
### Question

Edinburgh nowhere dense  $\subseteq$  Edinburgh meager?

### Theorem

Assume  $\kappa$  is inaccessible or  $\Diamond_{\kappa}$  holds.

Then every Edinburgh meager set is Edinburgh nowhere dense.



If  $f \in \mathbb{O}$  and  $|\text{dom}(f)| = \kappa$ , then [f] is closed nowhere dense.

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If  $f \in \mathcal{O}$  and  $|\operatorname{dom}(f)| = \kappa$ , then [f] is closed nowhere dense.



### Proposition

So there is a meager set which is not Edinburgh meager.

This was the easy direction.



... so what about the other direction?

Here the situation is more subtle and interesting:



#### Lemma

Assume  $\kappa$  is inaccessible or  $\Diamond_{\kappa}$  holds.

Then each co-meager set contains an Edinburgh cone [f]:

given 
$$(D_{\alpha})_{\alpha<\kappa}$$
 open dense  $\exists f\in \emptyset$   $\bigcap_{\alpha<\kappa}D_{\alpha}\supseteq [f]$ 

#### Theorem

Assume X has the **Baire property** and the conclusion of the lemma holds. Then "X Edinburgh meager" implies "X meager".

For  $a, y \in [\kappa]^{\kappa}$ , we say that a splits y if  $a \cap y$  and  $(\kappa \setminus a) \cap y$  are of size  $\kappa$ .

### Definition

A reaping family on  $\kappa$  is a set  $\mathcal{R} \subseteq [\kappa]^{\kappa}$  such that no  $a \in [\kappa]^{\kappa}$  splits all  $y \in \mathcal{R}$ .  $\mathfrak{r}(\kappa)$  is the smallest size of a reaping family on  $\kappa$ .

## Theorem ( $\kappa$ inacce<u>ssible)</u>

Assume  $\mathfrak{r}(\kappa) = 2^{\kappa}$ .

Then there is an Edinburgh meager set which is not meager.

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Let  $\mathfrak{S}$  denote the set of partial functions from  $\kappa$  to 2 with  $|\text{dom}(f)| = \kappa$ .

### **Definition**

 $\mathfrak{ph}(\kappa)$  is the smallest size of a family  $\mathcal{F}\subseteq\mathfrak{S}$  such that  $\bigcup_{f\in\mathcal{F}}[f]=2^{\kappa}.$ 

## Lemma ( $\kappa$ inaccessible)

 $\mathfrak{ph}(\kappa)$  is the smallest size of a family  $\mathcal{F}\subseteq\mathfrak{S}$  such that  $\bigcup_{f\in\mathcal{F}}[f]$  is co-meager.

## Lemma ( $|2^{<\kappa}| = \kappa$ )

 $\mathfrak{ph}(\kappa) = \mathfrak{r}(\kappa).$ 

### Thank you for your attention and enjoy Amsterdam...



Edinburgh, July 2017