A Note on *Inference Systems for Update Semantics* by Willem Groeneveld and Frank Veltman

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Although Section 2 of the paper introduces many useful general techniques, there still seems to be a large gap between what has been learned about abstract update semantics and the results about the concrete systems of Veltman. In this note, I will make one proposal that I hope will narrow this gap.

One thing that makes it difficult to compare Veltman's logic of *might* with a more general system is that the semantic clause for *might* is designed to apply only to a very special kind of model, with a distinguished bottom element. In Veltman's model, this is $\emptyset$, which is the empty set of possible worlds. To study an abstract *might* logic, it is desirable to be able to interpret *might* in any model (or frame, in the authors' terminology) that interprets atomic formulas as arbitray binary relations on the states. In fact, there is a natural way to achieve this, if we make the following simple observation. Let us recall Veltman's semantic clause for *might*:

$$[\![\mathit{might}\, \varphi]\!] = \{\, \langle \sigma, \emptyset \rangle \mid \sigma[\![\varphi]\!] = \emptyset \,\} \cup \{\, \langle \sigma, \sigma \rangle \mid \sigma[\![\varphi]\!] \neq \emptyset \,\}. \tag{1}$$

In Veltman's model, $\emptyset$ is like a black hole: once you get there, there's no way to get out of it. The interpretation of every formula loops at $\emptyset$. This is exactly like the dummy state $\perp$ that Definition 13 of the paper introduces to turn a partial functional model into a total functional one. Applying this technique backwards, one sees that the submodel of Veltman's model that you get by simply throwing away $\emptyset$ is equivalent to the original model. In this new model, all formulas denote partial functions. The semantic clause for *might* can then be simplified as follows:

$$[\![\mathit{might}\, \varphi]\!] = \{\, \langle \sigma, \sigma \rangle \mid \sigma \in \mathrm{dom}([\![\varphi]\!]) \,\}. \tag{2}$$

Clearly, the interpretation of *might* φ given by this definition in our new model is the same as the result of restricting the original interpretation of *might* φ to the states of the new model.

Unlike (1), (2) makes sense in all models, not only those in which atomic formulas denote partial functions. The operator *might* is now a simple domain operator. Since the clipped Veltman model is just one special model, the logical rules governing the new *might* operator should be a subset of the rules for *might* in Veltman's logic.

Let our language consist of atomic formulas $p_0, p_1, p_2, \ldots$ and formulas of the form $\mathit{might}\, p$, where p is atomic. Prohibition to iterate *might* is not important here, since $\mathit{might}\, \mathit{might}\, \varphi$ would be simply equivalent to $\mathit{might}\, \varphi$. By a translation to PDL by

$$\mathrm{tr}(\mathit{might}\, \varphi) = (\langle \mathrm{tr}(\varphi) \rangle \top)?,$$

one can see that the (global) Update-Test consequence relation

$$\Gamma \models X \Rightarrow \varphi$$

between finite sets of sequents Γ and sequents $X \Rightarrow \varphi$ in this language is recursive (see Kanazawa 1994). A complete calculus witnessing the recursive enumerability of this relation is not hard to find, and consists of the rules in Figure 1, in addition to Left Monotonicity and Cautious Cut. Let us call this calculus **AML**. Note that the rules

$$\mathit{might}\text{-Contraction}\ \frac{X_1\ \mathit{might}\, \varphi\ \mathit{might}\, \varphi\ X_2 \Rightarrow \psi}{X_1\ \mathit{might}\, \varphi\ X_2 \Rightarrow \psi}$$

$$\mathit{might}\text{-Permutation}\ \frac{X_1\ \mathit{might}\, \varphi\ \mathit{might}\, \psi\ X_2 \Rightarrow \chi}{X_1\ \mathit{might}\, \psi\ \mathit{might}\, \varphi\ X_2 \Rightarrow \chi}$$

are derivable in this calculus by *might*-Reflexivity, *might*-Monotonicity, and Cautious Cut.

$$\mathit{might}\text{-Reflexivity}\ \frac{}{X\ \mathit{might}\, \varphi \Rightarrow \mathit{might}\, \varphi}$$

$$\mathit{might}\text{-Monotonicity}\ \frac{X_1\ X_2 \Rightarrow \psi}{X_1\ \mathit{might}\, \varphi\ X_2 \Rightarrow \psi}$$

$$(\mathit{might} \Rightarrow)\ \frac{X_1\ \mathit{might}\, \varphi\ \varphi\ X_2 \Rightarrow \psi}{X_1\ \varphi\ X_2 \Rightarrow \psi} \qquad (\Rightarrow \mathit{might})\ \frac{X \Rightarrow \varphi}{X \Rightarrow \mathit{might}\, \varphi}$$

Figure 1: Rules for abstract *might* logic **AML**.

Since **AML** does not have idempotency for atomic formulas, Groeneveld's general 'Henkin-type' method is not applicable here. There is a straightforward 'representation' method, however, that shows completeness. We let X (possibly with subscripts) range over finite sequences of formulas, and Y (possibly with subscripts) range over finite sequences of formulas of the form $\mathit{might}\, p$ (*might*-formulas). We write $\Gamma \vdash X \Rightarrow \varphi$ to mean $X \Rightarrow \varphi$ is derivable from Γ by **AML**.

Definition 1 Let Γ be any finite set of sequents. M_Γ is the model $\langle |M_\Gamma|, [\![\cdot]\!]_{M_\Gamma} \rangle$, where

- $|M_\Gamma| = \{\, X \mid X$ is a finite sequence of formulas $\}$.
- $[\![p]\!]_{M_\Gamma} = \{\, \langle X, X\ p\ Y \rangle \mid \Gamma \vdash X \Rightarrow \mathit{might}\, p \wedge$ Y is a sequence of *might*-formulas $\} \cup \{\, \langle X, X \rangle \mid \Gamma \vdash X \Rightarrow p \,\}$.

Lemma 2 *For every finite sequence of formulas X and every formula φ,*

$$\langle X, X \rangle \in [\![\varphi]\!]_{M_\Gamma} \quad \textit{iff} \quad \Gamma \vdash X \Rightarrow \varphi.$$

Proof. For $\varphi = p$, it holds by definition. For $\varphi = \mathit{might}\, p$, this follows from ($\Rightarrow$ *might*) (for left-to-right) and the definition of M_Γ. ∎

Unfortunately, the 'Groeneveld equivalence'

$$M_\Gamma, X_1 \models X_2 \Rightarrow \varphi \quad \text{iff} \quad \Gamma \vdash X_1\ X_2 \Rightarrow \varphi \tag{3}$$

does not always hold. (If $\Gamma \not\vdash X_1 \Rightarrow \mathit{might}\, p$, then $M_\Gamma, X_1 \models p\ X_2 \Rightarrow \varphi$ for all X_2, φ.) We do have one half of (3), however, and it is enough to prove $M_\Gamma \models \Gamma$.

Lemma 3 $\Gamma \vdash X_1\ X_2 \Rightarrow \varphi$ *implies* $M_\Gamma, X_1 \models X_2 \Rightarrow \varphi$.

Proof. Induction on the length of X_2.

Induction Basis. $X_2 = \mathbf{\Lambda}$. By (one half of) Lemma 2.

Induction Step. $X_2 = \psi\ X_3$. The assumption is

$$\Gamma \vdash X_1\ \psi\ X_3 \Rightarrow \varphi. \tag{4}$$

To show $M_\Gamma, X_1 \models \psi\ X_3 \Rightarrow \varphi$, we show that for all Z such that $X_1 \xrightarrow{\psi} Z$, $M_\Gamma, Z \models X_3 \Rightarrow \varphi$. So assume $X \xrightarrow{\psi} Z$.

Case 1. $\psi = p$.

Case 1a. $Z = X_1\ p\ Y$. By (4) and *might*-Monotonicity, $\Gamma \vdash X_1\ p\ Y\ X_3 \Rightarrow \varphi$. By induction hypothesis, $M_\Gamma, X_1\ p\ Y \models X_3 \Rightarrow \varphi$.

Case 1b. $Z = X_1$. Then $\Gamma \vdash X_1 \Rightarrow p$ by the construction of M_Γ. By (4) and Cautious Cut, $\Gamma \vdash X_1\ X_3 \Rightarrow p$. By induction hypothesis, $M_\Gamma, X_1 \models X_3 \Rightarrow p$.

Case 2. $\varphi = \mathit{might}\, p$. Then $Z = X_1$ and by Lemma 2, $\Gamma \vdash X_1 \Rightarrow \mathit{might}\, p$. By (4) and Cautious Cut, $\Gamma \vdash X_1\ X_3 \Rightarrow \varphi$. By induction hypothesis, $M_\Gamma, X_1 \models X_3 \Rightarrow \varphi$. ∎

Lemma 4 $\Gamma \vdash X \Rightarrow \varphi$ *implies* $M_\Gamma \models X \Rightarrow \varphi$.

Proof. By Lemma 3 and Left Monotonicity. ∎

Let us now show the converse of Lemma 4. Let X be a finite sequence of formulas. By the *might-prefix* of X, $\mathrm{mprefix}(X)$, we mean the longest prefix of X that entirely consists of *might*-formulas. By the *fattening* of X, $\mathrm{fattening}(X)$, we mean the result of replacing each occurrence of an atomic formula p by $\mathit{might}\, p\ p$.

Lemma 5 *For every finite sequence of formulas X and every formula φ,*

$$\Gamma \vdash X \Rightarrow \varphi \quad \textit{iff} \quad \Gamma \vdash \mathrm{fattening}(X) \Rightarrow \varphi.$$

PROOF. By *might*-Monotonicity and (*might* $\Rightarrow$). ∎

Lemma 6 *For every finite sequence of atomic formulas X,*

$$\mathrm{mprefix}(\mathrm{fattening}(X)) \xrightarrow{X} \mathrm{fattening}(X)$$

in M_Γ.

PROOF. By *might*-Reflexivity and *might*-Monotonicity. ∎

Lemma 7 $M_\Gamma \models X \Rightarrow \varphi$ *implies* $\Gamma \vdash X \Rightarrow \varphi$.

PROOF. By Lemmas 2 and 6, $M_\Gamma, \mathrm{mprefix}(\mathrm{fattening}(X)) \models X \Rightarrow \varphi$ implies $\Gamma \vdash \mathrm{fattening}(X) \Rightarrow \varphi$. By (one half of) Lemma 5, $\Gamma \vdash X \Rightarrow \varphi$ follows. ∎

By Lemmas 4 and 7, we get

Theorem 8 *Let Γ be a finite set of sequents. For every sequent $X \Rightarrow \varphi$,*

$$\Gamma \vdash X \Rightarrow \varphi \quad \textit{iff} \quad M_\Gamma \models X \Rightarrow p.$$

$$\frac{X \Rightarrow \varphi}{X \Rightarrow \mathit{might}\,\varphi} \qquad \frac{X\ \varphi \Rightarrow \bot}{X\ \mathit{might}\,\varphi \Rightarrow \bot} \qquad \frac{X \Rightarrow \bot}{X \Rightarrow \psi}$$

$$\frac{X \Rightarrow \mathit{might}\,\psi \quad X\ \psi \Rightarrow \varphi}{X\ \varphi \Rightarrow \mathit{might}\,\psi} \qquad \frac{X\ \varphi \Rightarrow \psi}{X\ \mathit{might}\,\varphi \Rightarrow \psi}$$

Figure 2: Rules for concrete *might* logic.

Having shown the completeness of **AML**, let us compare it to the *might*-rules for Update-Test Consequence given in the paper (Figure 2).

The first rule is simply our ($\Rightarrow$ *might*). The second and the third rules can be seen to be valid in the abstract setting if we set $[\![\bot]\!] = \emptyset$, which is the interpretation we get by clipping the Veltman model. The last two rules are invalid in our abstract setting—it is easy to produce countermodels—and can be seen to depend on some special features of the concrete model. Our rules *might*-Reflexivity and *might*-Monotonicity are present in the concrete *might* logic in the form of general Reflexivity and Monotonicity. What about our rule (*might* $\Rightarrow$)? I conjecture that this is an admissible, but underivable rule of the concrete *might* logic.

References

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