Structural Control *

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September 8, 1995

Abstract

In this paper we study Lambek systems as *grammar logics*: logics for reasoning about structured linguistic resources. The structural parameters of precedence, dominance and dependency generate a cube of resource-sensitive categorial type logics. From the pure logic of residuation **NL**, one obtains **L**, **NLP** and **LP** in terms of Associativity, Commutativity, and their combination. Each of these systems has a dependency variant, where the product is split up into a left-headed and a right-headed version.

We develop a theory of systematic communication between these systems. The communication is two-way: we show how one can fully recover the structural discrimination of a weaker logic from within a system with a more liberal resource management regime, and how one can reintroduce the structural flexibility of a stronger logic within a system with a more articulate notion of structure-sensitivity.

In executing this programme we follow the standard logical agenda: the categorial formula language is enriched with extra control operators, so-called structural modalities, and on the basis of these control operators, we prove embedding theorems for the two directions of substructural communication. But our results differ from the Linear Logic style of embedding with $S4$-like modalities in that we realize the communication in both directions in terms of a *minimal* pair of structural modalities. The control devices $\Diamond, \Box^{\downarrow}$ used here represent the pure logic of residuation for a family of unary multiplicatives: they do not impose any restrictions on the binary accessibility relation interpreting the unary modalities, unlike the $S4$ operators which require a transitive and reflexive interpretation. With the more delicate control devices we can avoid the model-theoretic and prooftheoretic problems one encounters when importing the Linear Logic modalities in a linguistic setting.

1 Logics of structured resources

This paper is concerned with the issue of *communication* between categorial type logics of the Lambek family. Lambek calculi occupy a lively corner in the broader landscape of resource-sensitive systems of inference. We study these systems here as grammar logics. In line with the 'Parsing as Deduction' slogan, we present the key concept in grammatical analysis — *well-formedness* — in logical terms, i.e.

*An earlier version of this paper appeared under the title 'Controlling resource management' in Deliverable R1.1.B of the Esprit BRA 6852 Dyana-2 (Dynamic Interpretation of Natural Language). The present version is to appear, with slight modifications, in Blackburn & de Rijke (eds) *Logic, Structures and Syntax*. We thank Johan van Benthem and Dick Oehrle for comments. All errors are our own.

grammatical well-formedness amounts to *derivability* in our grammar logic. In the grammatical application, the resources we are talking about are linguistic expressions — multidimensional form-meaning complexes, or *signs* as they have come to be called in current grammar formalisms. These resources are structured in a number of grammatically relevant dimensions. For the sake of concreteness, we concentrate on three types of linguistic structure of central importance: linear order, hierarchical grouping (constituency) and dependency. The structure of the linguistic resources in these dimensions plays a crucial role in determining well-formedness: one cannot generally assume that changes in the structural configuration of the resources will preserve well-formedness. In logical terms, we are interested in structure-sensitive notions of linguistic inference.

Fig 1 charts the eight logics that result from the interplay of the structural parameters of precedence, dominance and dependency. The systems lower in the cube exhibit a more fine-grained sense of structure-sensitivity; their neighbours higher up loose discrimination for one of the structural parameters we distinguish here.

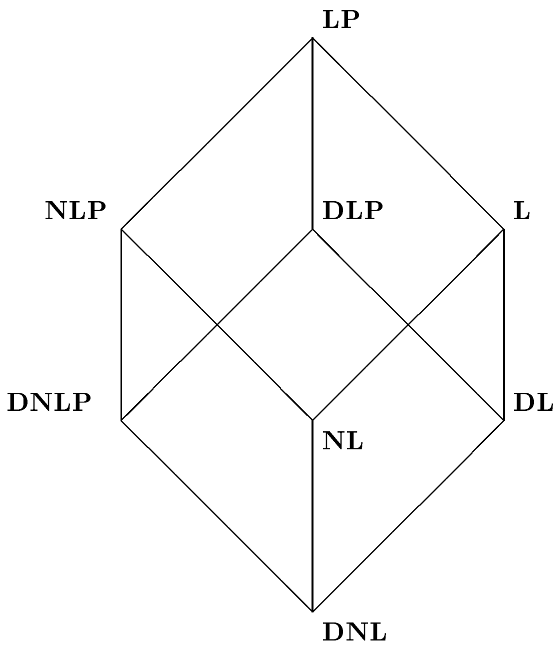


Figure 1: Resource-sensitive logics: precedence, dominance, dependency

Let us present the essentials (syntactically and semantically) of the framework we are assuming before addressing the communication problem. For a fuller treatment of multimodal categorial architecture, the reader can turn to [Moortgat 95, M & O 94, M & M 91, M & O 93, Morrill 94]. Consider the standard language of categorial type formulae $\mathcal{F}$ freely generated from a set of atomic formulae $\mathcal{A}$: $\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}/\mathcal{F} \mid \mathcal{F} \bullet \mathcal{F} \mid \mathcal{F}\backslash\mathcal{F}$. The most general interpretation for such a language can be given in terms of Kripke style relational structures — ternary relational structures $\langle W, R^3\rangle$ in the case of the binary connectives (cf. [Došen 92]). W here is to be understood as the set of linguistic resources (signs) and the accessibility relation R as representing linguistic composition. From a ternary frame we obtain a model by adding a valuation V sending prime formulae to subsets of W and satisfying the clauses below for compound formulae.

$$\begin{array}{rcl} V(A \bullet B) & = & \{z \mid \exists x \exists y [Rzxy \ \& \ x \in V(A) \ \& \ y \in V(B)]\} \\ V(C/B) & = & \{x \mid \forall y \forall z [(Rzxy \ \& \ y \in V(B)) \Rightarrow z \in V(C)]\} \\ V(A\backslash C) & = & \{y \mid \forall x \forall z [(Rzxy \ \& \ x \in V(A)) \Rightarrow z \in V(C)]\} \end{array}$$

With no restrictions on R, we obtain the pure logic of residuation known as **NL**.

$$\textsc{res}(2) \qquad A \to C/B \quad \Longleftrightarrow \quad A \bullet B \to C \quad \Longleftrightarrow \quad B \to A\backslash C$$

And with restrictions on the interpretation of R, and corresponding structural postulates, we obtain the systems **NLP**, **L** and **LP**. Below we give the structural postulates of Associativity (A) and Permutation (P) and the corresponding frame conditions $F(A)$ and $F(P)$. Notice that the structural discrimination gets coarser as we impose more constraints on the interpretation of R. In the presence of Permutation, well-formedness is unaffected by changes in the linear order of the linguistic resources. In the presence of Associativity, different groupings of the linguistic resources into hierarchical constituent structures has no influence on derivability.

$$\begin{array}{rc} (A) & A \bullet (B \bullet C) \longleftrightarrow (A \bullet B) \bullet C \\ F(A) & (\forall xyz \in W)\ \exists t.Rxyt \ \& \ Rtzu \Leftrightarrow \exists v.Rvyz \ \& \ Rxvu \\ (P) & A \bullet B \to B \bullet A \\ F(P) & (\forall xyz \in W)\ Rxyz \Leftrightarrow Rxzy \end{array}$$

What we have said so far concerns the upper face of the cube of Fig 1. To obtain the systems at the lower face, we split the connective $\bullet$ in left-headed $\bullet_l$ and right-headed $\bullet_r$, taking into account the asymmetry between heads and dependents. It is argued in [M & M 91] that the dependency dimension should be treated as orthogonal in principle to the functor/argument asymmetry. The distinction between left-headed $\bullet_l$ and right-headed $\bullet_r$ (and their residual implications) makes the type language articulate enough to discriminate between head/complement configurations, and modifier/head or specifier/head configurations. A determiner, for example, could be typed as $np/_r n$. Such a declaration naturally accounts for the fact that determiners act semantically as functions from n-type meanings to np-type meanings, whereas in the form dimension they should be treated as dependent on the common noun they are in construction with, so that they can derive their agreement properties from the head noun.

In the Kripke models, the lower plane of Fig 1 is obtained by moving from unimodal to multimodal (in this case: bimodal) frames $\langle W, R^3_l, R^3_r \rangle$, with a distinct accessibility relation for each product. Again, we have the pure (bimodal) logic of residuation **DNL**, with an arbitrary interpretation for R^3_l, R^3_r, and its relatives **DNLP**, **DL**, **DLP**, obtained by imposing associativity or (dependency-preserving!) commutativity constraints on the frames. The relevant structural postulates are given below. The distinction between the left-headed and right-headed connectives is destroyed by the postulate (D).

$$\begin{array}{rc} (A_l) & A \bullet_l (B \bullet_l C) \longleftrightarrow (A \bullet_l B) \bullet_l C \\ (A_r) & A \bullet_r (B \bullet_r C) \longleftrightarrow (A \bullet_r B) \bullet_r C \\ (P_{l,r}) & A \bullet_l B \longleftrightarrow B \bullet_r A \\ (D) & A \bullet_l B \longleftrightarrow A \bullet_r B \end{array}$$

It will be clear already from the foregoing that in presenting the grammar for a given language, we will in general not be in a position to restrict ourselves to one particular type logic — we want to have access to the combined inferential capacities of the different logics, without destroying their individual characteristics. For this to be possible we need a theory of systematic *communication* between type systems.

The structural postulates presented above do not have the required granularity for such a theory of communication: they globally destroy structure sensitivity in one of the relevant dimensions, whereas we would like to have *lexical control* over resource management. Depending on the direction of communication, one can develop two perspectives on controlled resource management. On the one hand, one would like to have control devices to license limited access to a more liberal resource management regime from within a system with a higher sense of structural discrimination. On the other hand, one would like to impose constraints on resource management in systems where such constraints are lacking by default. For discussion of linguistic phenomena motivating these two types of communication, the reader can turn to the papers in [Barry & Morrill 90] where the licensing perspective was originally introduced, and to [Morrill 94] where apart from licensing of structural relaxation one can also find discussion of constraints with respect to the associativity dimension. We give an illustration for each type of control, drawing on the references just mentioned.

LICENSING STRUCTURAL RELAXATION. For the licensing type of communication, consider type assignment to relative pronouns like *that* in the sentences below.

$$\begin{array}{rl} & \text{the book that John read} \\ & \text{the book that John read yesterday} \\ \mathbf{L} \vdash & r/(s/np), np, (np\backslash s)/np \Rightarrow r \\ \mathbf{L} \not\vdash & r/(s/np), np, (np\backslash s)/np, s\backslash s \Rightarrow r \\ \mathbf{NL} \not\vdash & (r/(s/np), (np, (np\backslash s)/np)) \Rightarrow r \end{array}$$

Suppose first we are dealing with the associative regime of $\mathbf{L}$, and assign the relative pronoun the type $r/(s/np)$, abbreviating $n\backslash n$ as r, i.e. the pronoun looks to its right for a relative clause body missing a noun phrase. The first example is derivable[1] (because 'John read np' indeed yields s), the second is not (because the hypothetical np assumption in the subderivation 'John read yesterday np' is not in the required position adjacent to the verb 'read'). We would like to refine the assignment to the relative pronoun to a type $r/(s/np^\sharp)$, where $np^\sharp$ is a noun phrase resource which has access to Permutation in virtue of its $\cdot^\sharp$ decoration. Similarly, if we change the default regime to $\mathbf{NL}$, already the first example fails on the assignment $r/(s/np)$ with the indicated constituent bracketing: although the hypothetical np in the subcomputation '((John read) np)' finds itself in the right position with respect to linear order requirements, it cannot satisfy the direct object role for 'read' being outside the clausal boundaries. A refined assignment $r/(s/np^\sharp)$ here could license the marked $np^\sharp$ a controlled access to the structural rule of Associativity which is absent in the $\mathbf{NL}$ default regime.

IMPOSING STRUCTURAL CONSTRAINTS. For the other direction of communication, we take an example from [Morrill 94] which again concerns relative clause formation, but this time in its interaction with coordination. Assume we are dealing with an associative default regime, and let the conjunction particle 'and' be polymorphically typed as $(X\backslash X)/X$. With the instantiation $X = s/np$ we can derive the first example. But, given Associativity and an instantiation $X = s$, nothing blocks the ungrammatical second example: 'Melville wrote Moby Dick and John read np' derives s, so that withdrawing the np hypothesis indeed gives s/np, the type required for the relative clause body.

$$\begin{array}{rl} & \text{the book that Melville wrote and John read} \\ \mathbf{L} \vdash & r/(s/np), np, (np\backslash s)/np, (X\backslash X)/X, np, (np\backslash s)/np \Rightarrow r \quad (X = s/np) \\ & \text{*the book that Melville wrote Moby Dick and John read} \\ \mathbf{L} \vdash & r/(s/np), np, (np\backslash s)/np, np, (X\backslash X)/X, np, (np\backslash s)/np \Rightarrow r \quad (X = s) \end{array}$$

[1]The Appendix gives axiomatic and Gentzen style presentation of the logics under discussion.

To block this violation of the so-called Coordinate Structure Constraint, while allowing Across-the-Board Extraction as exemplified by our first example, we would like to refine the type assignment for the particle 'and' to $(X\backslash X^\flat)/X$, where the intended interpretation for the marked $X^\flat$ now would be the following: after combining with the right and the left conjuncts, the $\cdot^\flat$ decoration makes the complete coordination freeze into an island configuration which is inaccessible to extraction under the default associative resource management regime.

Minimal structural modalities. Our task in the following pages is to give a logical implementation of the informal idea of decorating formulas with a label $(\cdot)^\sharp$ or $(\cdot)^\flat$, licensing extra flexibility or imposing a tighter regime for the marked formulae. The original introduction of the licensing type of communication in [Barry & Morrill 90] was inspired by the modalities '!,?' of Linear Logic — unary operators which give marked formulae access to the structural rules of Contraction and Weakening, thus making it possible to recover the full power of Intuitionistic or Classical Logic from within the resource sensitive linear variants. On the proof-theoretic level, the '!,?' operators have the properties of $S4$ modalities. It is not self-evident that $S4$ behaviour is appropriate for substructural systems *weaker* than Linear Logic — indeed [Venema 93] has criticised an $S4$ '!' in such settings for the fact that the proof rule for '!' has undesired side-effects on the meaning of other operators. On the semantic level it has been shown in [Versmissen 93] that the $S4$ regime is incomplete with respect to the linguistic interpretation which was originally intended for the structural modalities — a subalgebra interpretation in a general groupoid setting, cf. [Morrill 94] for discussion.

Given these model-theoretic and proof-theoretic problems with the use of Linear Logic modalities in linguistic analysis, we will explore a different route and develop an approach attuned to the specific domain of application of our grammar logics — a domain of structured linguistic resources.

[Moortgat 95] proposes an enrichment of the type language of categorial logics with *unary* residuated operators, interpreted in terms of a *binary* relation of accessibility. These operators will be the key devices in our strategy for controlled resource management. If we were talking about temporal organization, $\Diamond$ and $\Box^\downarrow$ could be interpreted as future possibility and past necessity, respectively. But in our grammatical application, R^2 just like R^3 is to be interpreted in terms of structural composition. Where a ternary configuration $(xyz) \in R^3$ interpreting the product connective abstractly represents putting together the components y and z into a structured configuration x in the manner indicated by R^3, a binary configuration $(xy) \in R^2$ interpreting the unary $\Diamond$ can be seen as the construction of the sign x out of a structural component y in terms of the building instructions referred to by R^2.

$$\text{RES}(1) \quad \Diamond A \rightarrow B \Longleftrightarrow A \rightarrow \Box^\downarrow B$$

$$V(\Diamond A) = \{x \mid \exists y (R^2 xy \ \wedge \ y \in V(A)\}$$
$$V(\Box^\downarrow A) = \{x \mid \forall y (R^2 yx \ \Rightarrow y \in V(A)\}$$

From the residuation laws $\text{RES}(1)$ one directly derives the monotonicity laws below and the properties of the compositions of $\Diamond$ and $\Box^\downarrow$:

$$A \rightarrow B \quad \text{implies} \quad \Diamond A \rightarrow \Diamond B \quad \text{and} \quad \Box^\downarrow A \rightarrow \Box^\downarrow B$$

$$\Diamond\Box^\downarrow A \rightarrow A \qquad A \rightarrow \Box^\downarrow\Diamond A$$

In the Appendix, we present the sequent logic for these unary operators. It is shown in [Moortgat 95] that the Gentzen presentation is equivalent to the axiomatic presentation, and that it enjoys Cut Elimination. For our examples later on we will use decidable sequent proof search.

Semantically, the pure logic of residuation for $\Diamond, \Box^{\downarrow}$ does not impose any restrictions on the interpretation of R^2. As in the case of the binary connectives, we can add structural postulates for $\Diamond$ and corresponding frame constraints on R^2. With a reflexive and transitive R^2, one obtains an $S4$ system. Our objective here is to show that one can develop a systematic theory of communication, both for the licensing and for the constraining perspective, in terms of the *minimal* structural modalities, i.e. the pure logic of residuation for $\Diamond, \Box^{\downarrow}$.

COMPLETENESS. The communication theorems to be presented in the following sections rely heavily on semantic argumentation. The cornerstone of the approach is the completeness of the logics compared, which guarantees that syntactic derivability $\vdash A \to B$ and semantic inclusion $V(A) \subseteq V(B)$ coincide for the classes of models we are interested in. For the $\mathcal{F}(/, \bullet, \backslash)$ fragment, [Došen 92] shows that **NL** is complete with respect to the class of all ternary models, and **L**, **NLP**, **LP** with respects to the classes of models satisfying the frame constraints for the relevant packages of structural postulates. The completeness results are obtained on the basis of a simple canonical model construction which directly accomodates bimodal dependency systems with $\mathcal{F}(/_i, \bullet_i, \backslash_i)$ $(i \in \{l, r\})$. And it is shown in [Moortgat 95] that the construction also extends unproblematically to the language enriched with $\Diamond, \Box^{\downarrow}$ as soon as one realizes that $\Diamond$ can be seen as a 'truncated' product and $\Box^{\downarrow}$ its residual implication.

Definition 1.1 Define the canonical model for mixed (2,3) frames as $\mathcal{M} = \langle W, R^2, R^3_i \rangle$, where

W is the set of formulae $\mathcal{F}(/_i, \bullet_i, \backslash_i, \Diamond, \Box^{\downarrow})$

$R^3_i(A, B, C)$ iff $\vdash A \to B \bullet_i C$, $R^2(A, B)$ iff $\vdash A \to \Diamond B$

$A \in V(p)$ iff $\vdash A \to p$.

The Truth Lemma then states that, for any formula ϕ, $\mathcal{M}, A \models \phi$ iff $\vdash A \to \phi$. Now suppose $V(A) \subseteq V(B)$ but $\not\vdash A \to B$. If $\not\vdash A \to B$ with the canonical valuation on the canonical frame, $A \in V(A)$ but $A \notin V(B)$ so $V(A) \not\subseteq V(B)$. Contradiction.

We have to check the Truth Lemma for the new compound formulae $\Diamond A$, $\Box^{\downarrow} A$. Below the direction that requires a little thinking.

($\Diamond$) Assume $A \in V(\Diamond B)$. We have to show $\vdash A \to \Diamond B$. $A \in V(\Diamond B)$ implies $\exists A'$ such that $R^2 A A'$ and $A' \in v(B)$. By inductive hypothesis, $\vdash A' \to B$. By Isotonicity for $\Diamond$ this implies $\vdash \Diamond A' \to \Diamond B$. We have $\vdash A \to \Diamond A'$ by (Def R^2) in the canonical frame. By Transitivity, $\vdash A \to \Diamond B$.

($\Box^{\downarrow}$) Assume $A \in V(\Box^{\downarrow} B)$. We have to show $\vdash A \to \Box^{\downarrow} B$. $A \in V(\Box^{\downarrow} B)$ implies that $\forall A'$ such that $R^2 A' A$ we have $A' \in V(B)$. Let A' be $\Diamond A$. $R^2 A' A$ holds in the canonical frame since $\vdash \Diamond A \to \Diamond A$. By inductive hypothesis we have $\vdash A' \to B$, i.e. $\vdash \Diamond A \to B$. By Residuation this gives $\vdash A \to \Box^{\downarrow} B$.

Apart from global structural postulates we will introduce in the remainder of this paper 'modal' versions of such postulates — versions which are relativized to the presence of $\Diamond$ control operators. The completeness results extend to these new structural postulates. Syntactically, they consist of formulas built up entirely in terms of the $\bullet$ operator and its truncated one-place variant $\Diamond$. This means they have the required shape for a generalized Sahlqvist-van Benthem theorem and frame completeness result which is proved in [Kurtonina 95]:

If $R_\diamond : A \to B$ is a modal version of a structural postulate, then there exists a first order frame condition effectively obtainable from $R_\diamond$, and any logic $\mathcal{L} + R_\diamond$ is complete if $\mathcal{L}$ is complete.

EMBEDDING THEOREMS: THE METHOD IN GENERAL. In the sections that follow, we consider pairs of logics $\mathcal{L}_0, \mathcal{L}_1$ where $\mathcal{L}_0$ is a 'southern' neighbour of $\mathcal{L}_1$. Let us write $\mathcal{L}\diamond$ for a system $\mathcal{L}$ extended with the unary operators $\diamond, \Box^\downarrow$ with their minimal residuation logic. For the 12 edges of the cube of Fig 1, we define embedding translations $(\cdot)^\flat : \mathcal{F}(\mathcal{L}_0) \mapsto \mathcal{F}(\mathcal{L}_1\diamond)$ which impose the structural discrimination of $\mathcal{L}_0$ in $\mathcal{L}_1$ with its more liberal resource management, and $(\cdot)^\sharp : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\diamond)$ which license relaxation of structure sensitivity in $\mathcal{L}_0$ in such a way that one fully recovers the flexibility of the the coarser $\mathcal{L}_1$.

Our strategy for obtaining the embedding results is quite uniform. It will be helpful to present the recipe first in abstract terms, so that in the following sections we can supply the particular ingredients with reference to the general scheme. The embedding theorems have the format shown below. We call $\mathcal{L}$ the source logic, $\mathcal{L}'$ the target.

$$\mathcal{L} \vdash A \to B \quad \text{iff} \quad \mathcal{L}'\diamond(+\mathcal{R}_\diamond) \vdash A^\natural \to B^\natural$$

For the constraining perspective, $(\cdot)^\natural$ is $(\cdot)^\flat$ with $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{L}' = \mathcal{L}_1$. For the licensing type of embedding, $(\cdot)^\natural$ is $(\cdot)^\sharp$ with $\mathcal{L} = \mathcal{L}_1$ and $\mathcal{L}' = \mathcal{L}_0$. The embedding translation $(\cdot)^\natural$ decorates critical subformulae in the target logic with the operators $\diamond, \Box^\downarrow$. The translations are defined on the product $\bullet$ of the source logic: their action on the implicational formulas is fully determined by the residuation laws. A $\bullet$ configuration of the source logic is mapped to the *composition* of $\diamond$ and the product of the target logic. The elementary compositions are given below (writing $\circ$ for the target product). They mark the product as a whole, or one of the subtypes with the $\diamond$ control operator.

$$\diamond(- \circ -) \qquad ((\diamond -) \circ -) \qquad (- \circ (\diamond -))$$

Sometimes the modal decoration in itself is enough to obtain the required structural control. We call these cases pure embeddings. In other cases realizing the embedding requires the addition of $\mathcal{R}_\diamond$ — the modalized version of a structural rule package discriminating $\mathcal{L}$ from $\mathcal{L}'$. Typically, this will be the case for communication in the licensing direction: the target logics lack an option for structural manipulation that is present in the source.

The proof of the embedding theorems comes in two parts.

($\Rightarrow$) *Soundness of the embedding.* The ($\Rightarrow$) half is the easy part. Using the Lambek-style axiomatization of A.1 we obtain this direction of the embedding by a straightforward induction on the length of derivations in $\mathcal{L}$.

($\Leftarrow$) *Completeness of the embedding.* For the proofs of the ($\Leftarrow$) part, we reason semantically and rely on the completeness of the logics compared. To show that $\vdash A^\natural \to B^\natural$ in $\mathcal{L}'\diamond$ implies $\vdash A \to B$ in $\mathcal{L}$ we proceed by contraposition. Suppose $\mathcal{L} \not\vdash A \to B$. By completeness, there is an $\mathcal{L}$ model $\mathcal{M} = \langle F, V\rangle$ falsifying $A \to B$, i.e. there is a point a such that $\mathcal{M}, a \models A$ but $\mathcal{M}, a \not\models B$. We obtain the proof for the ($\Leftarrow$) direction in two steps.

Model construction. From $\mathcal{M}$, we construct an $\mathcal{L}'\diamond$ model $\mathcal{M}' = \langle F', V'\rangle$. For the valuation, we set $V'(p) = V(p)$. For the frames, we define a mapping between the R^3 configurations in F and corresponding mixed $R^{2'}, R^{3'}$ configurations in F'. We make sure that the mapping reflects the properties of the translation schema, and that it takes into account the different frame conditions for F and F'.

Truth preservation lemma. We prove that for any $a \in W \cap W'$, $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^\natural$, i.e. that the construction of $\mathcal{M}'$ is truth preserving.

Now, if $\mathcal{M}$ is a countermodel for $A \to B$, so is $\mathcal{M}'$ for $A^\natural \to B^\natural$. Soundness then leads us to the conclusion that $\mathcal{L}'\Diamond \not\vdash A^\natural \to B^\natural$.

With this proof recipe in hand, the reader is prepared to tackle the sections that follow. Recovery of structural discrimination is the subject of §2. In §3 we turn to licensing of structural relaxation. In §4 we reflect on general logical and linguistic features of the proposed architecture, signaling some open questions and directions for future research.

2 Imposing structural constraints

Let us first look at the embedding of more discriminating logics within systems with a less fine-grained sense of structure sensitivity. Modal decoration, in this case, serves to block structural manipulation that would be available by default. The section is organised as follows. In §2.1, we give a detailed treatment of a representative case for each of the structural dimensions of precedence, dominance and dependency. This covers the edges connected to the pure logic of residuation, **NL**. With minor adaptions the embedding translations of §2.1 can be extended to the remaining edges, with the exception of the four associative logics at the right back face of the cube. We present these generalisations in §2.2. This time we refrain from fully explicit treatment where extrapolation from §2.1 is straightforward. The remaining systems are treated in §2.3. They share associative resource management but differ in their sensitivity for linear order or dependency structure. We obtain the desired embeddings in these cases via a tactical manoeuvre which combines the composition of simple translation schemata and the reinstallment of Associativity via *modally controlled* structural postulates.

2.1 Simple embeddings

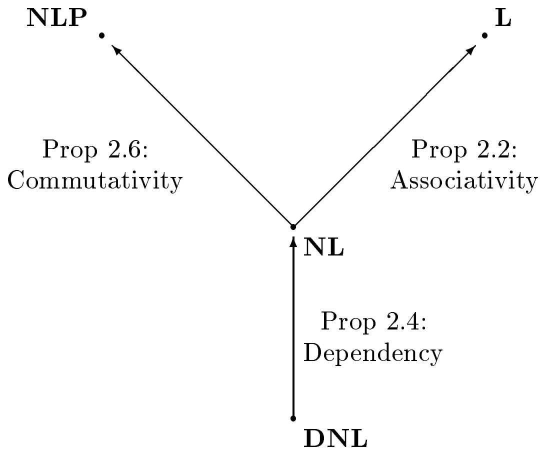


Figure 2: Imposing constraints: precedence, dominance, dependency

Associativity

Consider first the pair **NL** versus **L**$\Diamond$. Let us subscript the symbols for the connectives in **NL** with 0 and those of **L** with 1. The **L** family $/_1, \bullet_1, \backslash_1$ has an associative resource management. We extend **L** with the operators $\Diamond, \Box^\downarrow$ and recover control over associativity by means of the following translation.

Definition 2.1 Translation $\cdot^\flat : \mathcal{F}(\mathbf{NL}) \mapsto \mathcal{F}(\mathbf{L}\Diamond)$ as below.

$$\begin{array}{c} p^\flat = p \\ (A \bullet_0 B)^\flat = \Diamond(A^\flat \bullet_1 B^\flat) \\ (A/_0 B)^\flat = \Box^\downarrow A^\flat /_1 B^\flat \\ (B\backslash_0 A)^\flat = B^\flat \backslash_1 \Box^\downarrow A^\flat \end{array}$$

Proposition 2.2

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{L}\Diamond \vdash A^\flat \to B^\flat$$

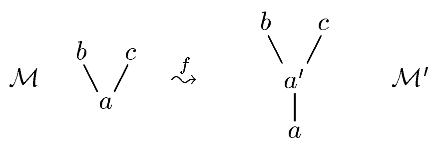
Proof. *($\Rightarrow$) Soundness of the embedding.* For the left-to-right direction we use induction on the length of derivations in **NL** on the basis of the Lambek-style axiomatization given in the Appendix, where apart from the identity axiom and Transitivity, the Residuation rules are the only rules of inference. Assume $A \bullet_0 B \to C$ is derived from $A \to C/_0 B$ in **NL**. By inductive hypothesis, $\mathbf{L} \vdash A^\flat \to (C/_1 B)^\flat$, i.e. (†) $A^\flat \to \Box^\downarrow C^\flat /_1 B^\flat$. We have to show (‡) $\mathbf{L} \vdash (A \bullet_1 B)^\flat \to C^\flat$, i.e. $\Diamond(A^\flat \bullet_1 B^\flat) \to C^\flat$. By RES(2) we have from (†) $A^\flat \bullet_1 B^\flat \to \Box^\downarrow C^\flat$ which derives (‡) by RES(1). For the other side of the residuation inferences, assume $A \to C/_0 B$ is derived from $A \bullet_0 B \to C$. By inductive hypothesis, $\mathbf{L} \vdash (A \bullet_1 B^\flat) \to C^\flat$, i.e. (‡) $\Diamond(A^\flat \bullet_1 B^\flat) \to C^\flat$. We have to show $\mathbf{L} \vdash A^\flat \to C/_1 B^\flat$, i.e. (†) $A^\flat \to \Box^\downarrow C^\flat /_1 B^\flat$. By RES(1) we have from (‡) $A^\flat \bullet_1 B^\flat \to \Box^\downarrow C^\flat$ which derives (†) by RES(2). The residual pair $(\bullet_0, \backslash_0)$ is treated in a fully symmetrical way. □

($\Leftarrow$) Completeness of the embedding. We apply the method outlined in §1. From a falsifying model $\mathcal{M} = \langle W, R_0^3, V\rangle$ for $A \to B$ in **NL** we construct $\mathcal{M}' = \langle W', R_1^3, R_\diamond^2, V'\rangle$. We prove that the construction is truth preserving, so that we can conclude from Soundness that $\mathcal{M}'$ falsifies $A^\flat \to B^\flat$ in $\mathbf{L}\Diamond$.

Model construction. Let W_1 be a set such that $W \cap W_1 = \emptyset$ and $f : R_0^3 \mapsto W_1$ a bijection associating each triple $(abc) \in R_0^3$ with a fresh point $f((abc)) \in W_1$. $\mathcal{M}'$ is defined as follows:

$$\begin{array}{rcl} W' & = & W \cup W_1 \\ R_1 & = & \{(a'bc) \mid \exists a. R_0 abc \wedge f((abc)) = a'\} \\ R_\diamond & = & \{(aa') \mid \exists bc. R_0 abc \wedge f((abc)) = a'\} \\ V'(p) & = & V(p) \end{array}$$

The following picture will help the reader to visualize how the model construction relates to the translation schema.



$$A \bullet_0 B \overset{\flat}{\rightsquigarrow} \Diamond(A^\flat \bullet_1 B^\flat)$$

We have to show that $\mathcal{M}'$ is an appropriate model for **L**, i.e. that the construction of $\mathcal{M}'$ realizes the frame condition for associativity:

$$F(A) \quad \forall xyzw \in W'(\exists t(R_1 wxt \wedge R_1 tyz) \Longleftrightarrow \exists t'(R_1 wt'z \wedge R_1 t'xy))$$

$F(A)$ is satisfied automatically because, by the construction of $\mathcal{M}'$, there are no $x, y, z, w \in W'$ that fulfill the requirements: for every triple $(xyz) \in R_1^3$, the point x is chosen fresh, which implies that no point of W' can be both the root of one triangle and a leaf in another one.

Lemma: Truth Preservation. By induction on the complexity of A we show that for any $a \in W$

$$\mathcal{M}, a \models A \quad \text{iff} \quad \mathcal{M}', a \models A^\flat$$

We prove the biconditional for the product and for one of the residual implications.

($\Rightarrow$). Suppose $\mathcal{M}, a \models A \bullet_0 B$. By the truth conditions for $\bullet_0$, there exist b, c such that (i) R_0abc and (ii) $\mathcal{M}, b \models A$, (iii) $\mathcal{M}, c \models B$. By inductive hypothesis, from (ii) and (iii) we have (ii') $\mathcal{M}', b \models A^\flat$ and (iii') $\mathcal{M}', c \models B^\flat$. By the construction of $\mathcal{M}'$, we conclude from (i) that there is a fresh $a' \in W_1$ such that (iv) $R_\diamond aa'$ and (v) $R_1a'bc$. Then, from (v) and (ii',iii') we have $\mathcal{M}', a' \models A^\flat \bullet_1 B^\flat$ and from (iv) $\mathcal{M}', a \models \diamond(A^\flat \bullet_1 B^\flat)$.

($\Leftarrow$). Suppose $\mathcal{M}', a \models \diamond(A^\flat \bullet_1 B^\flat)$. From the truth conditions for $\bullet_1, \diamond$, we know there are $x, y, z \in W'$ such that (i) $R_\diamond ax$, (ii) R_1xyz and (iii) $\mathcal{M}', y \models A^\flat$ and $\mathcal{M}', z \models B^\flat$. In he construction of $\mathcal{M}'$ the function f is a bijection, so that we can conclude that the configuration (i,ii) has a unique preimage, namely (iv) R_0ayz. By inductive hypothesis, we have from (iii) $\mathcal{M}, y \models A$, and $\mathcal{M}, z \models B$, which then with (iv) gives $\mathcal{M}, a \models A \bullet_0 B$.

($\Rightarrow$). Suppose (i) $\mathcal{M}, a \models A \backslash_0 B$. We have to show $\mathcal{M}', a \models A^\flat \backslash_1 \Box^\downarrow B^\flat$. Suppose we have (ii) R_1yxa such that $\mathcal{M}', x \models A^\flat$. It remains to be shown that $\mathcal{M}', y \models \Box^\downarrow B^\flat$. Suppose we have (iii) $R_\diamond zy$. It remains to be shown that $\mathcal{M}', z \models B^\flat$. The configuration (ii,iii) has a unique preimage by the construction of $\mathcal{M}'$, namely R_0zxa. By inductive hypothesis from (ii) we have $\mathcal{M}, x \models A$ which together with (i) leads to $\mathcal{M}, z \models B$ and, again by inductive hypothesis $\mathcal{M}', z \models B^\flat$, as required.

($\Leftarrow$). Suppose (i) $\mathcal{M}', a \models A^\flat \backslash_1 \Box^\downarrow B^\flat$. We have to show $\mathcal{M}, a \models A \backslash_0 B$. Suppose we have (ii) R_0cba such that $\mathcal{M}, b \models A$. To be shown is whether $\mathcal{M}, c \models B$. By the model construction and inductive hypothesis we have $R_\diamond cc'$, $R_1c'ba$ and $\mathcal{M}', b \models A^\flat$. Hence by (i) $\mathcal{M}', c \models \Box^\downarrow B^\flat$ and therefore $\mathcal{M}', c \models B^\flat$. By inductive hypothesis this leads to $\mathcal{M}, c \models B$ as required. $\Box$

ILLUSTRATION: ISLANDS. For a concrete linguistic illustration, we return to the Coordinate Structure Constraint violations of §1. The translation schema of Def 2.1 was originally proposed by [Morrill 92], who conjectured on the basis of this schema an embedding of **NL** into **L** extended with a pair of unary 'bracket' operators closely related to $\diamond, \Box^\downarrow$. Whether the conjecture holds for the bracket operators remains open. But it is easy to recast Morrill's analysis of island constraints in terms of $\diamond, \Box^\downarrow$. We saw above that on an assignment $(X\backslash X)/X$ to the particle 'and', both the grammatical and the illformed examples are **L** derivable. Within **L**$\diamond$, we can refine the assignment to $(X\backslash\Box^\downarrow X)/X$. The relevant sequent goals now assume the following form (omitting the associative binary structural punctuation, but keeping the crucial $(\cdot)^\diamond$):

(†) the book that Melville wrote and John read

L $\vdash$ $r/(s/np), (np, (np\backslash s)/np, (X\backslash\Box^\downarrow X)/X, np, (np\backslash s)/np)^\diamond \Rightarrow r \quad (X = s/np)$

(‡) *the book that Melville wrote Moby Dick and John read

L$\diamond$ $\not\vdash$ $r/(s/np), (np, (np\backslash s)/np, np, (X\backslash\Box^\downarrow X)/X, np, (np\backslash s)/np)^\diamond \Rightarrow r \quad (X = s)$

The $(X\backslash\Box^\downarrow X)/X$ assignment allows the particle 'and' to combine with the left and right conjuncts in the associative mode. The resulting coordinate structure is of type

$\Box^{\downarrow} X$. To eliminate the $\Box^{\downarrow}$ connective, we have to close off the coordinate structure with $\Diamond$ (or the corresponding structural operator $(\cdot)^{\diamond}$ in the Gentzen presentation) — recall that $\Diamond\Box^{\downarrow} X \to X$. The Accross-the-Board case of extraction (†) works out fine, the island violation (‡) fails because the hypothetical gap np assumption finds itself outside the scope of the $(\cdot)^{\diamond}$ operator.

Dependency

For a second straightforward application of the method, we consider the dependecy calculus **DNL** of [M & M 91] and show how it can be emdedded in **NL**. Recall that **DNL** is the pure logic of residuation for a bimodal system with asymmetric products $\bullet_l, \bullet_r$ for left-headed and right-headed composition respectively. The distinction between left- and right-headed products can be recovered within **NL**$\Diamond$, where we have the unary residuated pair $\Diamond, \Box^{\downarrow}$ next to a symmetric product $\bullet$ and its implications. For the embedding translation $(\cdot)^{\flat}$, we label the head subtype of a product with $\Diamond$. The residuation laws then determine the modal decoration of the implications.

Definition 2.3 The embedding translation $(\cdot)^{\flat} : \mathcal{F}(\mathbf{DNL}) \mapsto \mathcal{F}(\mathbf{NL}\Diamond)$ is defined as follows.

$$\begin{array}{ll} \multicolumn{2}{c}{p^{\flat} = p} \\ (A \bullet_l B)^{\flat} = \Diamond A^{\flat} \bullet B^{\flat} & (A \bullet_r B)^{\flat} = A^{\flat} \bullet \Diamond B^{\flat} \\ (A/_l B)^{\flat} = \Box^{\downarrow}(A^{\flat}/B^{\flat}) & (A/_r B)^{\flat} = A^{\flat}/\Diamond B^{\flat} \\ (B\backslash_l A)^{\flat} = \Diamond B^{\flat}\backslash A^{\flat} & (B\backslash_r A)^{\flat} = \Box^{\downarrow}(B^{\flat}\backslash A^{\flat}) \end{array}$$

Proposition 2.4

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\Diamond \vdash A^{\flat} \to B^{\flat}$$

Proof. *(⇒) Soundness of the embedding.* The soundness half is proved by induction on the length of the derivation of $A \to B$ in **DNL**. We trace the residuation inferences under the translation mapping for the pair $(\bullet_l, /_l)$. The remaining cases are completely parallel.

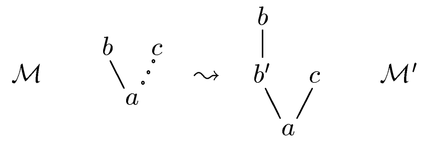
$$\mathbf{DNL} \quad \frac{A \bullet_l B \to C}{\overline{A \to C/_l B}} \quad \leadsto \quad \frac{(A \bullet_l B)^{\flat} \to C^{\flat}}{\overline{A^{\flat} \to (C/_l B)^{\flat}}} \quad \leadsto \quad \frac{\dfrac{\Diamond A^{\flat} \bullet B^{\flat} \to C^{\flat}}{\overline{\Diamond A^{\flat} \to C^{\flat}/B^{\flat}}}}{\overline{A^{\flat} \to \Box^{\downarrow}(C^{\flat}/B^{\flat})}} \quad \mathbf{NL}$$

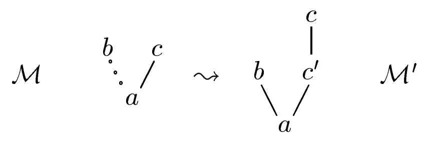
(⇐) Completeness of the embedding. Suppose $\mathbf{DNL} \not\vdash A \to B$. By completeness, there is a model $\mathcal{M} = \langle W, R^3_l, R^3_r, V\rangle$ falsifying $A \to B$. From $\mathcal{M}$, we want to construct a model $\mathcal{M}' = \langle W', R^3_{\bullet}, R^2_{\diamond}, V'\rangle$ which falsifies $A^{\flat} \to B^{\flat}$. Then from soundness we will be able to conclude $\mathbf{NL}\Diamond \not\vdash A^{\flat} \to B^{\flat}$.

Model construction. Let W, W_l, W_r be disjoint sets and $f : R^3_l \mapsto W_l$ and $g : R^3_r \mapsto W_r$ bijective functions. $\mathcal{M}'$ is defined as follows:

$$\begin{array}{rcl} W' & = & W \cup W_l \cup W_r \\ R_{\bullet} & = & \{(ab'c) \mid \exists b. R_l abc \wedge f((abc)) = b'\} \quad \cup \\ & & \{(abc') \mid \exists c. R_r abc \wedge g((abc)) = c'\} \\ R_{\diamond} & = & \{(c'c) \mid \exists ab. R_r abc \wedge g((abc)) = c'\} \quad \cup \\ & & \{(b'b) \mid \exists ac. R_l abc \wedge f((abc)) = b'\} \\ V'(p) & = & V(p) \end{array}$$

We comment on the frames. For every triple $(abc) \in R^3_l$, we introduce a fresh b' and put the worlds $a, b, b', c \in W'$, $(b'b) \in R^2_\diamond$ and $(ab'c) \in R^3_\bullet$. Similarly, for every triple $(abc) \in R^3_r$, we introduce a fresh c' and put the worlds $a, b, c, c' \in W'$, $(c'c) \in R^2_\diamond$ and $(abc') \in R^3_\bullet$. In a picture (with dotted lines for the dependent daughter for R_l, R_r):





Lemma: truth preservation. By induction on the complexity of A, we show that for any $a \in W$, $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^\flat$. We prove the biconditional for the left-headed product. The other connectives are handled in a similar way.

($\Rightarrow$). Suppose $\mathcal{M}, a \models A \bullet_l B$. By the truth conditions for $\bullet_l$, there exist b, c such that (i) $R_l abc$ and (ii) $\mathcal{M}, b \models A$, (iii) $\mathcal{M}, c \models B$. By the construction of $\mathcal{M}'$, we conclude from (i) that there is a fresh $b' \in W'$ such that (iv) $R^2_\diamond b'b$ and (v) $R^3_\bullet ab'c$. By inductive hypothesis, from (ii) and (iii) we have $\mathcal{M}', b \models A^\flat$ and $\mathcal{M}', c \models B^\flat$. Then, from (iv) we have $\mathcal{M}', b' \models \diamond A^\flat$ and from (v), $\mathcal{M}', a \models \diamond A^\flat \bullet B^\flat$.

($\Leftarrow$). Suppose $\mathcal{M}', a \models \diamond A^\flat \bullet B^\flat$. From the truth conditions for $\bullet, \diamond$, we know there are $d', d, e \in W'$ such that (i) $R^2_\diamond d'd$, (ii) $R^3_\bullet ad'e$ and (iii) $\mathcal{M}', d \models A^\flat$ and $\mathcal{M}', e \models B^\flat$. From the construction of $\mathcal{M}'$, we may conclude that $d' = b', d = b, e = c$, since every triple $(abc) \in R^3_l$ is keyed to a fresh world $b' \in W'$. So we actually have (i') $R^2_\diamond b'b$, (ii') $R^3_\bullet ab'c$ and (iii') $\mathcal{M}', b \models A^\flat$ and $\mathcal{M}', c \models B^\flat$. (i') and (ii') imply $R^3_l abc$. By inductive hypothesis, we have from (iii') $\mathcal{M}, b \models A$, and $\mathcal{M}, c \models B$. But then $\mathcal{M}, a \models A \bullet_l B$. $\Box$

Illustration. Below two instances of lifting in **DNL**. The left one is derivable, the right one is not.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^\flat \Rightarrow A^\flat}{(A^\flat)^\diamond \Rightarrow \diamond A^\flat}\,\diamond R \qquad B^\flat \Rightarrow B^\flat}{((A^\flat)^\diamond, \diamond A^\flat \backslash B^\flat)^\bullet \Rightarrow B^\flat}\,\backslash L}{(A^\flat)^\diamond \Rightarrow B^\flat/(\diamond A^\flat \backslash B^\flat)}\,/R}{A^\flat \Rightarrow \Box^\downarrow(B^\flat/(\diamond A^\flat \backslash B^\flat))}\,\Box^\downarrow R}{A \Rightarrow B/_l(A\backslash_l B)}\,\cdot^\flat
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{?}{((A^\flat)^\diamond, \Box^\downarrow(A^\flat \backslash B^\flat))^\bullet \Rightarrow B^\flat}}{(A^\flat)^\diamond \Rightarrow B^\flat/\Box^\downarrow(A^\flat \backslash B^\flat)}\,/R}{A^\flat \Rightarrow \Box^\downarrow(B^\flat/\Box^\downarrow(A^\flat \backslash B^\flat))}\,\Box^\downarrow R}{A \Rightarrow B/_l(A\backslash_r B)}\,\cdot^\flat$$

Commutativity

We can exploit the strategy for modal embedding of the dependency calculus to recover control over Permutation. Here we look at the pure case: the embedding of **NL** into **NLP**$\diamond$. In §2.2 we will generalize the result to the other cases where Permutation is involved. For the embedding, choose one of the (asymmetric) dependency product translations for $\bullet$ in **NL**. Permutation in **NLP** spoils the asymmetry of the product. Whereas one could read the $\diamond$ label in the cases of Def2.3 as a head marker, in the present case $\diamond$ functions as a marker of the first daughter.

Definition 2.5 The embedding translation $\cdot^\flat : \mathcal{F}(\mathbf{NL}) \mapsto \mathcal{F}(\mathbf{NLP}\Diamond)$ is defined as follows.

$$\begin{aligned} p^\flat &= p \\ (A \bullet B)^\flat &= \Diamond A^\flat \otimes B^\flat \\ (A/B)^\flat &= \Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) \\ (B\backslash A)^\flat &= \Diamond B^\flat -\!\!\circ A^\flat \end{aligned}$$

Proposition 2.6

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{NLP}\Diamond \vdash A^\flat \to B^\flat$$

Proof sketch. The ($\Rightarrow$) part again is proved straightforwardly by induction on the length of the derivation of $A \to B$ in **NL**. We leave this to the reader. For the ($\Leftarrow$) direction, suppose $\mathbf{NL} \not\vdash A \to B$. By completeness, there is a model $\mathcal{M} = \langle W, R^3_\bullet, V\rangle$ falsifying $A \to B$. From $\mathcal{M}$, we now have to construct a *commutative* model $\mathcal{M}' = \langle W', R^3_\otimes, R^2_\diamond, V'\rangle$ which falsifies $A^\flat \to B^\flat$. From soundness we will conclude that $\mathbf{NLP}\Diamond \not\vdash A^\flat \to B^\flat$.

The construction of the frame for $\mathcal{M}'$ in this case proceeds as follows. For every triple $(abc) \in R^3_\bullet$, we introduce a fresh b' and put the worlds $a, b, b', c \in W'$, $(b'b) \in R^2_\diamond$ and both $(ab'c), (acb') \in R^3_\otimes$. The construction makes the frame for $\mathcal{M}'$ commutative. But because every commutative triple $(ab'c)$ depends on a fresh $b' \in W' - W$, the commutativity of $\mathcal{M}'$ has no influence on $\mathcal{M}$. For the valuation, we set $V'(p) = V(p)$. Now for any $a \in W \cap W'$, we can show by induction on the complexity of A that $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^\flat$ which then leads to the proof of the main proposition in the usual way.

Illustration. Below first a theorem of **NL**, followed by a non-theorem. We compare their image under $\cdot^\flat$ in $\mathbf{NLP}\Diamond$. And we notice that the second example *is* derivable in **NLP**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^\flat \Rightarrow B^\flat \quad A^\flat \Rightarrow A^\flat}{(A^\flat \circ\!\!- B^\flat, B^\flat)^\otimes \Rightarrow A^\flat}\circ\!\!-L}{((\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond, B^\flat)^\otimes \Rightarrow A^\flat}\Box^{\downarrow}L}{(\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!\!- B^\flat}\circ\!\!-R}{\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!\!- B^\flat)}\Box^{\downarrow}R}{(\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond \Rightarrow \Diamond\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat)}\Diamond R \quad A^\flat \Rightarrow A^\flat}{((\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond, \Diamond\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) -\!\!\circ A^\flat)^\otimes \Rightarrow A^\flat}-\!\!\circ L}{(\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!\!- (\Diamond\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) -\!\!\circ A^\flat)}\circ\!\!-R}{\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!\!- (\Diamond\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) -\!\!\circ A^\flat))}\Box^{\downarrow}R}{\mathbf{NL} \vdash A/B \Rightarrow A/((A/B)\backslash A)}\cdot^\flat$$

$$\dfrac{\dfrac{\dfrac{?}{((\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond, \Box^{\downarrow}(A^\flat \circ\!\!- (\Diamond B^\flat -\!\!\circ A^\flat)))^\otimes \Rightarrow A^\flat}}{(\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!\!- (\Box^{\downarrow}(A^\flat \circ\!\!- (\Diamond B^\flat -\!\!\circ A^\flat))}\circ\!\!-R}{\Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!\!- (\Box^{\downarrow}(A^\flat \circ\!\!- (\Diamond B^\flat -\!\!\circ A^\flat)))}\Box^{\downarrow}R \over \mathbf{NL} \not\vdash A/B \Rightarrow A/(A/(B\backslash A))}\cdot^\flat$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{(A \circ\!\!- B, B)^{\otimes} \Rightarrow A}\ \circ\!\!-L}{(B, A \circ\!\!- B)^{\otimes} \Rightarrow A}\ P}{A \circ\!\!- B \Rightarrow B -\!\!\circ A}\ -\!\!\circ R \qquad A \Rightarrow A}{\dfrac{(A \circ\!\!- (B -\!\!\circ A), A \circ\!\!- B)^{\otimes} \Rightarrow A}{(A \circ\!\!- B, A \circ\!\!- (B -\!\!\circ A))^{\otimes} \Rightarrow A}\ P}\ \circ\!\!-L
$$

$$
\dfrac{(A \circ\!\!- B, A \circ\!\!- (B -\!\!\circ A))^{\otimes} \Rightarrow A}{\mathbf{NLP} \vdash A \circ\!\!- B \Rightarrow A \circ\!\!- (A \circ\!\!- (B -\!\!\circ A))}\ \circ\!\!-R
$$

2.2 Generalisations

The results of the previous section can be extended with minor modifications to the five edges that remain when we keep the Associativity face for §2.3.

What we have done in Prop 2.4 for the pair **DNL** versus **NL**$\Diamond$ can be adapted straightforwardly to the commutative pair **DNLP** versus **NLP**$\Diamond$. Recall that in **DNLP**, the dependency products satisfy head-preserving commutativity ($P_{l,r}$), whereas in **NLP** we have simple commutativity (P).

$$
\begin{array}{rl}
P_{l,r}: & A \otimes_l B \longleftrightarrow B \otimes_r A \\
P: & A \otimes B \rightarrow B \otimes A
\end{array}
$$

Accomodating the commutative products, the embedding translation is that of Prop 2.4: $\Diamond$ marks the head subtype.

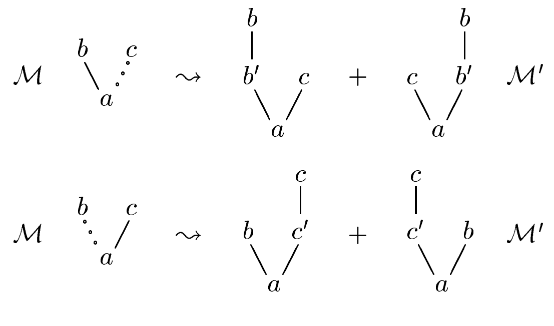
Definition 2.7 Translation $(\cdot)^\flat : \mathcal{F}(\mathbf{DNLP}) \mapsto \mathcal{F}(\mathbf{NLP}\Diamond)$:

$$
\begin{array}{ll}
\multicolumn{2}{c}{p^\flat = p} \\
(A \otimes_l B)^\flat = \Diamond A^\flat \otimes B^\flat & (A \otimes_r B)^\flat = A^\flat \otimes \Diamond B^\flat \\
(A \circ\!\!-_l B)^\flat = \Box^{\downarrow}(A^\flat \circ\!\!- B^\flat) & (A \circ\!\!-_r B)^\flat = A^\flat \circ\!\!- \Diamond B^\flat \\
(B -\!\!\circ_l A)^\flat = \Diamond B^\flat -\!\!\circ A^\flat & (B -\!\!\circ_r A)^\flat = \Box^{\downarrow}(B^\flat -\!\!\circ A^\flat)
\end{array}
$$

Proposition 2.8

$$
\mathbf{DNLP} \vdash A \rightarrow B \quad \text{iff} \quad \mathbf{NLP}\Diamond \vdash A^\flat \rightarrow B^\flat
$$

For the proof of the ($\Leftarrow$) direction, we combine the method of construction of Prop 2.4 with that of Prop 2.6. For a configuration $R_l^{\otimes}abc$ in $\mathcal{M}$, we take fresh b' and put the configurations $R_\diamond b'b, R_\otimes ab'c, R_\otimes acb'$ in $\mathcal{M}'$. Similarly, for a configuration $R_r^{\otimes}abc$ in $\mathcal{M}$, we take fresh c' and put the configurations $R_\diamond c'c, R_\otimes abc', R_\otimes ac'b$ in $\mathcal{M}'$. The commutativity property of $\otimes$ is thus realized by the construction.



Let us check the truth preservation lemma. This time a configuration $(\star)$ in $\mathcal{M}'$ does not have a unique pre-image: it can come from $R_l^{\otimes}xyz$ or $R_r^{\otimes}xzy$. But because of head-preserving commutativity (DP), these are both in $\mathcal{M}$.

$$(\star)\qquad \begin{array}{ccc} & & y \\ & & | \\ z & & y' \\ & \diagdown\ \diagup & \\ & x & \end{array}$$

Similarly, the embedding construction presented in Prop 2.6 for the pair **NL** versus **NLP**$\Diamond$ can be generalized directly to the related pair **DNL** versus **DNLP**$\Diamond$. This time, we want the embedding translation to block the structural postulate of *head-preserving* commutativity in **DNLP**. The translation below invalidates the postulate by uniformly decorating with $\Diamond$, say, the left subtype of a product.

Definition 2.9 Define $(\cdot)^\flat : \mathcal{F}(\mathbf{DNL}) \mapsto \mathcal{F}(\mathbf{DNLP}\Diamond)$ as follows.

$$\begin{array}{rcl} & p^\flat = p & \\ (A \bullet_l B)^\flat = \Diamond A^\flat \otimes_l B^\flat & & (A \bullet_r B)^\flat = \Diamond A^\flat \otimes_r B^\flat \\ (A/_l B)^\flat = \Box^\downarrow(A^\flat \circ\!-_l B^\flat) & & (A/_r B)^\flat = \Box^\downarrow(A^\flat \circ\!-_r B^\flat) \\ (B\backslash_l A)^\flat = \Diamond B^\flat -\!\circ_l A^\flat & & (B\backslash_r A)^\flat = \Diamond B^\flat -\!\circ_r A^\flat \end{array}$$

We then have the following proposition. The proof is entirely parallel to that of Prop 2.6 before.

Proposition 2.10

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{DNLP}\Diamond \vdash A^\flat \to B^\flat$$

The method of Prop 2.2 generalizes to the following cases with some simple changes.

Definition 2.11 Translation $(\cdot)^\flat : \mathcal{F}(\mathbf{NLP}) \mapsto \mathcal{F}(\mathbf{LP}\Diamond)$ as below.

$$\begin{array}{c} p^\flat = p \\ (A \otimes B)^\flat = \Diamond(A^\flat \otimes B^\flat) \\ (A \circ\!- B)^\flat = \Box^\downarrow A^\flat \circ\!- B^\flat \\ (B -\!\circ A)^\flat = B^\flat -\!\circ \Box^\downarrow A^\flat \end{array}$$

Proposition 2.12

$$\mathbf{NLP} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\Diamond \vdash A^\flat \to B^\flat$$

The only difference with Prop 2.2 is that the product in input and target logic are commutative. Commutativity is realized automatically by the construction of $\mathcal{M}'$.

Proposition 2.13

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{DL}\Diamond \vdash A^\flat \to B^\flat$$

Proposition 2.14

$$\mathbf{DNLP} \vdash A \to B \quad \text{iff} \quad \mathbf{DLP}\Diamond \vdash A^\flat \to B^\flat$$

2.3 Composed translations

The remaining cases concern the right back face of the cube, where we find the systems **DL**, **L**, **LP**, and **DLP**. These logics share associative resource management, but they differ with respect to one of the remaining structural parameters — sensitivity for linear order (**L** versus **LP**, **DL** versus **DLP**) or for dependency structure (**DL** versus **L**, and **DLP** versus **LP**). We already know how to handle each of the structural dimensions individually. We use this knowledge to obtain the embeddings for systems with shared Associativity. Our strategy has two components. First we neutralize direct appeal to Associativity by taking the composition of the translation schema blocking Associativity with the schema responsible for control in the structural dimension which discriminates between the source and target logics. This first move does *not* embed the source logic, but its non-associative neighbour. The second move then is to reinstall associativity in terms of $\diamond$ modally controlled versions of the Associativity postulates.

Associative dependency calculus. We work out the 'rear attack' manoeuvre first for the pair **DL** versus **L**. In **DNL** we have no restrictions on the interpretation of $\bullet_l, \bullet_r$. In **DL** we assume $\bullet_l, \bullet_r$ are interpreted on (bimodal) associative frames, and we have structural associativity postulates $A(l), A(r)$ on top of the pure logic of residuation for $\bullet_l, \bullet_r$. In **L** we cannot discriminate between $\bullet_l$ and $\bullet_r$ — there is just one $\bullet$ operator, which shares the associative resource management with its dependency variants. The objective of the embedding is to recover the distinction between left- and right-headed structures in a system which has only one product connective.

$$A(l) : (A \bullet_l B) \bullet_l C \longleftrightarrow A \bullet_l (B \bullet_l C)$$
$$A(r) : A \bullet_r (B \bullet_r C) \longleftrightarrow (A \bullet_r B) \bullet_r C$$

For the embedding translation, we *compose* the mappings of Def 2.3 embedding **DNL** into **NL** and Def 2.1 embedding **NL** into **L**.

Definition 2.15

$$\begin{array}{rlrl} & p^\flat = p & & \\ (A \bullet_l B)^\flat = & \diamond(\diamond A^\flat \bullet B^\flat) & (A \bullet_r B)^\flat = & \diamond(A^\flat \bullet \diamond B^\flat) \\ (A/_l B)^\flat = & \Box^\downarrow(\Box^\downarrow A^\flat / B^\flat) & (A/_r B)^\flat = & \Box^\downarrow A^\flat / \diamond B^\flat \\ (B\backslash_l A)^\flat = & \diamond B^\flat \backslash \Box^\downarrow A^\flat & (B\backslash_r A)^\flat = & \Box^\downarrow(B^\flat \backslash \Box^\downarrow A^\flat) \end{array}$$

From the proof of the embedding of **NL** into **L** we know that $\diamond$ neutralizes the effects of the associativity of $\bullet$ in the target logic **L**: the frame condition for Associativity is satisfied vacuously. To realize the desired embedding of **DL** into **L**, we reinstall modal versions of the associativity postulates.

$$A(l)^\diamond : \diamond(\diamond\diamond(\diamond A \bullet B) \bullet C \longleftrightarrow \diamond(\diamond A \bullet \diamond(\diamond B \bullet C))$$
$$A(r)^\diamond : \diamond(A \bullet \diamond\diamond(B \bullet \diamond C)) \longleftrightarrow \diamond(\diamond(A \bullet \diamond B) \bullet \diamond C)$$

Figure 3 is a graphical illustration of the interplay between the composed translation schema and the modal structural postulate. f is the translation schema $(\cdot)^\flat$ of Def 2.1, g that of Def 2.3.

Modalized structural postulates: frame completeness. The modalized structural postulates $A(l,r)^\diamond$ introduce a new element in the discussion. Semantically, these postulates require frame constraints correlating the binary and ternary relations of structural composition. Fortunately we know, from the generalized Sahlqvist-van Benthem Theorem and frame completeness result discussed in §1, that from $A(l,r)^\diamond$ we can effectively obtain the relevant first order frame conditions, and that completeness of $\mathbf{L}\diamond$ extends to the system augmented with $A(l,r)^\diamond$. We check

$$
\begin{tikzcd}
 & \mathbf{L} \\
\mathbf{NL} \arrow[ur, "f"] & \mathbf{DL} \arrow[u, "{h = f \circ g,\ A(l,r)^{\diamond}}"'] \arrow[dl, "A(l,r)^{\diamond}"] \\
\mathbf{DNL} \arrow[u, "g"] &
\end{tikzcd}
$$

Figure 3: Rear Attack Embedding **DL** into **L**.

completeness for $A(l)^\diamond$ here as an illustration — the situation for $A(r)^\diamond$ is entirely similar. Fig 4 gives the frame condition for $A(l)^\diamond$.

The models for $\mathbf{L}\Diamond$ are structures $\langle W, R^2_\diamond, R^3_\bullet, V\rangle$. Now consider $(\Rightarrow)$ in Figure 4 below. Given the canonical model construction of Def 1.1 the following are derivable by the definition of $R^2_\diamond, R^3_\bullet$:

$$
\begin{array}{ll}
a \to \Diamond b, & e \to \Diamond f, \\
b \to c \bullet d, & f \to g \bullet h, \\
c \to \Diamond e, & g \to \Diamond i.
\end{array}
$$

From these we can conclude $\vdash a \to \Diamond(\Diamond\Diamond(\Diamond i \bullet h) \bullet d))$, i.e. $a \in V(\Diamond(\Diamond\Diamond(\Diamond i \bullet h) \bullet d)))$, given the definition of the canonical valuation $(\star)$. For $(\ddagger)$ we have to find b', c', d', e', f' such that

$$
\begin{array}{ll}
a \to \Diamond b', & d' \to \Diamond e', \\
b' \to c' \bullet d', & e' \to f' \bullet d, \\
c' \to \Diamond i, & f' \to \Diamond h.
\end{array}
$$

Let us put

$$
\begin{array}{l}
f' = \Diamond h, \\
e' = f' \bullet d = \Diamond h \bullet d, \\
d' = \Diamond e' = \Diamond(\Diamond h \bullet d), \\
c' = \Diamond i, \\
b' = c' \bullet d' = \Diamond i \bullet \Diamond(\Diamond h \bullet d).
\end{array}
$$

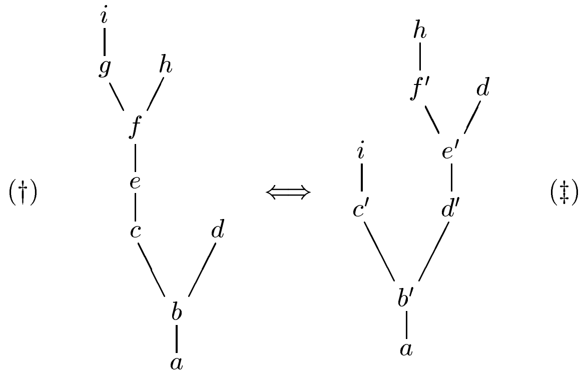
Together they imply $\vdash a \to \Diamond(\Diamond i \bullet \Diamond(\Diamond h \bullet d))$, i.e. $a \in V(\Diamond(\Diamond i \bullet \Diamond(\Diamond h \bullet d)))$ can be shown to follow from $(\star)$. Similarly for the other direction.

Now for the embedding theorem.

Proposition 2.16

$$
\mathbf{DL} \vdash A \to B \quad \text{iff} \quad \mathbf{L}\Diamond + A(l,r)^\diamond \vdash A^\flat \to B^\flat
$$

Model construction. Suppose $\mathbf{DL} \not\vdash A \to B$. Then there is a model $\mathcal{M} = \langle W, R_l, R_r, V\rangle$ where $A \to B$ fails. From $\mathcal{M}$ we construct $\mathcal{M}'$ as follows. For every triple $(abc) \in R_l$ we take fresh a', b' and put $(aa') \in R_\diamond, (a'b'c) \in R_\bullet, (b'b) \in R_\diamond$.



$$F(A(l)^{\diamond}): \quad \exists bcefg(R_{\diamond}ab \wedge R_{\bullet}bcd \wedge R_{\diamond}ce \wedge R_{\diamond}ef \wedge R_{\bullet}fgh \wedge R_{\diamond}gi) \Longleftrightarrow$$
$$\exists b'c'd'e'f'(R_{\diamond}ab' \wedge R_{\bullet}b'c'd' \wedge R_{\diamond}c'i \wedge R_{\diamond}d'e' \wedge R_{\bullet}e'f'd \wedge R_{\diamond}f'h)$$

Figure 4: Frame condition for $A(l)^{\diamond}$

Similarly, for every triple $(abc) \in R_r$ we take fresh a', c' and put $(aa') \in R_{\diamond}, (a'bc') \in R_{\bullet}, (c'c) \in R_{\diamond}$.

We have to check whether $\mathcal{M}'$ is an appropriate model for $\mathbf{L}\diamond + A(l,r)_{\diamond}$, specifically, whether the frame condition of Fig 4 is satisfied. Suppose (‡) holds, and let us check whether (†). Note that a configuration $R_{\diamond}ab', R_{\bullet}b'c'd', R_{\diamond}c'i$ can only hold in $\mathcal{M}'$ if in $\mathcal{M}$ we had $R_l aid'$ ($\star$). And a configuration $R_{\diamond}d'e', R_{\bullet}e'f'd, R_{\diamond}f'h$ can be in $\mathcal{M}'$ only if in $\mathcal{M}$ we had $R_l d'hd$ ($\star\star$). The frame for $\mathcal{M}$ is associative. Therefore, from ($\star, \star\star$) we can conclude $\mathcal{M}$ also contains a configuration $R_l aed, R_l eih$ for some $e \in W$. Applying the $\mathcal{M}'$ construction to that configuration we obtain (†). Similarly for the other direction.

From here on, the proof of Prop 2.16 follows the established path.

Generalisation. The rear attack strategy can be generalized to the remaining edges. Below we simply state the embedding theorems with the relevant composed translations and modal structural postulates. We give the salient ingredients for the construction of $\mathcal{M}'$, leaving the elaboration as an exercise to the reader.

Consider first embedding of $\mathbf{L}$ into $\mathbf{LP}$. The discriminating structural parameter is Commutativity. For the translation schema, we compose the translations of Def 2.11 and Def 2.5. Associativity is reinstalled in terms of the structural postulate $A^{\diamond}_{\otimes}$.

$$A^{\diamond}_{\otimes}: \quad \diamond(\diamond\diamond(\diamond A \otimes B) \otimes C \longleftrightarrow \diamond(\diamond A \otimes \diamond(\diamond B \otimes C))$$

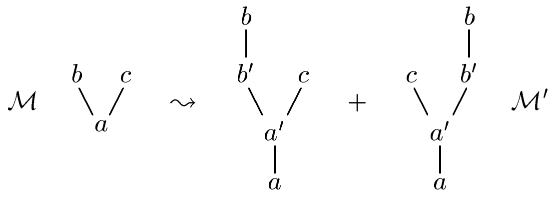
Definition 2.17 Embedding translation $(\cdot)^{\flat} : \mathcal{F}(\mathbf{L}) \mapsto \mathcal{F}(\mathbf{LP}\diamond)$.

$$\begin{aligned} p^{\flat} &= p \\ (A \bullet B)^{\flat} &= \diamond(\diamond A^{\flat} \otimes B^{\flat}) \\ (A/B)^{\flat} &= \Box^{\downarrow}(\Box^{\downarrow} A^{\flat} \circ\!- B^{\flat}) \\ (B\backslash A)^{\flat} &= \diamond B^{\flat} -\!\circ \Box^{\downarrow} A^{\flat} \end{aligned}$$

Proposition 2.18

$$\mathbf{L} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\diamond + A^{\diamond}_{\otimes} \vdash A^{\flat} \to B^{\flat}$$

Semantically, the commutativity of $R_\otimes$ is realized via the construction of $\mathcal{M}'$, as in the case of Prop 2.6:



For the pair **DL** versus **DLP**, again Commutativity is the discriminating structural parameter, but now in a bimodal setting. We compose the translations for the embedding of **DNLP** into **DLP** and **DNL** into **DNLP** The structural postulates $A^\diamond_{\otimes_l}$ and $A^\diamond_{\otimes_r}$ are the dependency variants of $A^\diamond_\otimes$ above.

$$
\begin{array}{ll}
A^\diamond_{\otimes_l}: & \diamond(\diamond\diamond(\diamond A \otimes_l B) \otimes_l C \longleftrightarrow \diamond(\diamond A \otimes_l \diamond(\diamond B \otimes_l C)) \\
A^\diamond_{\otimes_r}: & \diamond(\diamond\diamond(\diamond A \otimes_r B) \otimes_r C \longleftrightarrow \diamond(\diamond A \otimes_r \diamond(\diamond B \otimes_r C))
\end{array}
$$

Definition 2.19 Embedding translation $(\cdot)^\flat : \mathcal{F}(\mathbf{DL}) \mapsto \mathcal{F}(\mathbf{DLP}\diamond)$.

$$
\begin{array}{rr}
\multicolumn{2}{c}{p^\flat = p} \\
(A \bullet_l B)^\flat = \diamond(\diamond A^\flat \otimes_l B^\flat) & (A \bullet_r B)^\flat = \diamond(\diamond A^\flat \otimes_r B^\flat) \\
(A/_l B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \multimapinv_l B^\flat) & (A/_r B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \multimapinv_r B^\flat) \\
(B\backslash_l A)^\flat = \diamond B^\flat \multimap_l \Box^\downarrow A^\flat & (B\backslash_r A)^\flat = \diamond B^\flat \multimap_r \Box^\downarrow A^\flat
\end{array}
$$

Proposition 2.20

$$
\mathbf{DL} \vdash A \to B \quad \text{iff} \quad \mathbf{DLP}\diamond + (A^\diamond_{\otimes_l}, A^\diamond_{\otimes_r}) \vdash A^\flat \to B^\flat
$$

Finally, for the pair **DLP** versus **LP**, the objective of the embedding is to recapture the dependency distinctions. We compose the translations of Def 2.11 and Def 2.7. The modal structural postulates $A(l,r)^\diamond_\otimes$ are obtained from $A(l,r)^\diamond$ by replacing $\bullet$ by $\otimes$.

Definition 2.21 Embedding translation $\cdot^\flat : \mathcal{F}(\mathbf{DLP}) \mapsto \mathcal{F}(\mathbf{LP}\diamond)$.

$$
\begin{array}{ll}
\multicolumn{2}{c}{p^\flat = p} \\
(A \otimes_l B)^\flat = \diamond(\diamond A^\flat \otimes B^\flat) & (A \otimes_r B)^\flat = \diamond(A^\flat \otimes \diamond B^\flat) \\
(A \multimapinv_l B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \multimapinv B^\flat) & (A \multimapinv_r B)^\flat = \Box^\downarrow A^\flat \multimapinv \diamond B^\flat \\
(B \multimap_l A)^\flat = \diamond B^\flat \multimap \Box^\downarrow A^\flat & (B \multimap_r A)^\flat = \Box^\downarrow(B^\flat \multimap \Box^\downarrow A^\flat)
\end{array}
$$

Proposition 2.22

$$
\mathbf{DLP} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\diamond + A(l,r)^\diamond_\otimes \vdash A^\flat \to B^\flat
$$

2.4 Constraining embeddings: summary

We have completed the tour of the landscape and shown that the connectives $\diamond, \Box^\downarrow$ can systematically reintroduce structural discrimination in logics where on the level of the binary multiplicatives such discrimination is destroyed by global structural postulates. In Fig 5 we label the edges of the cube with the numbers of the embedding theorems.

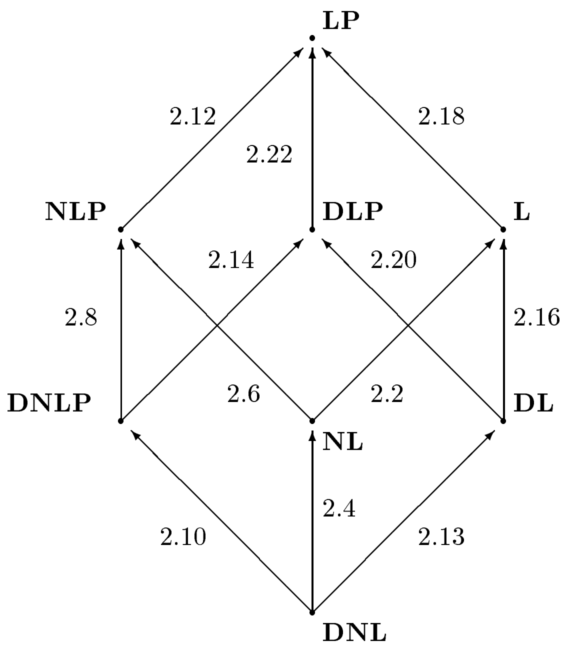


Figure 5: Embedding translations: recovering resource control

3 Licensing structural relaxation

In the present section we shift the perspective: instead of using modal decorations to *block* structural options for resource management, we now take the more discriminating logic as the starting point and use the modal operators to recover the flexibility of a neighbouring logic with a more liberal resource management regime from within a system with a more rigid notion of structure-sensitivity.

Licensing of structural relaxation has traditionally been addressed (both in logic [Došen 92] and in linguistics [Morrill 94]) in terms of a single universal $\Box$ modality with $S4$ type resource management. Here we stick to the minimalistic principles set out at the beginning of this paper, and realize also the licensing embeddings in terms of the pure logic of residuation for the pair $\Diamond, \Box^{\downarrow}$ plus modally controlled structural postulates. In §3.1 we present an external strategy for modal decoration: in the scope of the $\Diamond$ operator, products of the more discriminating logics gain access to structural rules that are inaccessible in the non-modal part of the logic. In §3.2 we develop a complementary strategy for internal modal decoration, where modal versions of the structural rules are accessible provided one or all of the immediate substructures are labelled with $\Diamond$. We present linguistic considerations that will affect the choice for the external or internal approach.

3.1 Modal labelling: external perspective

Licensing structural relaxation is simpler than recovering structural control: the target logics for the embeddings in this section *lack* an option for structural manipulation which can be reinstalled straightforwardly in terms of a modal version of the relevant structural postulate. We do not have to design specific translation strategies for the individual pairs of logics, but can do with one general translation schema.

Definition 3.1 General translation schema $(\cdot)^\sharp : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\Diamond)$ embedding a stronger logic $\mathcal{L}_1$ into a weaker logic $\mathcal{L}_0$ extended with $\Diamond, \Box^\downarrow$.

$$\begin{aligned} p^\sharp &= p \\ (A \bullet_1 B)^\sharp &= \Diamond(A^\sharp \bullet_0 B^\sharp) \\ (A/_1 B)^\sharp &= \Box^\downarrow A^\sharp /_0 B^\sharp \\ (B\backslash_1 A)^\sharp &= B^\sharp \backslash_0 \Box^\downarrow A^\sharp \end{aligned}$$

The embedding theorems we are interested in now have the general format shown below, where $\mathcal{R}_\diamond$ is (a package of) the modal translation(s) $A^\sharp \to B^\sharp$ of the structural rule(s) $A \to B$ which differentiate(s) $\mathcal{L}_1$ from $\mathcal{L}_0$.

$$\mathcal{L}_1 \vdash A \to B \quad \text{iff} \quad \mathcal{L}_0\Diamond + \mathcal{R}_\diamond \vdash A^\sharp \to B^\sharp$$

We look at the dimensions of dependency, precedence and dominance in general terms first, discussing the relevant aspects of the model construction. Then we comment on individual embedding theorems.

Relaxation of dependency sensitivity. For a start let us look at a pair of logics $\mathcal{L}_0$,$\mathcal{L}_1$, where $\mathcal{L}_0$ makes a dependency distinction between a left-dominant and a right-dominant product, whereas $\mathcal{L}_1$ cannot discriminate these two. There is two ways of setting up the coarser logic $\mathcal{L}_1$. Either we present $\mathcal{L}_1$ as a bimodal system where the distinction between right-dominant $\bullet_r$ and left-dominant $\bullet_l$ collapses as a result of the structural postulate (D).

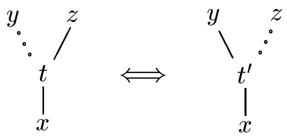
$$\mathcal{L}_1 : \quad A \bullet_r B \longleftrightarrow A \bullet_l B \quad (D)$$

Or we have a unimodal presentation for $\mathcal{L}_1$ and pick an arbitrary choice of the dependency operators for the embedding translation. We take the second option here, and realize the embedding translation as indicated below.

$$\begin{aligned} p^\sharp &= p \\ (A \bullet B)^\sharp &= \Diamond(A^\sharp \bullet_r B^\sharp) \\ (A/B)^\sharp &= \Box^\downarrow A^\sharp /_r B^\sharp \\ (B\backslash A)^\sharp &= B^\sharp \backslash_r \Box^\downarrow A^\sharp \end{aligned}$$

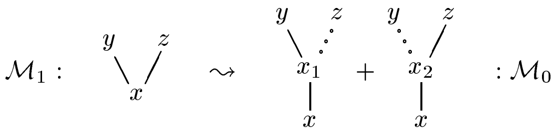
Relaxation of dependency sensitivity is obtained by means of a modally controlled version of (D). Corresponding to the structural postulate $(D_\diamond)$ we have the frame condition $F(D_\diamond)$ as a restriction on models for the more discriminating logic.

$$\mathcal{L}_0 : \quad \Diamond(A \bullet_r B) \longleftrightarrow \Diamond(A \bullet_l B) \quad (D_\diamond)$$



$$F(D_\diamond) : \quad (\forall xyz \in W_0)\ \exists t(R_\diamond xt \wedge R_r tyz) \Leftrightarrow \exists t'(R_\diamond xt' \wedge R_r t'yz)$$

Model construction. To construct an $\mathcal{L}_0$ model $\langle W_0, R^2_\diamond, R^3_l, R^3_r, V_0\rangle$ from a model $\langle W_1, R^3_1, V_1\rangle$ for $\mathcal{L}_1$ we proceed as follows. For every triple $(xyz) \in R_1$ we take fresh points x_1, x_2, put x, x_1, x_2, y, z in W_0 with $(xx_1) \in R_\diamond$, $(x_1yz) \in R_l$ and $(xx_2) \in R_\diamond$, $(x_2yz) \in R_r$.



To show that the generated model $\mathcal{M}_0$ satisfies the required frame condition $F(D_\diamond)$, assume there exists $b \in W_0$ such that $R_\diamond ab$ and $R_r bcd$. Such a configuration has a unique preimage in $\mathcal{M}_1$ namely $R_1 acd$. By virtue of the construction of $\mathcal{M}_0$ this means there exists $b' \in W_0$ such that $R_\diamond ab'$ and $R_l b'cd$, as required for $F(D_\diamond)$.

Truth preservation of the model construction is unproblematic. The proof of the following proposition then is routine.

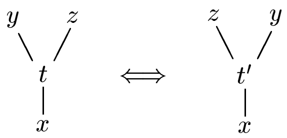
Proposition 3.2

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{DNL}\Diamond + D_\diamond \vdash A^\sharp \to B^\sharp$$

Relaxation of order sensitivity. Here we compare logics $\mathcal{L}_1$ and $\mathcal{L}_0$ where the structural rule of Permutation is included in the resource management package for $\mathcal{L}_1$, but not in that of $\mathcal{L}_0$. Controlled Permutation is reintroduced in $\mathcal{L}_0$ in the form of the modal postulate $(P_\diamond)$. The corresponding frame condition on $\mathcal{L}_0$ models $\mathcal{M}_0$ is given as $F(P_\diamond)$.

$$\mathcal{L}_1 : \quad A \bullet_1 B \longleftrightarrow B \bullet_1 A \quad (P)$$

$$\mathcal{L}_0 : \quad \Diamond(A \bullet_0 B) \longleftrightarrow \Diamond(B \bullet_0 A) \quad (P_\diamond)$$



$$F(P_\diamond) : \quad (\forall xyz \in W_0)\ \exists t(R_\diamond xt \wedge R_0 tyz) \Rightarrow \exists t'(R_\diamond xt' \wedge R_0 t'zy)$$

To generate the required model $\mathcal{M}_0$ from $\mathcal{M}_1$ we proceed as follows. If $(xyz) \in R_1$ we take fresh x_1, x_2 and put both $(xx_1) \in R_\diamond$ and $(x_1yz) \in R_0$ and $(xx_2) \in R_\diamond$ and $(x_2zy) \in R_0$.

We have to show that the generated model $\mathcal{M}_0$ satisfies $F(P_\diamond)$. Assume there exists $b \in W_0$ such that $R_\diamond ab$ and $R_0 bcd$. Because of the presence of Permutation in $\mathcal{L}_1$ this configuration has two preimages, $R_1 acd$ and $R_1 adc$. By virtue of the construction algorithm for $\mathcal{M}_0$ each of these guarantees there exists $b' \in W_0$ such that $R_\diamond ab'$ and $R_0 xdc$.

Proposition 3.3

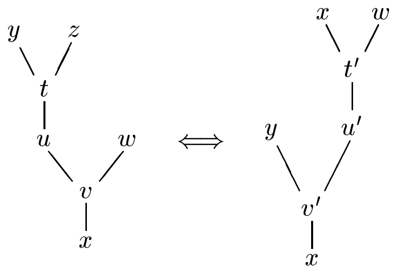
$$\mathbf{NLP} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\Diamond + P_\diamond \vdash A^\sharp \to B^\sharp$$

Relaxation of constituent sensitivity. Next compare a logic $\mathcal{L}_1$ where Associativity obtains with a more discriminating logic without global Associativity.

We realize the embedding by introducing a modally controlled form of Associativity ($A_\diamond$) with its corresponding frame condition $F(A_\diamond)$.

$$\mathcal{L}_1 : \quad A \bullet_1 (B \bullet_1 C) \longleftrightarrow (A \bullet_1 B) \bullet_1 C \quad (A)$$

$$\mathcal{L}_0 : \quad \diamond(A \bullet_0 \diamond(B \bullet_0 C)) \longleftrightarrow \diamond(\diamond(A \bullet_0 B) \bullet_0 C) \quad (A_\diamond)$$



$$F(A_\diamond) : \quad (\forall xyzw \in W_0)$$

$$\exists tuv(R_\diamond xv \wedge R_0 vuw \wedge R_\diamond ut \wedge R_0 tyz) \Leftrightarrow \exists t'u'v'(R_\diamond xv' \wedge R_0 v'yu' \wedge R_\diamond u't' \wedge R_0 t'zw)$$

The $\mathcal{M}_0$ model is generated from $\mathcal{M}_1$ in the familiar way. For every triple $(xyz) \in R_1$, we take a fresh point x', and put $x, x', y, z \in W_0$, with $(xx') \in R_\diamond$ and $(x'yz) \in R_0$.

We have to show that the frame condition $F(A_\diamond)$ holds in the generated model. Suppose (†) $R_\diamond ab$ and $R_0 bcd$ and (‡) $R_\diamond ce$ and $R_0 efg$. We have to show that there are $x, y, z \in W_0$ such that $R_\diamond ax$ and $R_0 xfy$ and $R_\diamond yz$ and $R_0 zgd$. Observe that the configurations (†) and (‡) both have unique preimages in $\mathcal{M}_1$, $R_1 acd$ and $R_1 cfg$ respectively. Because R_1 is associative, there exists $y \in W_1$ such that $R_1 afy$ and $R_1 ygd$. But then, by the construction of $\mathcal{M}_0$, also $y \in W_0$ and there exist $x, z \in W_0$ such that $R_\diamond ax$, $R_0 xfy$, $R_\diamond yz$ and $R_0 zgd$, as required.

Proposition 3.4

$$\mathbf{L} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\diamond + A_\diamond \vdash A^\sharp \to B^\sharp$$

Generalisations. The preceding discussion covers the individual dimensions of structural organisation. Generalizing the approach to the remaining edges of Fig 1 does not present significant new problems. Here are some suggestions to assist the tenacious reader who wants to work out the full details.

The embeddings for the lower plane of Fig 1 are obtained from the parallel embeddings in the upper plane by doubling the construction from a unimodal product setting to the bimodal situation with two dependency products.

Embeddings between logics sharing associative management, but differing with respect to order or dependency sensitivity require modal associativity $A_\diamond$ in addition to $P_\diamond$ or $D_\diamond$ for the more discriminating logic: as we have seen in §2, the external $\diamond$ decoration on product configurations pre-empts the conditions of application for the non-modal associativity postulate. We have already come across this interplay between the translation schema and modal structural postulates in §2.3. For the licensing type of embedding, concrete instances are the embedding of **LP** into $\mathbf{L}\diamond + A_\diamond + P_\diamond$, and the embedding of **L** into $\mathbf{DL}\diamond + A_\diamond + D_\diamond$.

External decoration: applications. Linguistic application for the external strategy of modal licensing will be found in areas where one wants to induce

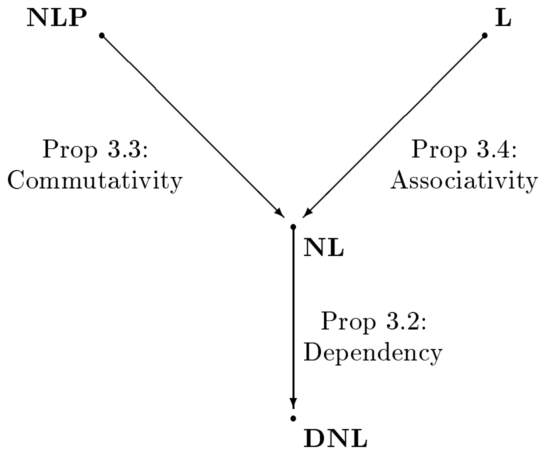


Figure 6: Licensing structural relaxation: precedence, dominance, dependency

structural relaxation in a configuration from the outside. The complementary view, where a subconfiguration induces structural relaxation in its context, is explored in §3.2 below. For the outside perspective, consider a non-commutative default regime with $P_\diamond$ for the modal extension. Collapse of the directional implications is underivable, $\nvdash A/B \longleftrightarrow B\backslash A$, but the modal variant below is. In general terms: a lexical assignment $A/\Box^{\downarrow}\Diamond B$ will induce commutativity for the argument subtype.

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad (A)^{\diamond} \Rightarrow \Diamond A}{((A/B,B)^{\bullet})^{\diamond} \Rightarrow \Diamond A}\,/L}{((B,A/B)^{\bullet})^{\diamond} \Rightarrow \Diamond A}\,P_\diamond}{(B,A/B)^{\bullet} \Rightarrow \Box^{\downarrow}\Diamond A}\,\Box^{\downarrow}R}{A/B \Rightarrow B\backslash\Box^{\downarrow}\Diamond A}\,\backslash R$$

Similarly, in the context of a non-associative default regime with $A_\diamond$ for the modal extension, one finds the following modal variant of the Geach rule, which remains underivable without the modal decoration.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow C \quad (\Box^{\downarrow}B)^{\diamond} \Rightarrow B}{(\Box^{\downarrow}B/C,C)^{\bullet})^{\diamond} \Rightarrow B}\,/L \quad (A)^{\diamond} \Rightarrow \Diamond A}{((A/B,((\Box^{\downarrow}B/C,C)^{\bullet})^{\diamond})^{\bullet})^{\diamond} \Rightarrow \Diamond A}\,/L}{((((A/B,\Box^{\downarrow}B/C)^{\bullet})^{\diamond},C)^{\bullet})^{\diamond} \Rightarrow \Diamond A}\,A_\diamond}{(((A/B,\Box^{\downarrow}B/C)^{\bullet})^{\diamond},C)^{\bullet} \Rightarrow \Box^{\downarrow}\Diamond A}\,\Box^{\downarrow}R}{((A/B,\Box^{\downarrow}B/C)^{\bullet})^{\diamond} \Rightarrow \Box^{\downarrow}\Diamond A/C}\,/R}{(A/B,\Box^{\downarrow}B/C)^{\bullet} \Rightarrow \Box^{\downarrow}(\Box^{\downarrow}\Diamond A/C)}\,\Box^{\downarrow}R}{A/B \Rightarrow \Box^{\downarrow}(\Box^{\downarrow}\Diamond A/C)/(\Box^{\downarrow}B/C)}\,/R$$

3.2 Modal labelling: the internal perspective

The embeddings discussed in the previous section license special structural behaviour by *external* decoration of product configurations: in the scope of the $\Diamond$ operator the product gains access to a structural rule which is unavailable in the default resource management of the logic in question. In view of the intended linguistic applications of structural modalities we would like to complement the external modalization strategy by an *internal* one where a structural rule is applicable to a product configuration provided one of its subtypes is modally decorated. In fact, the examples of modally controlled constraints we gave at the beginning of this paper were of this form. For the internal perspective, the modalized versions of Permutation and Associativity take the form shown below.

$$
\begin{array}{ll}
(P'_\diamond) & \diamond A \bullet B \longleftrightarrow B \bullet \diamond A \\
(A'_\diamond) & A_1 \bullet (A_2 \bullet A_3) \longleftrightarrow (A_1 \bullet A_2) \bullet A_3 \quad (\text{provided } A_i = \diamond A, 1 \leq i \leq 3)
\end{array}
$$

We prove embedding theorems for internal modal decoration in terms of the following translation mapping, which labels positive (proper) subformulae with the modal prefix $\diamond\Box^{\downarrow}$ and leaves negative subformulae undecorated.

Definition 3.5 Embedding translations $(\cdot)^+, (\cdot)^- : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\diamond)$ for positive and negative formula occurrences.

$$
\begin{array}{rcc rcc}
(p)^+ & = & p & (p)^- & = & p \\
(A \bullet_1 B)^+ & = & \diamond\Box^{\downarrow}(A)^+ \bullet_0 \diamond\Box^{\downarrow}(B)^+ & (A \bullet_1 B)^- & = & (A)^- \bullet_0 (B)^- \\
(A/_1 B)^+ & = & \diamond\Box^{\downarrow}(A)^+/_0(B)^- & (A/_1 B)^- & = & (A)^-/_0\diamond\Box^{\downarrow}(B)^+ \\
(B\backslash_1 A)^+ & = & (B)^-\backslash_0\diamond\Box^{\downarrow}(A)^+ & (B\backslash_1 A)^- & = & \diamond\Box^{\downarrow}(B)^+\backslash_0(A)^-
\end{array}
$$

The theorems embedding a stronger logic $\mathcal{L}_1$ into a more discriminating system $\mathcal{L}_0$ now assume the following general form, where $\mathcal{R}'_\diamond$ is the modal version of the structural rule package discriminating between $\mathcal{L}_1$ and $\mathcal{L}_0$.

Proposition 3.6

$$
\mathcal{L}_1 \vdash A \to B \quad \text{iff} \quad \mathcal{L}_0\diamond + \mathcal{R}'_\diamond \vdash A^+ \to B^-
$$

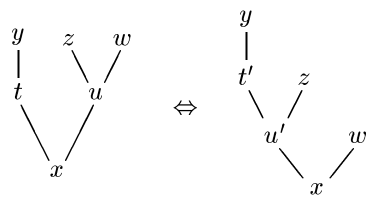
As an illustration we consider the embedding of **L** into **NL**$\diamond$ which involves licensing of Associativity in terms of the postulate $(A'_\diamond)$. The frame construction method we employ is completely general: it can be used unchanged for the other cases of licensing embedding one may want to consider.

The proof of the $(\Rightarrow)$ direction of Prop 3.6 is by easy induction. We present a Gentzen derivation of the Geach rule as an example. The type responsible for licensing $A'_\diamond$ in this case is $\diamond\Box^{\downarrow}(B/C)^+$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^+ \Rightarrow C^-}{(\Box^{\downarrow}C^+)^{\diamond} \Rightarrow C^-}\,L\Box^{\downarrow}}{\diamond\Box^{\downarrow}C^+ \Rightarrow C^-}\,L\diamond \quad \dfrac{\cdots}{\diamond\Box^{\downarrow}B^+ \Rightarrow B^-}}{\diamond\Box^{\downarrow}B^+/C^-, \diamond\Box^{\downarrow}C^+ \Rightarrow B^-}\,L/}{((\Box^{\downarrow}(\diamond\Box^{\downarrow}B^+/C^-))^{\diamond}, \diamond\Box^{\downarrow}C^+) \Rightarrow B^-}\,L\Box^{\downarrow} \quad \dfrac{\cdots}{\diamond\Box^{\downarrow}A^+ \Rightarrow A^-}}{(\diamond\Box^{\downarrow}A^+/B^-, ((\Box^{\downarrow}(\diamond\Box^{\downarrow}B^+/C^-))^{\diamond}, \diamond\Box^{\downarrow}C^+)) \Rightarrow A^-}\,L/}{((\diamond\Box^{\downarrow}A^+/B^-, (\Box^{\downarrow}(\diamond\Box^{\downarrow}B^+/C^-))^{\diamond}), \diamond\Box^{\downarrow}C^+) \Rightarrow A^-}\,A'_\diamond}{((\diamond\Box^{\downarrow}A^+/B^-, \diamond\Box^{\downarrow}(\diamond\Box^{\downarrow}B^+/C^-)), \diamond\Box^{\downarrow}C^+) \Rightarrow A^-}\,L\diamond}{\diamond\Box^{\downarrow}A^+/B^- \Rightarrow (A^-/\diamond\Box^{\downarrow}C^+)/\diamond\Box^{\downarrow}(\diamond\Box^{\downarrow}B^+/C^-)}\,R/, R/}{(A/B)^+ \Rightarrow ((A/C)/(B/C))^-}\,(\cdot)^+, (\cdot)^-
$$

For the $(\Leftarrow)$ direction, we proceed by contraposition. Suppose $\mathbf{L} \not\vdash A \to B$. Completeness tells us there exists an **L** model $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ with a point $a \in W$ such that $\mathcal{M}_1, a \models A$ but $\mathcal{M}_1, a \not\models B$. From $\mathcal{M}_1$ we want to construct an $\mathbf{NL}\diamond + A'_\diamond$ model $\mathcal{M}_0 = \langle W_0, R_\diamond, R_0, V_0\rangle$ such that $A^+ \to B^-$ fails. Recall that R_0 has to satisfy the frame conditions for the modal versions $A'_\diamond$ of the Associativity postulate. We give one instantiation below.

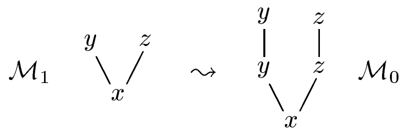
$$
(A'_\diamond) \quad \diamond A \bullet_0 (B \bullet_0 C) \longleftrightarrow (\diamond A \bullet_0 B) \bullet_0 C
$$



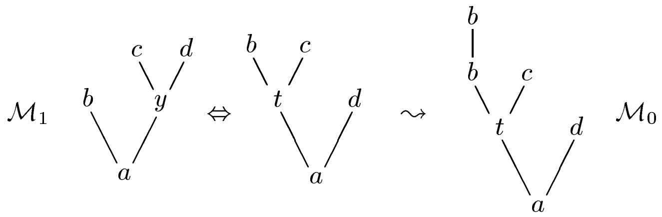
$$(\dagger) \quad (\forall xyzw \in W_0)$$

$$\exists tu(R_0xtu \wedge R_\diamond ty \wedge R_0uzw) \Leftrightarrow \exists t'u'(R_0xu'w \wedge R_0u't'z \wedge R_\diamond t'y)$$

The model construction proceeds as follows. We put the falsifying point $a \in W_0$, and for every triple $(xyz) \in R_1$ we put $x, y, z \in W_0$ and $(xyz) \in R_0$, $(yy) \in R_\diamond$, $(zz) \in R_\diamond$.



We have to show that the model construction realizes the frame condition ($\dagger$) (and its relatives) in $\mathcal{M}_0$. Suppose $\exists xy(R_0axy \wedge R_\diamond xb \wedge R_0ycd)$. By the model construction, $x = b$, so R_0aby which has the pre-image R_1aby. The pre-image of R_0ycd is R_1ycd. The combination of these two R_1 triangles satisfies the Associativity frame condition of **L**, so that we have a point t such that $R_1atd \wedge R_1tbc$. Again by the model construction, this means in $\mathcal{M}_0$ we have $\exists z, t(R_0tzc \wedge R_\diamond zb \wedge R_0atd)$, as required.

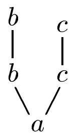


The central Truth Preservation Lemma now is that for any $a \in W_1 \cap W_0$,

$$\mathcal{M}_1, a \models A \quad \text{iff} \quad \mathcal{M}_0, a \models A^+ \quad \text{iff} \quad \mathcal{M}_0, a \models A^-$$

We concentrate on the $(\cdot)^+$ case — the $(\cdot)^-$ case is straightforward.

($\Rightarrow$) Suppose $\mathcal{M}_1, a \models A \bullet_1 B$. We have to show that $\mathcal{M}_0, a \models \diamond\Box^\downarrow A^+ \bullet_0 \diamond\Box^\downarrow B^+$. By assumption, there exist b, c such that R_1abc, and $\mathcal{M}_1, b \models A$, $\mathcal{M}_1, c \models B$. By inductive hypothesis and the model construction algorithm, we have in $\mathcal{M}_0$



$$\mathcal{M}_0, b \models A^+ \quad \mathcal{M}_0, c \models B^+$$

Observe that if x is the only point accessible from x via $R_\diamond$ (as is the case in $\mathcal{M}_0$), then for any formula ϕ, $x \models \phi$ iff $x \models \diamond\phi$ iff $x \models \Box^\downarrow\phi$ iff $x \models \diamond\Box^\downarrow\phi$. Therefore, from the above we can conclude $\mathcal{M}_0, b \models \diamond\Box^\downarrow A^+$ and $\mathcal{M}_0, c \models \diamond\Box^\downarrow B^+$, hence $\mathcal{M}_0, a \models \diamond\Box^\downarrow A^+ \bullet_0 \diamond\Box^\downarrow B^+$.

($\Leftarrow$) Suppose $\mathcal{M}_0, a \models \Diamond\Box^{\downarrow}A^+ \bullet_0 \Diamond\Box^{\downarrow}B^+$. We show that $\mathcal{M}_1, a \models A \bullet_1 B$. By assumption, there exist b, c such that R_0abc, and $\mathcal{M}_0, b \models \Diamond\Box^{\downarrow}A^+$, $\mathcal{M}_0, c \models \Diamond\Box^{\downarrow}B^+$. In $\mathcal{M}_0$ all triangles are such that the daughters have themselves and only themselves accessible via $R_\diamond$. Using our observation again, we conclude that $\mathcal{M}_0, b \models A^+$, $\mathcal{M}_0, c \models B^+$, and by inductive assumption $\mathcal{M}_1, a \models A \bullet B$.

We leave the implicational formulas to the reader.

COMMENT: FULL INTERNAL LABELING. Licensing of structural relaxation is implemented in the above proposal via modal versions of the structural postulates requiring at least *one* of the internal subtypes to be $\Diamond$ decorated. It makes good sense to consider a variant of internal licensing, where one requires *all* relevant subtypes of a structural configuration to be modally decorated — depending on the application one has in mind, one could choose one or the other. Embeddings with this property have been studied for algebraic models by [Venema 93, Versmissen 93]. In the terms of our minimalistic setting, modal structural postulates with full internal labeling would assume the following form.

$$
\begin{array}{ll}
(P''_\diamond) & \Diamond A \bullet \Diamond B \longleftrightarrow \Diamond B \bullet \Diamond A \\
(A''_\diamond) & \Diamond A \bullet (\Diamond B \bullet \Diamond C) \longleftrightarrow (\Diamond A \bullet \Diamond B) \bullet \Diamond C
\end{array}
$$

One obtains the variant of the embedding theorems for full internal labeling on the basis of the modified translation $(\cdot)^{++}$ which marks *all* positive subformulae with the modal prefix $\Diamond\Box^{\downarrow}$. (Below we abbreviate $\Diamond\Box^{\downarrow}$ to μ.) In the model construction, one puts $(xx) \in R_\diamond$ (and nothing more) for every point x that has to be put in W_0.

$$
\begin{array}{rcc@{\qquad}rcc}
(p)^{++} & = & \mu p & (p)^- & = & p \\
(A \bullet_1 B)^{++} & = & \mu(\mu(A)^{++} \bullet_0 \mu(B)^{++}) & (A \bullet_1 B)^- & = & (A)^- \bullet_0 (B)^- \\
(A/_1 B)^{++} & = & \mu(\mu(A)^{++}/_0(B)^-) & (A/_1 B)^- & = & (A)^-/_0\mu(B)^{++} \\
(B\backslash_1 A)^{++} & = & \mu((B)^-\backslash_0\mu(A)^{++}) & (B\backslash_1 A)^- & = & \mu(B)^{++}\backslash_0(A)^-
\end{array}
$$

Proposition 3.7

$$\mathcal{L}_1 \vdash A \to B \quad \text{iff} \quad \mathcal{L}_0\Diamond + \mathcal{R}''{}_\diamond \vdash A^{++} \to B^-$$

ILLUSTRATION: EXTRACTION. For a concrete linguistic illustration of $\Diamond\Box^{\downarrow}$ labeling licensing structural relaxation, we return to the example of extraction from non-peripheral positions in relative clauses. The example below becomes derivable in $\mathbf{NL}\Diamond + (A'_\diamond, P'_\diamond)$ given a modally decorated type assignment $r/(s/\Diamond\Box^{\downarrow}np)$ to the relative pronoun, which allows the hypothetical $\Diamond\Box^{\downarrow}np$ assumption to find its appropriate location in the relative clause body via controlled Associativity and Permutation. We give the relevant part of the Gentzen derivation, abbreviating $(np\backslash s)/np$ as tv.

$$
\begin{array}{ll}
 & \ldots \text{that ((John read) yesterday)} \\
\mathbf{NL}\Diamond + (A'_\diamond, P'_\diamond) \vdash & (r/(s/\Diamond\Box^{\downarrow}np), ((np, (np\backslash s)/np), s\backslash s)) \Rightarrow r
\end{array}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\ldots}{((np,(tv,np)),s\backslash s) \Rightarrow s}}{((np,(tv,(\Box^{\downarrow}np)^\diamond)),s\backslash s) \Rightarrow s}\,L\Box^{\downarrow}}{(((np,tv),(\Box^{\downarrow}np)^\diamond),s\backslash s) \Rightarrow s}\,A'_\diamond}{((np,tv),((\Box^{\downarrow}np)^\diamond,s\backslash s)) \Rightarrow s}\,A'_\diamond}{((np,tv),(s\backslash s,(\Box^{\downarrow}np)^\diamond)) \Rightarrow s}\,P'_\diamond}{(((np,tv),s\backslash s),(\Box^{\downarrow}np)^\diamond) \Rightarrow s}\,A'_\diamond}{(((np,tv),s\backslash s),\Diamond\Box^{\downarrow}np) \Rightarrow s}\,L\Diamond}{((np,tv),s\backslash s) \Rightarrow s/\Diamond\Box^{\downarrow}np}\,R/
$$

Comparing this form of licensing modal decoration with the treatment in terms of a universal $\Box$ operator with $S4$ structural postulates, one observes that on the proof-theoretic level, the $\Diamond\Box^{\downarrow}$ prefix is able to mimick the behaviour of the $S4$ $\Box$ modality, whereas on the semantic level, we are not forced to impose transitivity and reflexivity constraints on the interpretation of $R_{\Diamond}$. With a translation $(\Box A)^{\sim} = \Diamond\Box^{\downarrow}(A)^{\sim}$, the characteristic T and 4 postulates for $\Box$ become valid type transitions in the pure residuation system for $\Diamond, \Box^{\downarrow}$, as the reader can check.

$$\begin{array}{lll} T: & \Box A \to A & \rightsquigarrow \quad \Diamond\Box^{\downarrow}A \to A \\ 4: & \Box A \to \Box\Box A & \rightsquigarrow \quad \Diamond\Box^{\downarrow}A \to \Diamond\Box^{\downarrow}\Diamond\Box^{\downarrow}A \end{array}$$

4 Discussion

In this final section, we reflect on some general logical and linguistic aspects of the proposed architecture, and raise a number of questions for future research.

Linear Logic and the sublinear landscape. In order to obtain controlled access to Contraction and Weakening, Linear Logic extends the formula language with operators which on the proof-theoretic level are governed by an $S4$-like regime. The 'sublinear' grammar logics we have studied show a higher degree of structural organization: not only the multiplicity of the resources matters, but also the way they are put together into structured configurations. These more discriminating logics suggest more delicate instruments for obtaining structural control. We have presented embedding theorems for the licensing and for the constraining perspective on substructural communication in terms of the pure logic of residuation for a set of unary multiplicatives $\Diamond, \Box^{\downarrow}$. In the frame semantics setting, these operators make more fine-grained structural distinctions than their $S4$ relatives which are interpreted with respect to a transitive and reflexive accessibility relation. But they are expressive enough to obtain full control over grammatical resource management. Our minimalistic stance is motivated by linguistic considerations. For reasons quite different from ours, and for different types of models, a number of recent proposals in the field of Linear Logic proper have argued for a decomposition of the '!,?' modalities into more elementary operators. For comparison we refer the reader to [Bucalo 94, Girard 95].

The price of diamonds. We have compared logics with a 'standard' language of binary multiplicatives with systems where the formula language is extended with the unary logical constants $\Diamond, \Box^{\downarrow}$. The unary operators, one could say, are the price one has to pay to gain structural control. Do we really have to pay this price, or could one faithfully embed the systems of Fig 1 *as they stand*? For answers in a number of specific cases, one can turn to [van Benthem 91].

A question related to the above point is the following. Our embeddings compare the logics of Fig 1 pairwise, adding a modal control operator for each translation. This means that self-embeddings, from $\mathcal{L}$ to $\mathcal{L}'$ and back, end up two modal levels higher, a process which reaches equilibrium only in languages with infinitely many $\Diamond, \Box^{\downarrow}$ control operators. Can one stay within some finite modal repertoire? We conjecture the answer is positive, but a definitive result would require a deeper study of the residuation properties of the $\Diamond, \Box^{\downarrow}$ family.

Pure embeddings versus modal structural rules. The embedding results presented here are globally of two types. One type — what we have called the pure embeddings — obtains structural control solely in terms of the modal decoration added in the translation mapping. The other type adds a relativized structural postulate which can be accessed in virtue of the modal decoration of the translation. For the licensing type of communication, the second type of embedding is fully natural. The target logic, in these cases, does not allow a form of structural manipulation which is available in the source logic: in a *controlled* form, we want to regain this flexibility. But the distinction between the two types of embedding does not coincide with the shift from licensing to constraining communication. We have seen in §2.3 that imposing structural constraints for logics sharing associative resource management requires modalized structural postulates, in addition to the modal decoration of the translation mapping. In these cases, the $\Diamond$ decoration has accidentally damaged the potential for associative rebracketing: the modalized associativity postulates repair this damage. We leave it as an open question whether one could realize pure embeddings for some of the logics of §2.3. A related question can be raised for the same family of logics under the licensing perspective: in these cases, we find not just the modal structural postulate for the parameter which discriminates between the logics, but in addition modal associativity, again because the translation schema has impaired the normal rebracketing.

Uniform versus customized translations. Another asymmetry that may be noted here is our implementation of the licensing type of communication in terms of a uniform translation schema, versus the constraining type of embeddings where the translations are specifically tailored towards the particular structural dimension one wants to control. Could one treat the constraining embeddings of §2 also in terms of a uniform translation scheme? And if so, would such a scheme be cheaper or more costly than the individual schemes in the text?

Complexity. A final set of questions relates to issues of computational complexity. For many of the individual logics in the sublinear cube complexity results (pleasant or unpleasant) are known. Do the embeddings allow transfer of such results to systems where we still face embarrassing open questions (such as: the issue of polynomial complexity for **L**)? In other words: what is the computational cost of the translations and modal structural postulates proposed? We conjecture that modalized versions of structural rules have the same computational cost as corresponding structural rules themselves.

Embeddings: linguistic relevance. We close with a remark for the reader with a linguistics background. The embedding results presented in this paper may seem somewhat removed from the daily concerns of the working grammarian. Let us try to point out how our results can contribute to the foundations of grammar development work. In the literature of the past five years, a great variety of 'structural modalities' has been introduced, with different proof-theoretic behaviour and different intended semantics. It has been argued that the defects of particular type systems (either in the sense of overgeneration, or of undergeneration) can be overcome by refining type assignment in terms of these structural modalities. The accounts proposed for individual linguistic phenomena are often ingenious, but one may legitimately ask what the level of generality of the proposals is. The embedding results of this paper show that the operators $\Diamond, \Box^{\downarrow}$ provide a general logic of constraints in the dimensions of order, dominance and dependency.

A Appendix

A.1 Axiomatic and Gentzen presentation

In this section we juxtapose the axiomatic presentations and the Gentzen formulation of the logics under discussion. The Lambek and Došen style axiomatic presentations are two equivalent ways of characterizing $\Diamond, \Box^{\downarrow}$, $\bullet, /$ and $\bullet, \backslash$ as residuated pairs of operators. For the equivalence between the axiomatic and the Gentzen presentations, we refer to [Moortgat 95]. This paper also establishes a Cut Elimination result for the language extended with $\Diamond, \Box^{\downarrow}$.

Definition A.1 Lambek-style axiomatic presentation.

$$A \rightarrow A \qquad \frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$$

$$\Diamond A \rightarrow B \Longleftrightarrow A \rightarrow \Box^{\downarrow} B$$

$$A \rightarrow C/B \quad \Longleftrightarrow \quad A \bullet B \rightarrow C \quad \Longleftrightarrow \quad B \rightarrow A \backslash C$$

Definition A.2 Došen style axiomatization.

$$A \rightarrow A \qquad \frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$$

$$\begin{array}{cc} \Diamond\Box^{\downarrow} A \rightarrow A & A \rightarrow \Box^{\downarrow}\Diamond A \\ A/B \bullet B \rightarrow A & A \rightarrow (A \bullet B)/B \\ B \bullet B\backslash A \rightarrow A & A \rightarrow B\backslash(B \bullet A) \end{array}$$

$$\frac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B} \qquad \frac{A \rightarrow B}{\Box^{\downarrow} A \rightarrow \Box^{\downarrow} B}$$

$$\frac{A \rightarrow B \quad C \rightarrow D}{A \bullet C \rightarrow B \bullet D}$$

$$\frac{A \rightarrow B \quad C \rightarrow D}{A/D \rightarrow B/C} \qquad \frac{A \rightarrow B \quad C \rightarrow D}{D\backslash A \rightarrow C\backslash B}$$

The formulations of Def A.1 and Def A.2 give the pure residuation logic for the unary and binary families. The logics of Fig 1 are then obtained by adding different packages of structural postulates, as discussed in §1.

Definition A.3 Gentzen presentation. Sequents $\Gamma \Rightarrow A$ with Γ a structured database of linguistic resources, A a formula. Structured databases are inductively defined as terms $\mathcal{T} ::= \mathcal{F} \mid (\mathcal{T}, \mathcal{T})^m \mid (\mathcal{T})^{\diamond}$, with binary $(\cdot, \cdot)^m$ or unary $(\cdot)^{\diamond}$ structural connectives corresponding to the (binary, unary) logical connectives. We add resource management mode indexing for logical and structural connectives to keep families with different resource management properties apart. This strategy goes back to [Belnap 82] and has been applied to *modal* display logics in [Kracht 93, Wansing 92], two papers which are related in a number of respects to our own efforts.

$$[\mathrm{Ax}]\frac{}{A \Rightarrow A} \qquad \frac{\Gamma \Rightarrow A \quad \Delta[A] \Rightarrow C}{\Delta[\Gamma] \Rightarrow C}[\mathrm{Cut}]$$

$$[\mathrm{R}\diamond]\frac{\Gamma \Rightarrow A}{(\Gamma)^{\diamond} \Rightarrow \diamond A} \qquad \frac{\Gamma[(A)^{\diamond}] \Rightarrow B}{\Gamma[\diamond A] \Rightarrow B}[\mathrm{L}\diamond]$$

$$[\mathrm{R}\Box^{\downarrow}]\frac{(\Gamma)^{\diamond} \Rightarrow A}{\Gamma \Rightarrow \Box^{\downarrow} A} \qquad \frac{\Gamma[A] \Rightarrow B}{\Gamma[(\Box^{\downarrow} A)^{\diamond}] \Rightarrow B}[\mathrm{L}\Box^{\downarrow}]$$

$$[\mathrm{R}/_m]\frac{(\Gamma, B)^m \Rightarrow A}{\Gamma \Rightarrow A/_m B} \qquad \frac{\Gamma \Rightarrow B \qquad \Delta[A] \Rightarrow C}{\Delta[(A/_m B, \Gamma)^m] \Rightarrow C}[\mathrm{L}/_m]$$

$$[\mathrm{R}\backslash_m]\frac{(B, \Gamma)^m \Rightarrow A}{\Gamma \Rightarrow B\backslash_m A} \qquad \frac{\Gamma \Rightarrow B \qquad \Delta[A] \Rightarrow C}{\Delta[(\Gamma, B\backslash_m A)^m] \Rightarrow C}[\mathrm{L}\backslash_m]$$

$$[\mathrm{L}\bullet_m]\frac{\Gamma[(A, B)^m] \Rightarrow C}{\Gamma[A \bullet_m B] \Rightarrow C} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{(\Gamma, \Delta)^m \Rightarrow A \bullet_m B}[\mathrm{R}\bullet_m]$$

Structural postulates, in the axiomatic presentation, have been presented as transitions $A \to B$ where A and B are constructed out of formula variables $p_1, \ldots, p_n$ and logical connectives $\bullet_m$, $\diamond$. For structure variables $\Delta_1, \ldots, \Delta_n$ and structural connectives $(\cdot,\cdot)^m$, $(\cdot)^{\diamond}$, define the structural equivalent $\sigma(A)$ of a formula A as indicated below (cf [Kracht 93]):

$$\sigma(p_i) = \Delta_i \quad \sigma(A \bullet_m B) = (\sigma(A), \sigma(B))^m \quad \sigma(\diamond A) = (\sigma(A))^{\diamond}$$

The transformation of structural postulates into Gentzen rules allowing Cut Elimination then is straightforward: a postulate $A \to B$ translates as the Gentzen rule

$$\frac{\Gamma[\sigma(B)] \Rightarrow C}{\Gamma[\sigma(A)] \Rightarrow C}$$

In the cut elimination algorithm, one shows that if a structural rule precedes a Cut inference, the order of application of the inferences can be permuted, pushing the Cut upwards. See [Došen 89] for the case of global structural rules, [Moortgat 95] for the $\diamond$ cases.

In the multimodal setting, structural rules are relativized to the appropriate resource management modes, as indicated by the mode index. An example is given below (for k a commutative and l an associative regime). Where no confusion is likely to arise, in the text we use the conventional symbols for different families of operators, rather than the official mode indexing on one generic set of symbols.

$$\frac{\Gamma[(\Delta_2, \Delta_1)^k] \Rightarrow A}{\Gamma[(\Delta_1, \Delta_2)^k] \Rightarrow A}[\mathrm{P}] \qquad \frac{\Gamma[((\Delta_1, \Delta_2)^l, \Delta_3)^l] \Rightarrow A}{\Gamma[(\Delta_1, (\Delta_2, \Delta_3)^l)^l] \Rightarrow A}[\mathrm{A}]$$

$$cf\ A \bullet_k B \to B \bullet_k A \qquad cf\ A \bullet_l (B \bullet_l C) \to (A \bullet_l B) \bullet_l C$$

A.2 Trading ternary relations for binary ones

The embedding results developed above suggest the question whether Lambek systems can be embedded into standard modal logic with unary modalities. Moreover, it is at least of purely modal interest to see if the work of a ternary accessibility relation can be done by binary ones. So far, we have analyzed the Lambek Calculus **NL** and its neighbours as a modal logic with binary modalities. Can we also analyze it as a modal logic with only unary modalities? The answer is positive: we prove that **NL** can be faithfully embedded into minimal temporal logic K^t. The job will be done in two steps: first, we show how **NL** can be embedded into bimodal temporal logic $K^t_{1,2}$, and then we use Thomason's result on embedding bimodal systems into unary ones.

Definition A.4 *Minimal Bi-Tense Logic.* Let $K^t_{1,2}$ be a minimal tense logic having two 'forward looking operators' $\Box_1$, $\Box_2$ (with existential duals $\Diamond_1$, $\Diamond_2$) and two corresponding 'backward looking' operators $\Box^\downarrow_1$, $\Box^\downarrow_2$ (with existential duals $\Diamond^\downarrow_1$, $\Diamond^\downarrow_2$). The modal language of $K^t_{1,2}$ has its formulas built up from propositional letters according to the rule:

$$\phi ::= p|\neg\phi|\phi\&\psi|\Diamond_1\phi|\Diamond_2\phi|\Diamond^\downarrow_1\phi|\Diamond^\downarrow_2\phi|\Box_1\phi|\Box_2\phi|\Box^\downarrow_1\phi|\Box^\downarrow_2\phi$$

By standard methods, $K^t_{1,2}$ can be axiomatized using

Axioms

- all tautologies of classical propositional logic
- all modal distribution axioms
  $\Box_i(A \supset B) \supset (\Box_i A \supset \Box_i B)$ and $\Box^\downarrow_i(A \supset B) \supset (\Box^\downarrow_i A \supset \Box^\downarrow_i B)$
- all tense-logical conversion axioms
  $\Diamond_i\Box^\downarrow_i A \supset A$ and $A \supset \Box^\downarrow_i\Diamond_i A$

Rules

- modus ponens
- necessitation $A/\Box_i A$ and $A/\Box^\downarrow_i A$, where i=1,2.

A $K^t_{1,2}$ *model* is an ordinary bimodal model $M = \langle W, R^2_1, R^2_2, V\rangle$ with truth definition (i=1,2)

$$\begin{array}{lcl} M, a \models \Box_i A & \iff & \forall b(R_i ab \Rightarrow M, b \models A) \\ M, a \models \Box^\downarrow_i A & \iff & \forall b(R_i ab \Rightarrow M, b \models A) \end{array}$$

A faithful embedding of the non-associative Lambek Calculus into $K^t_{1,2}$ runs as follows:

$$\begin{array}{rcl} p^\sharp & = & p \\ (A \bullet B)^\sharp & = & \Diamond_1(\Diamond_1 A^\sharp \& \Diamond_2 B^\sharp) \\ (A\backslash B)^\sharp & = & \Box^\downarrow_2(\Diamond_1 A^\sharp \supset \Box^\downarrow_1 B^\sharp) \\ (B/A)^\sharp & = & \Box^\downarrow_1(\Diamond_2 A^\sharp \supset \Box^\downarrow_1 B^\sharp) \end{array}$$

Theorem A.5 *First Embedding Theorem.* The following assertions are equivalent:

(i) $A \vdash B$ is derivable in **NL**
(ii) $A^\sharp \vdash B^\sharp$ is derivable in $K^t_{1,2}$

Proof. The direction (i) $\Rightarrow$ (ii) can be proved by an easy induction on the length of the **NL**-derivation for $A \vdash B$. For example, consider the axiom

$$A \vdash B\backslash(B \bullet A)$$

Its translation

$$A^\sharp \vdash \Box^\downarrow_2(\Diamond_1 B^\sharp \supset \Box^\downarrow_1\Diamond_1(\Diamond_1 B^\sharp \& \Diamond_2 A^\sharp))$$

is derivable in the minimal bi-tense logic $K^t_{1,2}$. For the converse direction (ii) $\Rightarrow$ (i), we use a semantical representation argument. Let $A \vdash B$ be underivable in **NL**. By the above completeness of **NL** with respect to ternary semantics, there exists a ternary model M where $A \vdash B$ fails. So, there exists a world $k \in W$ which verifies A, but falsifies B. We construct a $K^t_{1,2}$ model $M = \langle W^*, R_1, R_2, V^*\rangle$ where $A^\sharp \vdash B^\sharp$ fails, as follows:

- put k in W^*
- if $\langle a, bc\rangle \in R$, then take a *fresh* object x, and put $\langle ax\rangle$ and $\langle xb\rangle$ in R_1 , $\langle xc\rangle$ in R_2, a, b, c, x in W^*
- set for all $a \in W \cap W^*$, $a \in V^*(p)$ iff $a \in V(p)$

□

Claim A.6 For all categorial formulas A, and all $a \in W \cap W^*$,

$$M^*, a \models A^\sharp \iff M, a \models A$$

Proof. Induction on the length of A. The basic case is a direct consequence of the definition of M^*. We demonstrate only one typical clause of the inductive step, to illustrate this kind of elementary semantic argument over ternary models.

(1) Suppose $M^*, a \models \Box_2^\downarrow(\Diamond_1 A^\sharp \supset \Box_1^\downarrow B^\sharp)$
We need to show that $M, a \models A \to B$.

(2) Suppose (a) $R^3 c, ba$ and (b) $M, b \models A$
We need to show that $M, c \models B$

(3) By the inductive hypothesis : $M^*, b \models A^\sharp$.
By the above construction of M^*, (2(a)) yields

(4) (a) $R_1 cx$ (b) $R_1 xb$ (c) $R_2 xa$

(5) By the truth definition:
$M^*, x \models \Diamond_1 A^\sharp$
From (1) and (4(c))
$M^*, x \models \Diamond_1 A\sharp \supset \Box_1^\downarrow B^\sharp$
Thus $M^*, x \models \Box_1^\downarrow B^\sharp$

(6) From (4(a)) we get $M^*, c \models B^\sharp$
and by inductive hypothesis $M, c \models B$.

Here is the converse argument. Again, we start by successively unpacking what needs to be shown.

(1) Suppose $M, a \models A \to B$.
We need to shaw that $M^*, a \models \Box_2^\downarrow(\Diamond_1 A^\sharp \supset \Box_1^\downarrow B^\sharp)$

(2) Suppose (a) $R_2 xa$ (b) $M^*, x \models \Diamond_1 A^\sharp$ (c) $R_1 bx$
We need to show that $M^*, b \models B^\sharp$

(3) By (2(b)) there exists c such that
(a) $R_1 xc$ and (b) $M^*, c \models A^\sharp$
By inductive assumption :
(c) $M, c \models A$

(4) Note, that by the construction of M^*,
$R_1 bx$, $R_1 xc$, $R_2 xa$
must 'come from' a unique triangle, namely Rb, ca.
Then, from (1) and (3(c)), $M \models B$, and therefore
$M^*, b \models B^\sharp$. □

The preceding Claim implies that any ternary **NL**-counter-model M for a sequent $A \vdash B$ can be transformed into $K^t_{1,2}$ model M^* where $A^\sharp \vdash B^\sharp$ fails. Hence $A^\sharp \vdash B^\sharp$ is not derivable in $K^t_{1,2}$. This proves the faithful embedding. □

Theorem A.7 *Second Embedding Theorem.* The following assertions are equivalent:

(i) $A \vdash B$ is derivable in **NL**
(ii) $A^\sharp \vdash B^\sharp$ is derivable in Kt

Proof. This is a direct consequence of S.K. Thomason's results on embeddings of bimodal systems into unary modal ones, see [Thomason 72]. $\square$

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