Meeting a Modality?

Restricted Permutation for the Lambek Calculus

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Abstract

This paper contributes to the theory of hybrid substructural logics, i.e. weak logics given by a Gentzen-style proof theory in which there are constraints on the application of some structural rules. In particular, we address the question how to add an operator to the Lambek Calculus in order to give it a restricted access to the rule of Permutation, an extension which is partly motivated by linguistic applications. In line with tradition, we use the operator (∇) as a label telling us how the marked formula may be used, qua structural rules. New in our approach is that we do not see ∇ as a modality. Rather, we treat a formula ∇A as the *meet* of A with a special type Q. In this way we can make the specific structural behaviour of marked formulas more explicit.

We define a minimal proof calculus for the system and prove some nice properties of it, like cut-elimination, decidability an an embedding result. The main motivation for our approach however is that we can supply the proof system with an intuitive semantics.

1 Introduction

Substructural logics are logics one can give by a Gentzen-style derivation system lacking some or all of the structural rules like associativity, permutation, weakening or contraction. Such logics have received a lot of attention in recent years, partly because of their interest for applications in e.g. computer science (linear logic, cf. Girard [7]) or linguistics (Lambek Calculus, cf. Lambek [12] for the original article, or Moortgat [13], van Benthem [2] or Morrill [14] for recent developments). There is a bewildering variety of substructural logics, as we may drop any subset of structural rules from a standard derivation system for let's say intuitionistic logic. Of this landscape, Wansing [21] draws a partial map in the form of a lattice, set-inclusion of the derivable sequents being the ordering.

Of both practical and theoretical interest now is the question, whether it is possible to define logics that are *hybrid* in the sense that they make a *restricted* use of one or more structural rules: one wants to travel in the substructural landscape.

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To illustrate how natural this question is for linguistic applications, we look at the Lambek Calculus L in some more detail (for definitions we refer to the next section). Consider a very simple example: the relative clause `that Mary read`, which should function as a noun modifier (CN/CN), like in `the books that Mary read`. We assume that we have assigned the following types already: NP to `Mary` and $(NP\backslash S)/NP$ to `read`, and are about to assign a type to `that`. In the relative clause, the object (`that`) is not in the expected place (after `read`). We might solve this puzzle by assigning the type $R/(S/NP)$ to `that` (where R denotes the noun-modifier $CN\backslash CN$), but this solution is not very satisfactory, as it does not take care of the similar example `the books that Mary read last year`. One way out is given by Moortgat [13], who formalizes the concept of a discontinous constituent in a Categorial Grammar. Another option is to relax the order sensitivity of the Lambek Calculus, by allowing application of the rule of Permutation to *some* sequents.

In this approach, it was a quite natural move to look for inspiration at linear logic, which also has devices built in to encapsulate stronger logics: the so-called exponential operators (! and ?). The idea was taken up by Morrill et alii in [15], who added an operator $\Box$ to the Lambek calculus, with basic rules

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Box A, \Gamma_2 \Rightarrow B}\,[\Box L] \qquad \text{and} \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,[\Box R]$$

where $\Box\Gamma$ denotes $\Box X_1, \ldots, \Box X_n$ if $\Gamma = X_1, \ldots, X_n$. The rules allowing permutation of boxed formulas then are

$$\frac{\Gamma_1, B, \Box A, \Gamma_2 \Rightarrow C}{\Gamma_1, \Box A, B, \Gamma_2 \Rightarrow C}\,[P\Box]$$

(The double bar indicates that we have both the downward and the upward rule.)

Independently, Došen has addressed this issue (cf. [5, 6]) in a more general way and concentrating on proof-theoretical properties like embeddability.

The problem however is to give a nice *semantics* for $L\Box$. The Lambek calculus L itself is known to have a nice semantics: it is sound and complete with respect to semigroup semantics, its product-free version even with respect to *free* semigroups, cf. Buszkowski [3] (again, for definitions, see section 2). The $S4$-like character of $[\Box L]$ and $[\Box R]$ lead people to see the operators as *modalities*, and in this line of thinking an interpretation for $\Box$ would use some accessibility relation. Some results are known in this direction, cf. Kurtonina [11] for a completeness result of $L\Box$ with respect to models consisting of a semigroup-like structure expanded with an accessibility relation. In de Paiva [16], a category-theoretic interpretation is given which was inspired, again, by linear logic. However, it is not immediately clear what *intuitive* meaning one can assign to these proposed interpretations for the $\Box$-operator.

Indeed, more natural from the applicational (linguistic) point of view seems to be the *subalgebra interpretation* of Hepple [8] and Morrill [14]: the boxed Lambek calculus $L\Box$ is interpreted in semigroups $\mathcal{G}$ having a designated 'commuting subalgebra' $\mathcal{G}'$ (i.e. consisting of elements g' satisfying $(\forall x \in G) \;\; g' \cdot x = x \cdot g'$). Here the meaning function assigns to a boxed formula $\Box A$ the intersection of the meaning of A with the universe of the subalgebra. In other words, boxed formulas are special pieces of information, with a special commutative semantic behaviour.

Unfortunately, the rules given above, although sound, are not sufficient to prove completeness with respect to this subalgebra semantics. This was shown by Versmissen [20]; replacing $[\Box R]$ by

$$\frac{\Gamma_1 \Rightarrow \Box B_1 \quad \ldots \quad \Gamma_n \Rightarrow \Box B_n \quad \Gamma_1, \ldots, \Gamma_n \Rightarrow A}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \Box A} [\Box R']$$

he can prove completeness for the subalgebra interpretation.

To analyze the rule $[\Box R']$, let us drop for a moment the association of $\Box$ with modal logic, and read $\Box A$ as 'a special A'. Now $[\Box R']$ says the following: if Γ proves an A, it proves that A is special if it can be decomposed into sequences proving (some other[1]) formulas to be special. Our idea is now to make this 'specialness' *explicit* by adding a special type Q to the language, and reading $\Box A$ as some sort of *meet* of Q and A. In the semigroup semantics, Q is then assigned a special subset of the semigroup, and $[\Box R']$ can be decomposed into

$$\frac{\Gamma \Rightarrow Q \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \text{and} \qquad \frac{\Gamma_1 \Rightarrow Q \quad \Gamma_2 \Rightarrow Q}{\Gamma_1, \Gamma_2 \Rightarrow Q}$$

where intuitively, the latter rule states that the Q-elements of the semigroup indeed form a subalgebra. Adding the $\Box$-permutation rule $[P\Box]$ ensures that this subalgebra consists of commuting elements.

Note that the structural *behaviour* of a formula $\Box A$ is in some sense independent of the formula A; therefore we call operators in this approach *outward strengtheners*. This paper is a first investigation into some logical consequences of this idea. We will confine ourselves here to discussing the Lambek Calculus, but of course, the idea is not limited to this particular site in the substructural landscape: in fact it can be applied to every combination of a substructural logic and a structural derivation rule. In the extended version [19] of this paper, we develop our approach in more/full generality.

To give an overview of the paper: in the next section we state some preliminary facts concerning the Lambek calculus. In section 3 we provide the formal definition of our approach and we discuss some applications. The sections 4, 5 and 6 then are devoted to logical aspects of our calculus, viz. cut-elimination, semantics and embeddings. We finish with giving our conclusions and some directions for further research.

ACKNOWLEDGEMENTS I would like to thank Natasha Kurtonina, Michael Moortgat, Glyn Morrill, Valeria de Paiva, Koen Versmissen and Heinrich Wansing for encouragement and stimulating discussions. Special thanks are due to Dirk Roorda, who detected a crucial error in one of the proofs.

[1] One may restrict the rule by demanding that the B_i's are subformulas of formulas in the Γ_i's and A.

2 Preliminaries

In this section we give some technical background needed for understanding the paper. In particular, we will define what the Lambek calculus is.

Let X be a finite set of basic types or atomic formulas. The set $T(X)$ of *types* (or *formulas*) in X is defined as the least set satisfying (i) $X \subset T$ and (ii) if A and B are in T, then so are A/B, $A\backslash B$ and $A \bullet B$. A *term* is a sequence of formulas. A *sequent* is a pair consisting of a term Γ and a formula A, and is usually denoted as $\Gamma \Rightarrow A$. We understand as well-known the notion of *subformula, subterm*, *substitution* of terms for subterms in terms, etc.

The (associative) *Lambek Calculus* L is a Gentzen-style proof system of sequents $\Gamma \Rightarrow A$ (where Γ is not empty). Its logical rules are the axiom of Identity and the rule of Cut:

$$[Id]: \quad A \Rightarrow A \qquad\qquad \frac{\Gamma \Rightarrow A \quad \Delta_1, A, \Delta_2 \Rightarrow B}{\Delta_1, \Gamma, \Delta_2 \Rightarrow B}\,[Cut].$$

The *operational* rules of L are

$$\frac{\Gamma \Rightarrow A \quad \Delta, B, \Delta' \Rightarrow C}{\Delta, B/A, \Gamma, \Delta' \Rightarrow C}\,[/L] \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}\,[/R]$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B, \Delta' \Rightarrow C}{\Delta, \Gamma, A\backslash B, \Delta' \Rightarrow C}\,[\backslash L] \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}\,[\backslash R]$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \bullet B, \Delta \Rightarrow C}\,[\bullet L] \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \bullet B}\,[\bullet R]$$

The notion of a *derivation* is defined as usual; here we are only interested in *theorems* of the system, i.e. sequents that can be derived without using premisses.

Note that L does *not* have the structural rule of Permutation

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}\,[P],$$

nor the rules of contraction, weakening or expansion. The *Lambek-van Benthem* calculus LP is obtained by adding the rule of permutation to L.

In this paper we will investigate some mathematical properties of extensions of L and LP. The first of these is *cut-elimination*: the rule of $[Cut]$, though one of the most characteristic principles of the concept of a logic, is not very attractive from a computational perspective. So, a desirable property of any logic for which one has applications in mind, is that the system can do without $[Cut]$, because it is *admissible* in the 'poorer' system. With respect to the basic systems L and LP, we state the following fact (cf. Lambek [12] for a proof).

Theorem 2.1 (Lambek)
The $[Cut]$-rule can be eliminated from both L and LP, i.e. every theorem of L (resp. LP) can be derived in L (resp. LP) without using $[Cut]$.

The second logical aspect of our extended Lambek calculi will be the semantics; therefore we will need some terminology concerning the interpretation of L in semi-groups and monoids. A *semi-group* is a structure $(G, \cdot)$ where $\cdot$ is a binary associative operator on the set G. A semigroup is *free* if it is (isomorphic to) an algebra of which the universe is formed by all strings over some language and the operation $\cdot$ is string concatenation. A *semigroup model* for L consists of a semigroup $\mathcal{G}$, together with a *valuation* $[\![\cdot]\!]$ mapping atomic formulas to subsets of G. Such a valuation can be extended to mapping arbitrary formulas and even terms to $Sb(G)$ by putting

$$\begin{array}{rcl} [\![B/A]\!] & = & \{c \in G \mid (\forall a \in [\![A]\!])c \cdot a \in [\![B]\!]\} \\ [\![B\backslash A]\!] & = & \{c \in G \mid (\forall b \in [\![B]\!])b \cdot c \in [\![A]\!]\} \\ [\![B \bullet A]\!] & = & \{a \cdot b \mid a \in [\![A]\!] b \in [\![B]\!]\} \\ [\![A_1, \ldots, A_n]\!] & = & \{g_1 \cdot \ldots \cdot g_n \mid g_i \in [\![A_i]\!]\}. \end{array}$$

A sequent $\Gamma \Rightarrow A$ is said to be *true* in a model, if $[\![\Gamma]\!] \subseteq [\![A]\!]$, *valid* in a class C of semigroups if it is true in every model based on a semigroup in the class.

Theorem 2.2 *(Buszkowski)*

1. *A sequent is derivable in L iff it is valid in the class of all semigroup models.*
2. *A product-free sequent is derivable in L iff it is valid in the class of all free semigroup models.*

3 The basic idea

Definition 3.1 *Let X be a set of basic types such that $Q, \nabla \notin X$. We define the set of types $T\nabla(X)$ as the smallest set Y satisfying: (i) $X \cup \{Q\} \subseteq Y$ and (ii) if A and B are in Y, then so are A/B, $A\backslash B$, $A \bullet B$ and ∇A.*

$L\nabla$ is the Lambek Calculus over $T\nabla$, extended with the following logical rules for Q and ∇.

The operator ∇ has two left rules:

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \nabla A, \Gamma_2 \Rightarrow B}\,[\nabla L, 1] \qquad \text{and} \qquad \frac{\Gamma_1, Q, \Gamma_2 \Rightarrow B}{\Gamma_1, \nabla A, \Gamma_2 \Rightarrow B}\,[\nabla L, 2]$$

and the following right rule:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow \nabla A}\,[\nabla R]$$

For Q, we only have one rule:

$$\frac{\Gamma_1 \Rightarrow Q \quad \Gamma_2 \Rightarrow Q}{\Gamma_1, \Gamma_2 \Rightarrow Q}\,[Q']$$

$L\nabla_P$ is $L\nabla$ extended with the following restricted permutation rule $[\nabla P]$:

$$\frac{\Gamma_1, \nabla A, B, \Gamma_2 \Rightarrow C}{\Gamma_1, B, \nabla A, \Gamma_2 \Rightarrow C}\,[P\nabla]$$

The rules for ∇ hardly need explanation: clearly ∇A is to be read as the intersection[2] of Q and A. The rule $[P\nabla]$ is equally perspicuous, indicating the intuitive meaning of ∇, namely to license permutation of the formulas that it marks.

Note that these four rules together already constitute a hybrid system $L\nabla_P^-$ between L and LP. To explain why we added the rule $[Q]$ to the basic system, let us have a look at the semigroup semantics for L. According to $L\nabla_P^-$, in a semigroup $(G, \cdot, [\![ ]\!])$, Q could be interpreted as any subset S of G satisfying $(\forall s \in S)(\forall g \in G) s \cdot g = g \cdot s$. However, we have a preference for the sub*algebra* interpretation: the product of two Q-elements should be a Q-element itself. For example, if A and B are both permutable types, then so should be their product $A \bullet B$. (Note the analogy with $\perp$ in intuitionistic logic here: one may view $\perp$ as just one of the atomic types, or as a special type denoting falsum. With respect to the semantics, the analogon is that in the second view one wants to interpret $\perp$ as a *special* subset of a Kripke model, namely the empty set.) The intuitive rule corresponding to this closure property (i.e. that the product of two Q's is a Q) is $[Q]$.

Now, an example of how to *use* the system: consider the phrase from the introduction: `the books that Mary read last year`. Analogous to Morrill [14, section VI.1], we assign the type $R/(S/\nabla N)$ to the lexical element `that`. To show that we can reduce the relative clause to type R, we derive

$$\texttt{Mary read last year} \Rightarrow S/\nabla N.$$

as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S \Rightarrow S \quad N \Rightarrow N}{N \quad (N\backslash S) \quad \Rightarrow S}[\backslash L] \qquad (N\backslash S) \Rightarrow (N\backslash S)}{N \quad (N\backslash S) \quad (N\backslash S)\backslash(N\backslash S) \quad \Rightarrow S}[\backslash L] \qquad N \Rightarrow N}{N \quad (N\backslash S)/N \quad N \quad (N\backslash S)\backslash(N\backslash S) \quad \Rightarrow S}[/L]}{N \quad (N\backslash S)/N \quad \nabla N \quad (N\backslash S)\backslash(N\backslash S) \quad \Rightarrow S}[\nabla L]}{N \quad (N\backslash S)/N \quad (N\backslash S)\backslash(N\backslash S) \quad \nabla N \quad \Rightarrow S}[P\nabla]}{\frac{N}{\texttt{Mary}} \quad \frac{(N\backslash S)/N}{\texttt{read}} \quad \frac{(N\backslash S)/(N\backslash S)}{\texttt{last year}} \quad \Rightarrow S/\nabla N}[/R]}{}$$

Note that $L\nabla_P$ cannot handle relative clauses with parasitic gaps as in

`the paper that John filed without reading`,

where in the relative clause `that John filed without reading`, semantically `paper` is supposed to be object of both `files` and of `reading`. However, again following Morril [14], we may extend $L\nabla_P$ by adding the restricted rule of Contraction

$$\frac{\Gamma_1, \nabla A, \nabla A, \Gamma_2 \Rightarrow B}{\Gamma_1, \nabla A, \Gamma_2 \Rightarrow B}\ [C\nabla]$$

[2] If we have a meet-operator in the language (with standard logical rules), then the equivalence of ∇A and $Q \wedge A$ is easily provable, and ∇A may be read as an abbreviation. There may be reasons however, where having an unrestricted meet-operator in the system is less attractive. For instance, adding a meet-operator to the Lambek calculus will pump up the recognizing power: recently M. Pentus proved the Chomsky Conjecture, namely that every language recognized by a Lambek Grammar is context-free. By a result of Kanazawa [10] it then follows that adding the meet operator makers languages recognizable that are strictly more complex.

and in the resulting calculus $L\nabla_{PE}$ one can easily show that the relative clause, `that John filed without reading`, will get the required modifier type $(CN\backslash CN)$.

4 Cut-elimination

A desirable property for substructural logics, is that applications of $[Cut]$ can be eliminated from the system. Unfortunately, our systems LQ_P and $L\nabla_P$ do *not* have this property. An example is given by the $L\nabla_P$-theorem $\nabla A, Q \Rightarrow Q \circ \nabla A$, proving which involves a necessary application of the Cut-rule. However, we can modify our logic in such a way that $[Cut]$ becomes eliminable. The idea is to move the effects of $[Cut]$ into the $[Q]$-rule and $[P\nabla]$ itself:

Definition 4.1 *LQQ is the system LQ where $[Q]$ is replaced by*

$$\frac{\Gamma_1 \Rightarrow Q \quad \Gamma_2 \Rightarrow Q \quad \Delta_1, Q, \Delta_2 \Rightarrow A}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \Rightarrow A}\,[Q']$$

If we add the following rules to LQQ, we obtain the system LQQ_P.

$$\frac{\Gamma_1 \Rightarrow Q \quad \Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \Rightarrow A}{\Delta_1, \Gamma_2, \Gamma_1, \Delta_2 \Rightarrow A}\,[P^l Q] \qquad \frac{\Gamma_2 \Rightarrow Q \quad \Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \Rightarrow A}{\Delta_1, \Gamma_2, \Gamma_1, \Delta_2 \Rightarrow A}\,[P^r Q]$$

Theorem 4.2 *All LQQ_P-theorems can be derived without applications of* $[Cut]$.

Proof.
We first treat LQQ, of which system we want to prove that every application of $[Cut]$ is eliminable from any proof, and thereto it suffices to show that applications of $[Cut]$ in a proof can always be replaced by cuts of a smaller degree, or permuted upwards. For brevity, the occurrence of the formula A in the antecedent of a sequent is not denoted by $\Gamma_1, A, \Gamma_2 \Rightarrow B$, but by $\Gamma[A] \Rightarrow B$.

Assume that we have in our LQQ-proof, an occurrence of [Cut] in the form

$$\frac{\Gamma \Rightarrow A \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma] \Rightarrow B}\,[Cut].$$

We will distinguish cases:

I First we look at the rule LR used to prove the left premisse:

a If LR is an axiom or one of the operational rules of L (i.e. for $/, \backslash$ or $\bullet$), proceed like in the standard cut-elimination proof of Lambek.

b If the Q'-rule was the last one applied on the left, the proof looks like the one below:

$$\frac{\dfrac{\Gamma_1 \Rightarrow Q \quad \Gamma_2 \Rightarrow Q \quad \Gamma[Q] \Rightarrow A}{\Gamma[(\Gamma_1, \Gamma_2)] \Rightarrow A}\,[Q'] \qquad \Delta[A] \Rightarrow B}{\Delta[\Gamma[(\Gamma_1, \Gamma_2)]] \Rightarrow B}\,[Cut]$$

and should be repaced by:

$$\dfrac{\Gamma_1 \Rightarrow Q \quad \Gamma_2 \Rightarrow Q \quad \dfrac{\Gamma[Q] \Rightarrow A \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma[Q]] \Rightarrow B}[Cut]}{\Delta[\Gamma[(\Gamma_1, \Gamma_2)]] \Rightarrow B}[Q']$$

c LR is $[\nabla L1]$. Convert the proof, as indicated below:

$$\dfrac{\dfrac{\Gamma[C] \Rightarrow A}{\Gamma[\nabla C] \Rightarrow A}[\nabla L1] \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma[\nabla C]] \Rightarrow B}[Cut] \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma[C] \Rightarrow A \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma[C]] \Rightarrow B}[Cut]}{\Delta[\Gamma[\nabla C]] \Rightarrow B}[\nabla L1]$$

d The case where LR is $[\nabla L2]$ is similar.

e The case where $\Gamma \Rightarrow A$ is proved by $[\nabla R]$, is taken care of by eliminating $[Cut]$ upwards into the derivation tree of the *right* premisse.

II Now, we have a look at how the right premisse was proved:

a If the right rule was a logical rule of L, proceed as in the cut-elimination proof for L.

b $\Delta[A] \Rightarrow B$ was proved by the Q'-rule; we make a subdistinction as to where in $\Delta[(\Delta_1, \Delta_2)]$ the formula A occurs. First assume that A occurs inside of Δ_1. Then the conversion is from

$$\dfrac{\Gamma \Rightarrow A \quad \dfrac{\Delta_1[A] \Rightarrow Q \quad \Delta_2 \Rightarrow Q \quad \Delta[Q] \Rightarrow B}{\Delta[(\Delta_1[A], \Delta_2)] \Rightarrow B}[Q']}{\Delta[(\Delta_1[\Gamma], \Delta_2)] \Rightarrow B}[Cut]$$

into

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad \Delta_1[A] \Rightarrow Q}{\Delta_1[\Gamma] \Rightarrow Q}[Cut] \quad \Delta_2 \Rightarrow Q \quad \Delta[Q] \Rightarrow B}{\Delta[(\Delta_1[\Gamma], \Delta_2)] \Rightarrow B}[Q']$$

The case where A occurs in Δ_2 is of course analogous. If A occurs in Δ, but outside of Δ_1 and Δ_2, we obtain the following:

$$\dfrac{\Gamma \Rightarrow A \quad \dfrac{\Delta_1 \Rightarrow Q \quad \Delta_2 \Rightarrow Q \quad \Delta[A][Q] \Rightarrow B}{\Delta[A][(\Delta_1, \Delta_2)] \Rightarrow B}[Q']}{\Delta[\Gamma][(\Delta_1, \Delta_2)] \Rightarrow B}[Cut]$$

which we transform into

$$\dfrac{\Delta_1 \Rightarrow Q \quad \Delta_2 \Rightarrow Q \quad \dfrac{\Gamma \Rightarrow A \quad \Delta[A][Q] \Rightarrow B}{\Delta[\Gamma][Q] \Rightarrow B}[Cut]}{\Delta[\Gamma][(\Delta_1, \Delta_2)] \Rightarrow B}[Q']$$

c The rule used to proved the right premisse, was $[\nabla R]$, or $[\nabla Li]$ in such a way that the cut-formula A is not the major formula of this rule; these cases cause no problems and are left to the reader.

III So, we are left with the main case of the Theorem, viz. where A is the major formula in both the left and the right premisse of the Cut, and in such a way that the rule applied left was $[\nabla R]$, and the rule applied right was $[\nabla Li]$. In this case we get

$$\dfrac{\dfrac{\Gamma \Rightarrow C \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow \nabla C}[\nabla R] \quad \dfrac{\Delta[P] \Rightarrow B}{\Delta[\nabla C] \Rightarrow B}[\nabla Li]}{\Delta[\Gamma] \Rightarrow B}[Cut] \quad \rightsquigarrow \quad \dfrac{\Gamma \Rightarrow P \quad \Delta[P] \Rightarrow B}{\Delta[\Gamma] \Rightarrow B}[Cut]$$

where P is either C or Q.

Now we consider the case of the logic LQQ_P. We have to consider a few extra cases:

If The left rule was $[P^lQ]$. Tranform the proof from

$$\dfrac{\dfrac{\Gamma[\Gamma_1, \Gamma_2] \Rightarrow B \quad \Gamma_1 \Rightarrow Q}{\Gamma[\Gamma_2, \Gamma_1] \Rightarrow B}[P^lQ] \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma[\Gamma_2, \Gamma_1]] \Rightarrow B}[Cut]$$

into

$$\dfrac{\Gamma_1 \Rightarrow Q \quad \dfrac{\Gamma[\Gamma_1, \Gamma_2] \Rightarrow B \quad \Delta[A] \Rightarrow B}{\Delta[\Gamma[\Gamma_1, \Gamma_2]] \Rightarrow B}[Cut]}{\Delta[\Gamma[\Gamma_2, \Gamma_1]] \Rightarrow B}[P^lQ]$$

IId The right rule was $[PQ]$; make a subdistinction, as to where in $\Delta[\Delta_1, \Delta_2]$ the formula A occurs (just like in case IIb). We only treat the case where A occurs inside of Δ_1 and the rule applied was $[P^lQ]$.

$$\dfrac{\Gamma \Rightarrow A \quad \dfrac{\Delta_1[A] \Rightarrow Q \quad \Delta[\Delta_1[A], \Delta_2] \Rightarrow B}{\Delta[\Delta_2, \Delta_1[A]] \Rightarrow B}[P^lQ]}{\Delta[\Delta_2, \Delta_1[\Gamma]] \Rightarrow B}[Cut]$$

becomes

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad \Delta_1[A] \Rightarrow Q}{\Delta_1[\Gamma] \Rightarrow Q}[Cut] \quad \dfrac{\Gamma \Rightarrow A \quad \Delta[\Delta_1[A], \Delta_2] \Rightarrow B}{\Delta[\Delta_1[\Gamma], \Delta_2] \Rightarrow B}[Cut]}{\Delta[\Delta_2, \Delta_1[\Gamma]] \Rightarrow B}[P^lQ]$$

□

Corollary 4.3 *LQQ_P is decidable i.e. there is an effective algorithm deciding whether a given sequent $\Gamma \Rightarrow A$ is provable or not.*

Proof.
If we stretch the notion of 'subformula' to the extent that Q is a subformula of any formula ∇A, then $L\nabla_P$ can be shown to have the subformula property: any proof for $\Gamma \Rightarrow A$ will only use subformulas of Γ and A. The decidability of $L\nabla_P$ then follows by a standard argument (note that the premisses of the $[Q]$-rule are *shorter* than its conclusion). $\square$

5 Semantics.

As we have already mentioned in the introduction, our idea to use a designated *constant type* to strengthen substructural logics stems from semantic considerations. Let us put it in another way: Došen [4] describes different kinds of algebras corresponding to different substructural logics. We need not go into details here, the point that we want to make is that these correspondences are such that when we consider two substructural logics, X and Y, of which Y is stronger than X, then the algebras for Y form a subclass of those for X. Now if we want to have 'parts' of X that do allow all structural rules of Y, what could be more natural than look at *subalgebras* of X-algebras that are themselves algebras for Y? In this sense, the linguistic motivations for the subalgebra interpretation of the strenthening operator, has a nice mathematical counterpart.

In our paper [19] we will discuss Došen's groupoid semantics; here we concentrate on extensions of Buszkowski's completeness result of the Lambek Calculus with respect to (free) semigroup semantics, cf. Buszkowski [3].

Definition 5.1 *A* (free) Q-semigroup *is a structure $\mathcal{G}' = (G, \cdot, S)$, where $\mathcal{G} = (G, \cdot)$ and $\mathcal{S} = (S, \cdot)$ are semigroups, (the first one being free), such that $\mathcal{S}$ is a subalgebra of $\mathcal{G}$. Q-semigroups can be considered to be models for the $L\nabla$-calculus by adding to the definition given in section 2, the clauses*

$$\begin{array}{rcl} [\![Q]\!] & = & S \\ [\![\nabla A]\!] & = & S \cap [\![A]\!]. \end{array}$$

Proposition 5.2 *The product-free fragment of $L\nabla$ is sound and complete with respect to the class of free Q-semigroups.*

Proof. We skip the soundness part, and concentrate on completeness. Building on Buszkowski's proof, we define a *canonical model* $\mathcal{G} = (G, \cdot, S)$ with G consisting of non-empty strings over the alphabet of formulas. The canonical interpretation sends an atomic formula A to the set of all sequences Γ for which $L\nabla \vdash \Gamma \Rightarrow A$.

The essential lemma of the proof is the *Canonical Lemma*:

(CL) For all Γ, A: $\Gamma \in [\![A]\!]$ iff $L\nabla \vdash \Gamma \Rightarrow A$.

We prove the induction step of (CL) where A is of the form ∇B. ¿From left to right: if Γ is in $[\![\nabla B]\!]$, then by definition of $[\![\nabla B]\!]$, we find $\Gamma \in [\![B]\!]$ and $\Gamma \in [\![Q]\!]$. By the induction hypothesis then, we get $\Gamma \Rightarrow B$ and $\Gamma \Rightarrow Q$, so one application of [∇R] gives $\Gamma \Rightarrow \nabla B$. For the converse, assume $\Gamma \Rightarrow \nabla B$; we then find $\Gamma \Rightarrow B$ by applying [Cut] to $\Gamma \Rightarrow \nabla B$ and $\nabla B \Rightarrow B$. Likewise we obtain $\Gamma \Rightarrow Q$, so by the induction hypothesis, Γ is in both $[\![B]\!]$ and $[\![Q]\!]$. But by definition this gives $\Gamma \in [\![\nabla B]\!]$.

It is left to prove that $([\![Q]\!], \cdot)$ is a subalgebra of $\mathcal{G}$, i.e. that $[\![Q]\!]$ is closed under $\cdot$. To show this, let Γ_1, Γ_2 be elements of $[\![Q]\!]$. This means $\Gamma_1 \Rightarrow Q$, $\Gamma_2 \Rightarrow Q$. Then an application of the Q-rule yields $\Gamma_1, \Gamma_2 \Rightarrow Q$. But by definition, this gives $\Gamma_1 \cdot \Gamma_2 \in [\![Q]\!]$. □

Of course, we are not interested in the system $L\nabla$ as such, but in its structural extensions like $L\nabla_P$. The main motivation of our approach towards strengthening substructural logics is the proposition below. (Note that it does not make sense to talk about e.g. commutative subalgebras of a *free* semigroup!)

Proposition 5.3 *The product-free version of $L\nabla_P$ is sound and complete with respect to the class of Q-semigroups satisfying $(\forall g \in G)(\forall s \in S) g \cdot s = s \cdot g$.*

Proof.
Soundness is straightforward; with respect to completeness, we only treat the product-free fragment of the Lambek Calculus[3].We start with the canonical frame $\mathcal{G}$ (cf. the previous proof), on which we define the following equivalence relation: $\Gamma \equiv \Gamma'$ if for all formulas A we have $\Gamma \Rightarrow A$ iff $\Gamma' \Rightarrow A$. We omit the proof that $\equiv$ is a congruence relation, and only prove that the quotient algebra $\mathcal{G}/\!\equiv\ = (G/\!\equiv, \cdot, S/\!\equiv)$ has the desired property: take arbitrary $g \in G/\!\equiv$ and $s \in S/\!\equiv$. We have to show that $g \cdot s = s \cdot g$ and for this, it suffices to consider $\Gamma \in g$, $\Sigma \in s$ and prove that $\Gamma, \Sigma \equiv \Sigma, \Gamma$.

So assume that $\Gamma, \Sigma \Rightarrow A$. As $\Sigma \in s$, we have $\Sigma \Rightarrow Q$. By the following proof:

$$\frac{\Sigma \Rightarrow Q \quad \Gamma, \Sigma \Rightarrow A}{\Sigma, \Gamma \Rightarrow A}\ [P^r Q]$$

we find $\Sigma, \Gamma \Rightarrow A$. As A was arbitrary, we have established the equivalence of Γ, Σ and Σ, Γ.

This proves that the Q-part of the quotient algebra indeed consists of elements that commute with any other element of the algebra. □

In the extended version [19] of this paper we will give similar results with respect to Došen's *groupoid semantics*, cf. [4]. It seems that results like the above proposition come more natural in Došen's framework.

[3]For the version with product, one can adapt the material in section 4.1 of Buszkowski [3].

6 Embedding LP in $L\nabla_P$

In the same way that intuitionistic and classical logic can be faithfully embedded in linear logic, we can show that our hybrid logic $L\nabla_P$ is not weaker in expressive power than the Lambek-van Benthem Calculus LP. First we define a translation from the formulas of LP into the formulas of $L\nabla_P$, then we prove the embedding theorem.

Definition 6.1 *Let $(\cdot)^\nabla$ be the translation of LP-formulas into $L\nabla$-formulas which marks all subformulas with a ∇. It is extended to terms in the obvious way, ie. Γ^∇ is Γ with every formula replaced by A^∇.*

Theorem 6.2 *For any LP-sequent $\Gamma \Rightarrow A$, we have*

$$LP \vdash \Gamma \Rightarrow A \;\; \textit{iff} \;\; L\nabla_P \vdash \Gamma^\nabla \Rightarrow A^\nabla.$$

Proof.

$\boxed{\Rightarrow}$ We prove the $\Rightarrow$-direction of the theorem by induction to the LP-derivation of $\Gamma \Rightarrow A$ in LP. If $\Gamma \Rightarrow A$ is an axiom of LP, then Γ is a one-element sequence consisting of A, so $\Gamma^\nabla \Rightarrow A^\nabla$ is an $L\nabla_P$-axiom.

So, suppose that $\Gamma \Rightarrow A$ was obtained as an LP-theorem by applying some rule $[R]$. We distinguish two cases:

First assume that $[R]$ is one of the logical rules; for instance, suppose that we found $\Gamma \Rightarrow A$ as a result of applying $[/L]$:

$$\frac{\Delta_0 \Rightarrow B \quad \Delta_1, C, \Delta_2 \Rightarrow D}{\Delta_1, C/B, \Delta_0, \Delta_2 \Rightarrow D}\,[/L]$$

By the induction hypothesis we have $\Delta_0^\nabla \Rightarrow B^\nabla$ and $\Delta_1^\nabla, C^\nabla, \Delta_2^\nabla \Rightarrow D^\nabla$ as $L\nabla_P$-theorems. Now we find $L\nabla_P \vdash \Gamma^\nabla \Rightarrow A^\nabla$ by

$$\frac{\dfrac{\Delta_0^\nabla \Rightarrow B^\nabla \quad \Delta_1^\nabla, C^\nabla, \Delta_2^\nabla \Rightarrow D^\nabla}{\Delta_1^\nabla, C^\nabla/B^\nabla, \Delta_0^\nabla, \Delta_2^\nabla \Rightarrow D^\nabla}\,[/L]}{\Delta_1^\nabla, \nabla(C^\nabla/B^\nabla), \Delta_0^\nabla, \Delta_2^\nabla \Rightarrow D^\nabla}\,[\nabla L]$$

The cases of the other logical rules are similar.

Second, assume that $[R]$ is the permutation rule $[P]$, in which case the LP-derivation is of the form

$$\frac{\Delta_0, B, C, \Delta_1 \Rightarrow D}{\Delta_0, C, B, \Delta_1 \Rightarrow D}\,[P]$$

By the induction hypothesis we have $L\nabla_P \vdash \Delta_0^\nabla, C^\nabla, B^\nabla, \Delta_1^\nabla \Rightarrow D^\nabla$ so one application of $[Cut]$ yields the desired result.

$\boxed{\Leftarrow}$ The basic idea for the other direction is that in a certain sense, every $L\nabla_P$-deriation 'is' an LP-derivation. To make this idea precise, we introduce an auxiliary system S which is nothing more than the extension of LP with the connective $\top$, together with its usual logical rule (axiom): $\Delta \Rightarrow \top$.

Furthermore, we define a translation from $L\nabla_P$-formulas into S-formulas by setting $P^\circ = P$ for atoms, $Q^\circ = \top$, and inductively: $(\nabla A)^\circ = A^\circ$ and $(A \heartsuit B)^\circ = A^\circ \heartsuit B^\circ$ for any other connective $\heartsuit$. For terms we set $(X_1, \ldots, X_n)^\circ = X_1^\circ \ldots X_n^\circ$. We now state the following claims, of which we omit the rather straightforward proofs:

1. For any LP-formula A, we have $(A^\nabla)^\circ = A$.
2. $L\nabla_P \vdash \Delta \Rightarrow B$ implies $S \vdash \Delta^\circ \Rightarrow B^\circ$.
3. S enjoys the subformula property, i.e. any S-theorem $\Gamma \Rightarrow A$ has a proof in which only the connectives appear that also occur in $\Gamma \Rightarrow A$.

Now, to prove the right-left direction of the theorem, let $L\nabla_P \vdash \Gamma^\nabla \Rightarrow A^\nabla$. By claim (2), $S \vdash (\Gamma^\nabla)^\circ \Rightarrow (A^\nabla)^\circ$, which is by (1) nothing more than saying that $S \vdash \Gamma \Rightarrow A$. Then (3) tells us that we have an S-proof for $\Gamma \Rightarrow A$ in which the new symbol $\top$ does not appear. Such a proof is thus completely performed within LP. $\square$

In the more general setting of [19] we will show that we can obtain the same result with a more economical embedding than $(\cdot)^\nabla$.

7 Conclusions

Accepting the idea to use operators for the task of strengthening a substructural logic, viz. adding a restricted version of the rule of Permutation to the Lambek Calculus, we have asked ourselves the question what the *meaning* of a formula ∇A (∇ the operator) in a resource-bounded derivation system might be. Our answer was, that a formula ∇A is like a labelled formula: the label (∇, but in fact a special type Q) tells us that the information proper, A, may be *used*, qua structural rules, in a way extending the default character of the logic. The novelty of this paper (as far as we know) lies in the fact that we have *implemented* this idea in a fashion inspired by the wish to give a natural *semantics* for the arising hybrid logic. We have separated the information of a formula from its structural behaviour, thus being able to make the structural properties of marked formulas *explicit* (by manipulating the proof rules involving the special type Q.

We believe our approach to be intuitive and compatible with the paradigm of resource-consciousness in substructural logics. Besides, it enjoys nice mathematical properties, like cut-elimination for the basic systems, and interesting applications.

A lot of research remains to be done — we mention a few questions:

1. A huge part of the research into linear logic is of a category-theoretic nature. Recently, the use of modalities in weaker logics has been studied from such a perspective as well, cf. de Paiva [16]. What is the category-theoretic side of our approaches?

2. Substructural logics have a type-theoretical side, via (adaptations of) the Curry-Howard interpretation, cf. Wansing [21], van Benthem [2]. (How) can we assign terms to proofs in our calculi?

3. Besides linear logic itself, Girard also invented a new proof method for it, viz. via proofnets. In his dissertation [18], Roorda extended this method to the Lambek calculus. Can we also find proof nets for the extended logic discussed here?

4. As mentioned before, in a separate paper [19] we will discuss the idea of a special meet-operator as a device to strengthen substructural logics, in more generality.

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