Lambek Semantics

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. . . ὁδὸς ἄνω κάτω μία καὶ ὡυτή.
. . . [the] way up [and] down [is] one and [the] same.

Heraclitus, *Diels–Kranz 22 B 60*

When Lambek (1958) introduced his syntactic calculus, he showed that it is equivalent to a sequent axiomatization, the Lambek-Gentzen sequent calculus (henceforth: **L**). The calculus **L** defines a general notion of derivability in the following sense: an expression consisting of the lexical items $e_1, \ldots, e_n$ of respective categories $c_1, \ldots, c_n$ is parsed as belonging to a certain category c if and only if the statement '$c_1, \ldots, c_n$ is a c' (written as a so-called *sequent* $c_1, \ldots, c_n \vdash c$) can be derived as a theorem of the system. Thus, grammatical derivations are reduced to logical deductions, giving rise to the slogan 'parsing as deduction'. The calculus **L** contains, among other rules, the so-called *Cut* rule. By proving that the set of theorems of **L** is not increased by adding *Cut*, Lambek established the decidability of **L**: for an arbitrary sequent the proof procedure is guaranteed to answer the question whether the sequent is valid after a finite number of steps. The proof of this fact is constructive: Lambek defined an algorithm which enables one to eliminate all occurrences of *Cut* in proofs.

The semantic interpretation procedure of Moortgat (1988) for product-free **L** is a sequent implementation of Van Benthem's term construction algorithm for the natural deduction formulation of **L** (1986, Chapter 7; 1991). In this semantics, the interpretations of a grammatical expression (derivable sequent) are directly determined by the proofs of its validity in the syntactic calculus. The present paper provides an alternative formulation of the Van Benthem/Moortgat semantics for **L** (which is introduced in Section 1) and uses it to prove the following results. Section 2 contains a straightforward proof of a semantic version of Lambek's *Cut* elimination theorem which entails that **L** is semantically decidable as well: the result of applying Lambek's *Cut* elimination algorithm is a derivation which is semantically equivalent to the original derivation. Section 3 is devoted to further normalization of

L, resulting in a sequent calculus **L*** which is put to use in the subsequent sections. First, we show in Section 4 that **L*** provides a solution to the so-called 'spurious ambiguity problem' – the problem that different proofs of a given sequent may yield one and the same semantic interpretation. In **L***, each interpretation of a sequent corresponds to exactly one proof. In Section 5, we use **L*** to prove that an explicit elaboration of the suggestions in Moortgat's 'Unambiguous Proof Representations for the Lambek Calculus' (1990) leads to representations of **L**-proofs of sequents with atomic goal categories which 'summarize' **L*** derivations and are, hence, indeed 'unambiguous', i.e., devoid of spurious ambiguity. A comparison of **L*** with the proof nets proposed in Roorda (1991), on the other hand, shows that every **L*** proof corresponds to a non-singleton *set* of proof nets. Finally, with the help of **L*** we are able to show in Section 6 that syntactic recognition within **L** can be mimicked using one basic category only, thus extending a result of Ponse (1988) for **LP**, the Lambek calculus with permutation, to **L** proper.

1 Introduction

We will employ the following definitions of the notions of category and sequent:

(1) Let ATOM be some finite set of atomic categories (none of which is of the form (a/b) or $(b\backslash a)$). Then CAT, the set of *categories based on* ATOM, is the smallest set such that (i) ATOM $\subseteq$ CAT, and (ii) if $a \in$ CAT and $b \in$ CAT, then $(a/b) \in$ CAT and $(b\backslash a) \in$ CAT.

(2) A *sequent* is an expression $T \vdash c$, where T is a finite non-empty sequence of categories and $c \in$ CAT. (So, $T = c_1, \ldots, c_n$, where $n > 0$ and for all i, $1 \leq i \leq n$: $c_i \in$ CAT.)

Outermost brackets of categories will be omitted. The sequence of categories $c_1, \ldots, c_n$ will be called the *left-hand side* of a sequent $c_1, \ldots, c_n \vdash c$, and the single category c its *right-hand side* or *goal*.

The product-free Lambek-Gentzen sequent calculus **L** consists of a set of axioms plus five inference rules: $/L$, $\backslash L$, $/R$, $\backslash R$ and *Cut*. The axioms and rules are listed in (3) and (4), respectively. In (4), a, b, and c denote arbitrary categories and T, U, and V arbitrary finite sequences of categories, of which T is non-empty.

(3) AXIOM, the set of *axioms* of **L**, is the set $\{c \vdash c \mid c \in \text{CAT}\}$.

(4)

$$\frac{T \vdash b \quad U, a, V \vdash c}{U, a/b, T, V \vdash c}\,[/L] \qquad \frac{T \vdash b \quad U, a, V \vdash c}{U, T, b\backslash a, V \vdash c}\,[\backslash L]$$

$$\frac{T, b \vdash a}{T \vdash a/b}\,[/R] \qquad \frac{b, T \vdash a}{T \vdash b\backslash a}\,[\backslash R]$$

$$\frac{T \vdash a \quad U, a, V \vdash c}{U, T, V \vdash c}\,[Cut]$$

Moortgat (1988) proposes a semantics for **L** in which every category occurrence c in a derivation is assigned some typed lambda term γ. The type of γ depends on c. For categories $c \in$ ATOM, some type assignment TYPE(c) is assumed, and the function TYPE is extended to non-atomic categories by the stipulation that TYPE$(a/b) =$ TYPE$(b\backslash a) =$ (TYPE(b), TYPE(a)). In this semantics, which is given in (5) below, the axioms of **L** correspond to *identity*, the rules $/L$ and $\backslash L$ to *functional application*, the rules $/R$ and $\backslash R$ to *lambda abstraction*, and the *Cut* rule to *substitution*.[1] The respective assignment of terms $\gamma_1, \ldots, \gamma_n, \gamma$ to categories $c_1, \ldots, c_n, c$ in a derivable sequent $c_1, \ldots, c_n \vdash c$ is usually written as $c_1 \!:\! \gamma_1, \ldots, c_n \!:\! \gamma_n \vdash c \!:\! \gamma$.

(5)

SYNTAX:	SEMANTICS:
$c \vdash c$ [AXIOM]	$\gamma \vdash \gamma$
$\dfrac{T \vdash b \quad U, a, V \vdash c}{U, a/b, T, V \vdash c}\,[/L]$	$\dfrac{T' \vdash \beta \quad U', \phi(\beta), V' \vdash \gamma}{U', \phi, T', V' \vdash \gamma}$
$\dfrac{T \vdash b \quad U, a, V \vdash c}{U, T, b\backslash a, V \vdash c}\,[\backslash L]$	$\dfrac{T' \vdash \beta \quad U', \phi(\beta), V' \vdash \gamma}{U', T', \phi, V' \vdash \gamma}$
$\dfrac{T, b \vdash a}{T \vdash a/b}\,[/R]$	$\dfrac{T', v \vdash \alpha}{T' \vdash \lambda v.\alpha}$
$\dfrac{b, T \vdash a}{T \vdash b\backslash a}\,[\backslash R]$	$\dfrac{v, T' \vdash \alpha}{T' \vdash \lambda v.\alpha}$
$\dfrac{T \vdash a \quad U, a, V \vdash c}{U, T, V \vdash c}\,[Cut]$	$\dfrac{T' \vdash \alpha \quad U', \alpha, V' \vdash \gamma}{U', T', V' \vdash \gamma}$

[1] Viz., the substitution of T' (which yields α) for α in $U', \alpha, V' \vdash \gamma$.

In (5), only the terms of the 'active' categories are indicated separately. The expression v denotes a variable of type $\textsc{type}(b)$, while α, β, γ and ϕ denote terms of type $\textsc{type}(a)$, $\textsc{type}(b)$, $\textsc{type}(c)$, and $(\textsc{type}(b), \textsc{type}(a))$, respectively. The expressions T', U' and V' refer to the sequences of terms assigned to the sequences of 'inactive' categories T, U, and V. Thus, the terms assigned to the categories in T, U, V are identical in premise and conclusion.

The interpretation which a proof with conclusion $c_1 : \gamma_1, \ldots, c_n : \gamma_n \vdash c : \gamma$ assigns to a sequence of lexical expressions $e_1, \ldots, e_n$ of categories $c_1, \ldots, c_n$ with interpretations $e'_1, \ldots, e'_n$ is obtained by performing the simultaneous substitution $[\gamma_1 := e'_1, \ldots, \gamma_n := e'_n]$ in the term γ assigned to the goal of the conclusion sequent: $\gamma[\gamma_1 := e'_1, \ldots, \gamma_n := e'_n]$.

For example: let the lexical expressions *someone*, *bores* and *everyone* be assigned the categories $(s/n)\backslash s$, $(n\backslash s)/n$ and $(s/n)\backslash s$, as well as the interpretations $\lambda P.\exists z\, P(z)$, $\textsc{bore}$ and $\lambda P.\forall z\, P(z)$ of type $((e,t),t)$, $(e,(e,t))$ and $((e,t),t)$, respectively. Then the following derivation shows that *Someone bores everyone* is of category s:

(6)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \qquad \dfrac{n \vdash n \quad s \vdash s}{n, n\backslash s \vdash s}\,[\backslash L]}{n, (n\backslash s)/n, n \vdash s}\,[/L]}{n, (n\backslash s)/n \vdash s/n}\,[/R] \qquad s \vdash s}{n, (n\backslash s)/n, (s/n)\backslash s \vdash s}\,[\backslash L]}{(n\backslash s)/n, (s/n)\backslash s \vdash n\backslash s}\,[\backslash R] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s}\,[/L]$$

An interpretation of derivation (6) is given in (7), where u, v, w, x, and y represent variables of type e, e, $((e,t),t)$, $(e,(e,t))$ and $((e,t),t)$, respectively:

(7)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v \vdash v \qquad \dfrac{u \vdash u \quad x(v)(u) \vdash x(v)(u)}{u, x(v) \vdash x(v)(u)}}{u, x, v \vdash x(v)(u)}}{u, x \vdash \lambda v.x(v)(u)} \qquad y(\lambda v.x(v)(u)) \vdash y(\lambda v.x(v)(u))}{u, x, y \vdash y(\lambda v.x(v)(u))}}{x, y \vdash \lambda u.y(\lambda v.x(v)(u))} \qquad w(\lambda u.y(\lambda v.x(v)(u))) \vdash w(\lambda u.y(\lambda v.x(v)(u)))}{w, x, y \vdash w(\lambda u.y(\lambda v.x(v)(u)))}$$

Performing the simultaneous substitution (8) in $w(\lambda u.y(\lambda v.x(v)(u)))$, the term assigned to the conclusion goal in (7), supplies the sentence *Someone bores everyone* with an interpretation equivalent to $\exists u \forall v\, \textsc{bore}(v)(u)$.

(8) $[w := \lambda P.\exists z\, P(z), x := \textsc{bore}, y := \lambda P.\forall z\, P(z)]$

Note, first, that semantics (5) needs to be supplemented with at least the following condition, which guarantees that the abstractions introduced by $\backslash R$ and $/R$ bind only one variable occurrence:

(9) If $T{:}T' \vdash c{:}\lambda v.\gamma$ is the conclusion of a $\backslash R$ or $/R$ inference, then v does not occur freely in T'.

For instance, if – violating this condition but in keeping with (5) – all occurrences of v in (7) are replaced by u, then the conclusion goal of (6) is assigned the term $w(\lambda u.y(\lambda u.x(u)(u)))$, which results in the incorrect interpretation $\exists u \forall u\, \textsc{bore}(u)(u)$, 'Everyone bores himself', for *Someone bores everyone*.

And, second, note that simultaneous substitution is not always well-defined for semantics (5), which allows complex terms and different occurrences of the same term in the left-hand side of sequents. For (i) it is unclear what is meant by a substitution of a term for a complex term (this requires at least some extension of the notion of 'free occurrence of' to terms other than variables), and (ii) simultaneous substitution does not make sense when the same term is to be substituted twice. The problem reappears at deeper levels, e.g., when a left-hand side category c_i is assigned a term γ_i which also occurs as a subterm of a term γ_j assigned to another left-hand side category c_j. Moreover, this substitution problem causes a complication in the proof of 'semantic *Cut* elimination'.[2]

[2] Consider Moortgat's treatment (1990, p. 151) of the crucial case **5** of Lambek's *Cut* elimination algorithm (cf. section 2 below), in which:

$$\dfrac{\dfrac{{}^{[\mathrm{A}]}T, d{:}v \vdash b{:}\beta}{T \vdash b/d{:}\lambda v.\beta}\,[/R] \qquad \dfrac{V_1 \vdash d{:}\delta \qquad {}^{[\mathrm{B}]}U, b{:}[\lambda v.\beta](\delta), V_2 \vdash c{:}\gamma}{U, b/d{:}\lambda v.\beta, V_1, V_2 \vdash c{:}\gamma}\,[/L]}{U, T, V_1, V_2 \vdash c{:}\gamma}\,[Cut]$$

is replaced by:

$$\dfrac{V_1 \vdash d{:}\delta \qquad \dfrac{{}^{[\mathrm{A}']}T, d{:}\delta \vdash b{:}\beta[v{:=}\delta] \qquad {}^{[\mathrm{B}']}U, b{:}\beta[v{:=}\delta], V_2 \vdash c{:}\gamma[[\lambda v.\beta](\delta){:=}\beta[v{:=}\delta]]}{U, T, d{:}\delta, V_2 \vdash c{:}\gamma[[\lambda v.\beta](\delta){:=}\beta[v{:=}\delta]]}\,[Cut]}{U, T, V_1, V_2 \vdash c{:}\gamma[[\lambda v.\beta](\delta){:=}\beta[v{:=}\delta]]}\,[Cut]$$

(The terms assigned to U, T, V_1 and V_2 have been omitted.) In order to be able to make the transition from [A] and [B] to [A′] and [B′], respectively, Moortgat needs something

These considerations suggest that it is more natural to have a semantics which always yields proofs of sequents $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ where the terms $v_1, \ldots, v_n$ constitute a sequence of different variables. Therefore, the alternative semantics (10) which will be used in the following sections is based on the same category-to-type assignment as semantics (5), but differs from (5) as regards the terms it assigns to the categories in a derivable sequent $c_1, \ldots, c_n \vdash c$. For the left-hand side categories c_i $(1 \leq i \leq n)$, this interpretation is some variable v_i of type $\text{TYPE}(c_i)$, whereas for the right-hand side category c, it is a (possibly complex) term γ of type $\text{TYPE}(c)$: $c_1 \!:\! v_1, \ldots, c_n \!:\! v_n \vdash c \!:\! \gamma$.

In (10), u, v, x and w represent variables of type $\text{TYPE}(a)$, $\text{TYPE}(b)$, $\text{TYPE}(c)$ and $(\text{TYPE}(b), \text{TYPE}(a))$; α, β and γ denote terms of type $\text{TYPE}(a)$, $\text{TYPE}(b)$ and $\text{TYPE}(c)$; and T', U' and V' refer to the sequences of variables assigned to the sequences of categories T, U and V. Again, only the terms of the active categories are indicated separately, and the terms of other categories are identical in premise and conclusion. The expression $\gamma[u \!:=\! \alpha]$

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like: ($) If $\pi[U \!:\! U', a \!:\! \alpha, V \!:\! V' \vdash_{(5)} c \!:\! \gamma]$, then for all terms α' of type $\text{TYPE}(a)$:
$\pi[U \!:\! U', a \!:\! \alpha', V \!:\! V' \vdash_{(5)} c \!:\! \gamma[\alpha \!:=\! \alpha']]$.

(We let $\pi[c_1 \!:\! \gamma_1, \ldots, c_n \!:\! \gamma_n \vdash_{(5)} c \!:\! \gamma]$ represent that semantics (5) assigns the respective terms $\gamma_1, \ldots \gamma_n, \gamma$ to the categories $c_1, \ldots, c_n, c$ in the conclusion sequent $c_1, \ldots, c_n \vdash c$ of π. Note that $[\alpha \!:=\! \alpha']$ is more than mere substitution for free variable occurrences; it also involves replacement of compound terms – cf. the transition from [B] to [B′].) But such a lemma is not available, as can be seen from the following derivations – where α, α', β and γ are terms of type $\text{TYPE}(a)$, $\text{TYPE}(a)$, $\text{TYPE}((a\backslash b)/a)$ and $(\text{TYPE}(a), \text{TYPE}(a))$, respectively:

$$(i)\qquad \frac{a \!:\! \alpha \vdash a \!:\! \alpha \qquad \dfrac{a \!:\! \gamma(\alpha) \vdash a \!:\! \gamma(\alpha) \qquad b \!:\! \beta(\alpha)(\gamma(\alpha)) \vdash b \!:\! \beta(\alpha)(\gamma(\alpha))}{b/a \!:\! \beta(\alpha), a \!:\! \gamma(\alpha) \vdash b \!:\! \beta(\alpha)(\gamma(\alpha))}[/L]}{a \!:\! \alpha, a\backslash(b/a) \!:\! \beta, a \!:\! \gamma(\alpha) \vdash b \!:\! \beta(\alpha)(\gamma(\alpha))}[\backslash L]$$

$$(ii)\qquad \frac{a \!:\! \alpha' \vdash a \!:\! \alpha' \qquad \dfrac{a \!:\! \gamma(\alpha) \vdash a \!:\! \gamma(\alpha) \qquad b \!:\! \beta(\alpha')(\gamma(\alpha)) \vdash b \!:\! \beta(\alpha')(\gamma(\alpha))}{b/a \!:\! \beta(\alpha'), a \!:\! \gamma(\alpha) \vdash b \!:\! \beta(\alpha')(\gamma(\alpha))}[/L]}{a \!:\! \alpha', a\backslash(b/a) \!:\! \beta, a \!:\! \gamma(\alpha) \vdash b \!:\! \beta(\alpha')(\gamma(\alpha))}[\backslash L]$$

Since no variable-binding operator is present, it seems plausible to assume that both occurrences of α in the term assigned to the conclusion goal of derivation (i) are free and that, hence, $[\beta(\alpha)(\gamma(\alpha))][\alpha \!:=\! \alpha']$ is the term $\beta(\alpha')(\gamma(\alpha'))$. But, refuting ($), the term assigned to the conclusion goal of (ii) is $\beta(\alpha')(\gamma(\alpha))$.

If, however, condition (9) is met, then in every sequent $T \!:\! T' \vdash c \!:\! \gamma$ the sequence T' of left-hand side terms has the same free variables as γ, and the following can be established (where v is a variable and α is a term, both of type $\text{TYPE}(a)$):

(#) If $\pi[T \!:\! T' \vdash_{(5)} c \!:\! \gamma]$, then $\pi[T \!:\! T'[v \!:=\! \alpha] \vdash_{(5)} c \!:\! \gamma[v \!:=\! \alpha]]$.

denotes the result of substituting term α for all free occurrences of u in γ.

(10) SYNTAX: SEMANTICS:

$c \vdash c$ [AXIOM] $\qquad x \vdash x$

$$\frac{T \vdash b \quad U, a, V \vdash c}{U, a/b, T, V \vdash c}\,[/L] \qquad \frac{T' \vdash \beta \qquad U', u, V' \vdash \gamma}{U', w, T', V' \vdash \gamma[u := w(\beta)]}$$

$$\frac{T \vdash b \quad U, a, V \vdash c}{U, T, b\backslash a, V \vdash c}\,[\backslash L] \qquad \frac{T' \vdash \beta \qquad U', u, V' \vdash \gamma}{U', T', w, V' \vdash \gamma[u := w(\beta)]}$$

$$\frac{T, b \vdash a}{T \vdash a/b}\,[/R] \qquad \frac{T', v \vdash \alpha}{T' \vdash \lambda v.\alpha}$$

$$\frac{b, T \vdash a}{T \vdash b\backslash a}\,[\backslash R] \qquad \frac{v, T' \vdash \alpha}{T' \vdash \lambda v.\alpha}$$

$$\frac{T \vdash a \quad U, a, V \vdash c}{U, T, V \vdash c}\,[Cut] \qquad \frac{T' \vdash \alpha \quad U', u, V' \vdash \gamma}{U', T', V' \vdash \gamma[u := \alpha]}$$

In the context of a proof, we will assume that all variables x and w assigned to axiom instances or introduced in the conclusions of $/L$ or $\backslash L$ inferences are different. Observe that, consequently, (i) the variables $v_1, \ldots v_n$ assigned to the left-hand side of a sequent $c_1, \ldots, c_n \vdash c$ are all different; (ii) these variables $v_1, \ldots v_n$ make up the free variables of the term γ assigned to the goal c; and (iii) each variable $v_1, \ldots v_n$ occurs exactly once in γ.

Note that semantics (10) is more 'compositional' than (5), in that the complexity of the term γ assigned to the goal of a sequent $c_1, \ldots, c_n \vdash c$ increases with the length of the proof of that sequent. This property makes semantics (10) more perspicuous, which facilitates proving things about it. In semantics (5), on the other hand, we see that the rule $/L$ assigns a term ϕ to its conclusion category a/b which is less complex than the term $\phi(\beta)$ assigned to the category a in its right-hand side premise. (The same holds for $\backslash L$.) Moreover, semantics (10) is easier to use in practice, since it entails considerably less duplication of semantic information than (5), witness its interpretation of derivation (6):

(11)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v \vdash v \qquad \dfrac{u \vdash u \quad z \vdash z}{u, z' \vdash z'(u)}}{u, x, v \vdash x(v)(u)}}{u, x \vdash \lambda v.x(v)(u)} \qquad z'' \vdash z''}{u, x, y \vdash y(\lambda v.x(v)(u))}}{x, y \vdash \lambda u.y(\lambda v.x(v)(u))} \qquad z''' \vdash z'''}{w, x, y \vdash w(\lambda u.y(\lambda v.x(v)(u)))}$$

Still, to the extent that semantics (5) is well-behaved, it is essentially the same as semantics (10), on account of the following relationship. Suppose that the assignment of terms by (5) is restricted by condition (9), and, for $i \in \{5, 10\}$, let $\pi[c_1 : \gamma_1, \ldots, c_n : \gamma_n \vdash_{(i)} c : \gamma]$ represent that semantics (i) assigns the respective terms $\gamma_1, \ldots, \gamma_n, \gamma$ to the categories $c_1, \ldots, c_n, c$ in the conclusion sequent $c_1, \ldots, c_n \vdash c$ of proof π. Then the following holds:

(12) $\pi[c_1 : v_1, \ldots, c_n : v_n \vdash_{(10)} c : \gamma]$ if and only if
$\pi[c_1 : \gamma_1, \ldots, c_n : \gamma_n \vdash_{(5)} c : \gamma[v_1 := \gamma_1 \ldots, v_n := \gamma_n]]$.

From (12), which is proven by induction on the length of π (where fact (#) – see footnote 2 – is needed as a lemma for the cases $\backslash R$ and $/R$), it follows that if the terms $\gamma_1, \ldots, \gamma_n$ of the left-hand side categories $c_1, \ldots, c_n$ in the conclusion sequent $c_1, \ldots, c_n \vdash c$ of π constitute a sequence $v_1, \ldots, v_n$ of different variables, then semantics (10) assigns a term to the goal category c which is syntactically identical to the term assigned by semantics (5).

2 Cut Elimination and Semantics

Van Benthem (1986) proved that the number of distinct readings for an **L**-derivable sequent is finite. However, in addition one would like to have an effective method for generating all readings. The availability of such a method could serve practical purposes by providing a solution to the 'parsing problem', which is:

> ...to find all readings for a given string. This is a necessary step if one wants to give a satisfactory parsing algorithm for **L**. It should be noticed, however, that it is not at all clear which members of the infinite set of derivations for a sentence S in **L**

> lead to different readings, and thus, finding all possible readings seems to be problematic to begin with, let alone finding them efficiently. (Bouma (1989))

A possible way of solving this problem is to prove that the result of applying Lambek's *Cut* elimination algorithm is a derivation which is semantically equivalent to the original derivation. Then for every derivation of an **L**-derivable sequent there is a *Cut*-free derivation with the same interpretation, and we can safely restrict our attention to the (finite number of) *Cut*-free derivations. Each of the inference rules $/L$, $\backslash L$, $/R$ and $\backslash R$ derives its conclusion from one or more premises with a strictly smaller number of occurrences of / and \. Hence establishing the derivability of the premise(s) is more simple than establishing the derivability of the conclusion, and it follows that every sequent has only finitely many *Cut*-free derivations.

Adapting the proof of Lambek (1958), we will show in this section that sequents in the proof of which the *Cut* inference rule has been used can also be derived without using *Cut*, while keeping their semantics the same. That is, *Cut* is a derived rule of inference which does not yield new theorems, nor new interpretations. This is done in a constructive way: it is possible to transform every proof which makes use of the *Cut* rule into a *Cut*-free proof which determines the same interpretation as the original one.

The two base cases of this transformation procedure (case **1** and **2** below) involve *Cut* inferences where one of the premises is an instance of the axiom scheme $c \vdash c$. Here the conclusion coincides with the other premise, and the *Cut* inference can be eliminated immediately. The recursive cases of the procedure (case **3**, **4** and **5** below) work by reduction of what is called the 'degree' of the *Cut* inference. Degree is defined as follows:

(13) (*i*) The degree $d(c)$ of a category c is defined inductively: $d(c) = 0$ for $c \in \textsc{atom}$; $d(a/b) = d(b\backslash a) = d(a) + d(b) + 1$.

(*ii*) The degree $d(c_1, \ldots, c_n)$ of a finite sequence of categories $c_1, \ldots, c_n$ equals $d(c_1) + \ldots + d(c_n)$.

(*iii*) The degree $d(T \vdash c)$ of a sequent $T \vdash c$ equals $d(T) + d(c)$.

(*iv*) The degree $d(\alpha)$ of a *Cut* inference $\alpha =$

$$\frac{T \vdash a \quad U, a, V \vdash c}{U, T, V \vdash c}$$

equals $d(T) + d(U) + d(V) + d(a) + d(c)$.

Thus, the degree of a category, a sequence of categories and a sequent is equal to the number of slashes it contains.

Any *Cut* inference of which at least one premise has been proven without *Cut* can be turned into one or two new *Cut* inferences of strictly smaller degree. Since the minimal degree of an inference is zero, these proof transformations will ultimately converge on the base case.

We will now prove

1. that in any application of *Cut*,

$$\frac{T \vdash a \quad U, a, V \vdash c}{U, T, V \vdash c}\,[Cut],$$

   of which at least one premise has been proven without *Cut*,

   (a) either the conclusion is identical with one of the premises so that the application of *Cut* can be eliminated,

   (b) or the application of *Cut* can be replaced by one or two applications of *Cut* of smaller degree; and

2. that execution of this *Cut* elimination procedure always leads to semantically equivalent proofs.

There are five main cases:

1: $T \vdash a$ is an axiom; then $T = a$ and the conclusion coincides with the premise $U, a, V \vdash c$.

$$\frac{a \vdash a \quad U, a, V \vdash c}{U, a, V \vdash c}\,[Cut] \qquad \frac{u' \vdash u' \quad U', u, V' \vdash \gamma}{U', u', V' \vdash \gamma[u := u']}\,[Cut]$$

Cut can be eliminated, for note that the variable u' does not occur in γ. Hence we also have the interpretation $U', u', V' \vdash \gamma[u := u']$ for (the subproof of) the premise $U, a, V \vdash c$, due to the following fact (which is proven by a straightforward induction on the length of π):

> If $\pi[U : U', a : u, V : V' \vdash c : \gamma]$ and u' is a variable of type $\textsc{type}(a)$ not occurring in γ, then $\pi[U : U', a : u', V : V' \vdash c : \gamma[u := u']]$.

2: $U, a, V \vdash c$ is an axiom; then U and V are empty, $c = a$, and the conclusion coincides with the premise $T \vdash a$. *Cut* can be eliminated because the terms α and $u[u := \alpha]$ assigned to the coinciding premise and conclusion coincide as well:

$$\frac{T \vdash a \quad a \vdash a}{T \vdash a}\,[Cut] \qquad\qquad \frac{T' \vdash \alpha \quad u \vdash u}{T' \vdash u[u := \alpha]}\,[Cut]$$

3: The last step in the proof of $T \vdash a$ uses a rule but does not introduce the main connective of a. Then $T \vdash a$ is derived by $/L$ (or $\backslash L$) from two sequents, one of which has the form $T' \vdash a$, with $d(T') < d(T)$. Hence we can reverse the order of the rules $/L$ (or $\backslash L$) and *Cut*, resulting in a new *Cut* inference which has smaller degree than the old one. We only consider $/L$. ($\backslash L$ is analogous.) Then $T = P, b/d, Q, R$, and we can replace:

$$\frac{\dfrac{Q \vdash d \quad P, b, R \vdash a}{P, b/d, Q, R \vdash a}\,[/L] \quad U, a, V \vdash c}{U, P, b/d, Q, R, V \vdash c}\,[Cut]$$

by:

$$\frac{Q \vdash d \quad \dfrac{P, b, R \vdash a \quad U, a, V \vdash c}{U, P, b, R, V \vdash c}\,[Cut]}{U, P, b/d, Q, R, V \vdash c}\,[/L]$$

Semantically, this amounts to replacing:

$$\frac{\dfrac{Q' \vdash \delta \qquad P', v, R' \vdash \alpha}{P', w, Q', R' \vdash \alpha[v := w(\delta)]}\,[/L] \quad U', u, V' \vdash \gamma}{U', P', w, Q', R', V \vdash \gamma[u := \alpha[v := w(\delta)]]}\,[Cut]$$

by:

$$\frac{Q' \vdash \delta \quad \dfrac{P', v, R' \vdash \alpha \quad U', u, V' \vdash \gamma}{U', P', v, R', V' \vdash \gamma[u := \alpha]}\,[Cut]}{U', P', w, Q', R', V' \vdash [\gamma[u := \alpha]][v := w(\delta)]}\,[/L]$$

The terms $\gamma[u := \alpha[v := w(\delta)]]$ and $[\gamma[u := \alpha]][v := w(\delta)]$ are identical because v does not occur in γ; hence this replacement of *Cut* preserves interpretation.

4: The last step in the proof of $U, a, V \vdash c$ uses one of the rules $/L$, $\backslash L$, $/R$, $\backslash R$, but does not introduce the main connective of a. Then $U, a, V \vdash c$ is inferred from one or two sequents, one of which has the form $U', a, V' \vdash c'$ with $d(U')+d(V')+d(c') < d(U)+d(V)+d(c)$ (since the inference introduces one occurrence of a connective). Again, we can reverse the order of the rules; the new *Cut* inference has smaller degree than the given one. We only consider $\backslash L$ and $\backslash R$. ($/L$ and $/R$ are analogous.)

4.1: $U, a, V \vdash c$ is obtained by $\backslash L$. Then $U, a, V = P, Q, d\backslash b, R$. Note that a and $d\backslash b$ are distinct category occurrences (for $\backslash L$ does not introduce the main connective of a). Therefore, there are three possibilities: a is part of P, Q or R (The last subcase, which is analogous to the first one, will not be treated.)

4.1.a: The category a is part of P, so $P = P_1, a, P_2$. Replace:

$$\dfrac{T \vdash a \qquad \dfrac{Q \vdash d \quad P_1, a, P_2, b, R \vdash c}{P_1, a, P_2, Q, d\backslash b, R \vdash c}[\backslash L]}{P_1, T, P_2, Q, d\backslash b, R \vdash c}[Cut]$$

by:

$$\dfrac{Q \vdash d \qquad \dfrac{T \vdash a \quad P_1, a, P_2, b, R \vdash c}{P_1, T, P_2, b, R \vdash c}[Cut]}{P_1, T, P_2, Q, d\backslash b, R \vdash c}[\backslash L]$$

On the semantic side, this entails the replacement of:

$$\dfrac{T' \vdash \alpha \qquad \dfrac{Q' \vdash \delta \qquad P_1', u, P_2', v, R' \vdash \gamma}{P_1', u, P_2', Q', w, R' \vdash \gamma[v := w(\delta)]}[\backslash L]}{P_1', T', P_2', Q', w, R' \vdash [\gamma[v := w(\delta)]][u := \alpha]}[Cut]$$

by:

$$\dfrac{Q' \vdash \delta \qquad \dfrac{T' \vdash \alpha \quad P_1', u, P_2', v, R' \vdash \gamma}{P_1', T', P_2', v, R' \vdash \gamma[u := \alpha]}[Cut]}{P_1', T', P_2', Q', w, R' \vdash [\gamma[u := \alpha]][v := w(\delta)]}[\backslash L]$$

The terms $[\gamma[v := w(\delta)]][u := \alpha]$ and $[\gamma[u := \alpha]][v := w(\delta)]$ are identical, because u and v are different variables, u does not occur in $w(\delta)$, and v does not occur in α.

4.1.b: The category a is part of Q, so $Q = Q_1, a, Q_2$. Replace:

$$\dfrac{T \vdash a \qquad \dfrac{Q_1, a, Q_2 \vdash d \quad P, b, R \vdash c}{P, Q_1, a, Q_2, d\backslash b, R \vdash c}[\backslash L]}{P, Q_1, T, Q_2, d\backslash b, R \vdash c}[Cut]$$

by:

$$\dfrac{\dfrac{T \vdash a \quad Q_1, a, Q_2 \vdash d}{Q_1, T, Q_2 \vdash d}\,[Cut] \qquad P, b, R \vdash c}{P, Q_1, T, Q_2, d\backslash b, R \vdash c}\,[\backslash L]$$

This entails a semantic replacement of:

$$\dfrac{T' \vdash \alpha \quad \dfrac{Q'_1, u, Q'_2 \vdash \delta \qquad P', v, R' \vdash \gamma}{P', Q'_1, u, Q'_2, w, R' \vdash \gamma[v := w(\delta)]}\,[\backslash L]}{P', Q'_1, T', Q'_2, w, R' \vdash [\gamma[v := w(\delta)]][u := \alpha]}\,[Cut]$$

by:

$$\dfrac{\dfrac{T' \vdash \alpha \quad Q'_1, u, Q'_2 \vdash \delta}{Q'_1, T', Q'_2 \vdash \delta[u := \alpha]}\,[Cut] \qquad P', v, R' \vdash \gamma}{P', Q'_1, T', Q'_2, w, R' \vdash \gamma[v := w(\delta[u := \alpha])]}\,[\backslash L]$$

Again, the terms $[\gamma[v := w(\delta)]][u := \alpha]$ and $\gamma[v := w(\delta[u := \alpha])]$ are identical, for u does not occur in γ, and u and w are different variables.

4.2: $U, a, V \vdash c$ is obtained by $\backslash R$. Then $c = d\backslash b$, and we can replace:

$$\dfrac{T \vdash a \quad \dfrac{d, U, a, V \vdash b}{U, a, V \vdash d\backslash b}\,[\backslash R]}{U, T, V \vdash d\backslash b}\,[Cut]$$

by:

$$\dfrac{\dfrac{T \vdash a \quad d, U, a, V \vdash b}{d, U, T, V \vdash b}\,[Cut]}{U, T, V \vdash d\backslash b}\,[\backslash R]$$

This is reflected in the semantic replacement of:

$$\dfrac{T' \vdash \alpha \quad \dfrac{v, U', u, V' \vdash \beta}{U', u, V' \vdash \lambda v.\beta}\,[\backslash R]}{U', T', V' \vdash [\lambda v.\beta][u := \alpha]}\,[Cut]$$

by:

$$\dfrac{\dfrac{T' \vdash \alpha \quad v, U', u, V' \vdash \beta}{v, U', T', V' \vdash \beta[u := \alpha]}\,[Cut]}{U', T', V' \vdash \lambda v.[\beta[u := \alpha]]}\,[\backslash R]$$

Since u and v are different variables, the term $[\lambda v.\beta][u := \alpha]$ is identical to $\lambda v.[\beta[u := \alpha]]$.

5: The last steps in the proofs of both $T \vdash a$ and $U, a, V \vdash c$ introduce the main connective, / or \, of a. So $a = b/d$ or $a = d\backslash b$. (Only the /-case is treated here.) We can replace:

$$\dfrac{\dfrac{T, d \vdash b}{T \vdash b/d}[/R] \quad \dfrac{V_1 \vdash d \quad U, b, V_2 \vdash c}{U, b/d, V_1, V_2 \vdash c}[/L]}{U, T, V_1, V_2 \vdash c}[Cut]$$

by:

$$\dfrac{V_1 \vdash d \quad \dfrac{T, d \vdash b \quad U, b, V_2 \vdash c}{U, T, d, V_2 \vdash c}[Cut]}{U, T, V_1, V_2 \vdash c}[Cut]$$

Semantically, this involves a replacement of:

$$\dfrac{\dfrac{T', v \vdash \beta}{T' \vdash \lambda v.\beta}[/R] \quad \dfrac{V_1' \vdash \delta \qquad U', u, V_2' \vdash \gamma}{U', w, V_1', V_2' \vdash \gamma[u := w(\delta)]}[/L]}{U', T', V_1', V_2' \vdash [\gamma[u := w(\delta)]][w := \lambda v.\beta]}[Cut]$$

by:

$$\dfrac{V_1' \vdash \delta \quad \dfrac{T', v \vdash \beta \quad U', u, V_2' \vdash \gamma}{U', T', v, V_2' \vdash \gamma[u := \beta]}[Cut]}{U', T', V_1', V_2' \vdash [\gamma[u := \beta]][v := \delta]}[Cut]$$

Because w does not occur in γ, the term $[\gamma[u := w(\delta)]][w := \lambda v.\beta]$ is identical to $\gamma[u := [w(\delta)][w := \lambda v.\beta]]$. The latter term is identical to $\gamma[u := [\lambda v.\beta](\delta)]$, because w does not occur in δ. Since the free variables V_1' of δ do not occur in β, they are free for v in β and we can apply β-conversion, with the result $[\gamma[u := \beta[v := \delta]]]$.[3] As v does not occur in γ, this is the same term as $[\gamma[u := \beta]][v := \delta]$. □

3 Further Normalization

We have shown that if $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ is derivable in **L**, then there is a *Cut*-free **L** derivation of $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma'$ such that γ and γ' are

[3]Thus, case **5** does not involve identity of terms, but equivalence under *β-conversion* (also called λ-conversion): $[\lambda v.\gamma'](\gamma'') = \gamma'[v := \gamma'']$ if γ'' is free for v in γ'.

It takes an induction on the length of the derivation to see that terms γ assigned to conclusion goals in *Cut*-free proofs are always in *β-normal form*, i.e., they do not have subterms of the form $[\lambda v.\gamma'](\gamma'')$.

equivalent. Consequently, we only have to consider the *Cut*-free derivations of a sequent to obtain all its semantic interpretations. But in spite of the fact that a given sequent has only finitely many *Cut*-free derivations, *Cut*-less **L** still suffers from what has been called the 'spurious ambiguity problem' (König (1989)), viz., the problem that different proofs of a given sequent may yield one and the same semantic interpretation. When the calculus is actually used for parsing and interpreting expressions, this may entail superfluous work: interpretations will be computed n times instead of once. For example, the sequent $c/c, c/c, c \vdash c$ has two semantically equivalent *Cut*-free proofs, viz., (14) and (15), in which the symbol $\sharp$ is used to keep track of the active category:

(14)
$$\dfrac{\dfrac{c \vdash c \quad c \vdash c}{c/c^\sharp, c \vdash c}[/L] \quad c \vdash c}{c/c^\sharp, c/c, c \vdash c}[/L] \qquad \dfrac{\dfrac{x \vdash x \quad u \vdash u}{y, x \vdash y(x)}[/L] \quad v \vdash v}{z, y, x \vdash z(y(x))}[/L]$$

(15)
$$\dfrac{c \vdash c \quad \dfrac{c \vdash c \quad c \vdash c}{c/c^\sharp, c \vdash c}[/L]}{c/c, c/c^\sharp, c \vdash c}[/L] \qquad \dfrac{x \vdash x \quad \dfrac{u \vdash u \quad v \vdash v}{z, u \vdash z(u)}[/L]}{z, y, x \vdash z(y(x))}[/L]$$

Of course, not all ambiguity in **L** is spurious. Some sequents are really ambiguous. The sequent $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$, for example, has six *Cut*-free derivations, which are *not* all equivalent. These derivations are listed below. Each derivation is followed by the interpretation it assigns to its conclusion sequent $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$:

(16)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \quad \dfrac{n \vdash n \quad s \vdash s}{n, n\backslash s^\sharp \vdash s}[\backslash L]}{n, (n\backslash s)/n^\sharp, n \vdash s}[/L]}{n, (n\backslash s)/n \vdash s/n^\sharp}[/R] \quad s \vdash s}{n, (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]}{(n\backslash s)/n, (s/n)\backslash s \vdash n\backslash s^\sharp}[\backslash R] \quad s \vdash s}{s/(n\backslash s)^\sharp, (n\backslash s)/n, (s/n)\backslash s \vdash s}[/L]$$

$$u, v, w \vdash u(\lambda u'.w(\lambda w'.v(w')(u')))$$

(17)

$$\dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \quad n\backslash s \vdash n\backslash s}{(n\backslash s)/n^\sharp, n \vdash n\backslash s}[/L] \qquad s \vdash s}{s/(n\backslash s)^\sharp, (n\backslash s)/n, n \vdash s}[/L]}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^\sharp}[/R] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]$$

$u, v, w \vdash w(\lambda w'.u(v(w')))$

(18)

$$\dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \quad \dfrac{\dfrac{n \vdash n \quad s \vdash s}{n, n\backslash s^\sharp \vdash s}[\backslash L]}{n\backslash s \vdash n\backslash s^\sharp}[\backslash R]}{(n\backslash s)/n^\sharp, n \vdash n\backslash s}[/L] \qquad s \vdash s}{s/(n\backslash s)^\sharp, (n\backslash s)/n, n \vdash s}[/L]}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^\sharp}[/R] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]$$

$u, v, w \vdash w(\lambda w'.u(\lambda u'.v(w')(u')))$

(19)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \quad \dfrac{n \vdash n \quad s \vdash s}{n, n\backslash s^\sharp \vdash s}[\backslash L]}{n, (n\backslash s)/n^\sharp, n \vdash s}[/L]}{(n\backslash s)/n, n \vdash n\backslash s^\sharp}[\backslash R] \qquad s \vdash s}{s/(n\backslash s)^\sharp, (n\backslash s)/n, n \vdash s}[/L]}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^\sharp}[/R] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]$$

$u, v, w \vdash w(\lambda w'.u(\lambda u'.v(w')(u')))$

(20)

$$\dfrac{\dfrac{\dfrac{n \vdash n \quad \dfrac{n\backslash s \vdash n\backslash s \quad s \vdash s}{s/(n\backslash s)^\sharp, n\backslash s \vdash s}[/L]}{s/(n\backslash s), (n\backslash s)/n^\sharp, n \vdash s}[/L]}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^\sharp}[/R] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]$$

$u, v, w \vdash w(\lambda w'.u(v(w')))$

(21)

$$\dfrac{\dfrac{n \vdash n \qquad \dfrac{\dfrac{\dfrac{\dfrac{n \vdash n \quad s \vdash s}{n, n\backslash s^\sharp \vdash s}[\backslash L]}{n\backslash s \vdash n\backslash s^\sharp}[\backslash R] \qquad s \vdash s}{s/(n\backslash s)^\sharp, n\backslash s \vdash s}[/L]}{}}{\dfrac{s/(n\backslash s), (n\backslash s)/n^\sharp, n \vdash s}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^\sharp}[/R]}[/L] \qquad s \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^\sharp \vdash s}[\backslash L]$$

$$u, v, w \vdash w(\lambda w'.u(\lambda u'.v(w')(u')))$$

The terms assigned to the conclusions of the derivations (18), (19) and (21) are identical. Moreover, because the subterm $\lambda u'.v(w')(u')$ of the term $w(\lambda w'.u(\lambda u'.v(w')(u')))$ assigned to the conclusion goal of these derivations is equivalent to $v(w')$ by η-conversion,[4] the interpretation of these conclusions is equivalent to $u, v, w \vdash w(\lambda w'.u(v(w')))$, the interpretation assigned to the conclusions of (17) and (20). So, five of the six *Cut*-free derivations of the sequent $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash \vdash s$ are semantically equivalent.

However, the interpretation $u(\lambda u'.w(\lambda w'.v(w')(u')))$ of the conclusion of derivation (16) is *not* equivalent to the interpretation of the other conclusions. This can be seen as follows. If the expressions *someone*, *bores* and *everyone* are assigned categories and translations as in (22), then performing the simultaneous substitution (23) in the term of the conclusion goal in (16) supplies the sentence *Someone bores everyone* with the wide scope-subject interpretation $\exists x \forall y\, \text{BORE}(y)(x)$, whereas performing this substitution in the terms of the conclusion goals in (17) through (21) yields the wide scope-object interpretation $\forall y \exists x\, \text{BORE}(y)(x)$.

(22)

EXPRESSION	CATEGORY	INTERPRETATION
someone	$s/(n\backslash s)$	$\lambda P.\exists x\, P(x)$
bores	$(n\backslash s)/n$	BORE
everyone	$(s/n)\backslash s$	$\lambda P.\forall y\, P(y)$

[4] By *η-conversion*, the following equivalence is meant: $\lambda v.\gamma'(v) = \gamma'$, provided that v does not occur freely in γ'. A term γ is in *η-normal form* iff it does not have subterms of the form $\lambda v.\gamma'(v)$. Equivalence in typed λ-calculus amounts to identity of $\beta\eta$-normal forms in the following sense: for every term γ there is a unique term $\gamma^{\beta\eta}$ which is in $\beta\eta$-normal form (i.e., it is both in β-normal form and in η-normal form) and which can be obtained from γ by performing a finite number of β-conversions and η-conversions (where renaming of bound variables – i.e., the replacement of subterms $\lambda v.\alpha$ by $\lambda v'.[\alpha[v{:=}v']]$ if v' is free for v in α – is allowed), and two terms γ and δ are equivalent iff their $\beta\eta$-nfs $\gamma^{\beta\eta}$ and $\delta^{\beta\eta}$ are syntactically identical (again modulo renaming bound variables).

(23) $[u := \lambda P.\exists x\, P(x), v := \text{BORE}, w := \lambda P.\forall y\, P(y)]$

This example gives a nice illustration of what spurious ambiguity comes down to: the wide scope-subject interpretation is derived only once, but the wide-scope-object interpretation is associated with five derivations. Clearly, if we want to put the calculus **L** to use in parsing and interpreting expressions, this is going to be a problem: the latter interpretation will be computed five times instead of once.

König (1989) is the first attempt to solve the problem, but it does not succeed completely. For instance, König's system allows the two derivations (14) and (15) for the unambiguous sequent $c/c, c/c, c \vdash c$. In the sequel we will show how this problem can be solved for **L** by further restricting the *Cut*-free calculus. The resulting system **L***, which is equivalent to the system which was indepently proposed by Hepple (1990), is a solution to the spurious ambiguity problem: it provides exactly one proof per interpretation.

The 'non-spuriously ambiguous' calculus **L*** which will be presented below is based on the following three observations:

A: First, each non-atomic axiom instance $a/b \vdash a/b$ and $b\backslash a \vdash b\backslash a$ can be decomposed into a proof with two less complex axiom premises, $a \vdash a$ and $b \vdash b$. For instance, axioms $b\backslash a \vdash b\backslash a$ with interpretation $w \vdash w$ can be replaced by:

$$\frac{\dfrac{b \vdash b \quad a \vdash a}{b, b\backslash a \vdash a}[\backslash L]}{b\backslash a \vdash b\backslash a}[\backslash R] \qquad\qquad \frac{\dfrac{u \vdash u \quad v \vdash v}{u, w \vdash w(u)}[\backslash L]}{w \vdash \lambda u.w(u)}[\backslash R]$$

Due to η-conversion, the term $\lambda u.w(u)$ is equivalent to w. (Axioms $a/b \vdash a/b$ can be treated analogously.)

B: Second, if a $\backslash R$ or $/R$ inference yields the right-hand side premise of a $/L$ or $\backslash L$ inference, we can always reverse the order of the rules, whereas the semantics remains the same. We only consider the $\langle /R, /L\rangle$-case, where the rules involved are $/R$ and $/L$, respectively. (The cases $\langle /R, \backslash L\rangle$, $\langle \backslash R, /L\rangle$ and $\langle \backslash R, \backslash L\rangle$ are analogous.) We can replace:

(24)
$$\frac{T \vdash d \quad \dfrac{U, c, V, b \vdash a}{U, c, V \vdash a/b}[/R]}{U, c/d, T, V \vdash a/b}[/L]$$

by:

$$\frac{\dfrac{T \vdash d \quad U, c, V, b \vdash a}{U, c/d, T, V, b \vdash a}\,[/L]}{U, c/d, T, V \vdash a/b}\,[/R]$$

Semantically, this entails a replacement of:

$$\frac{T' \vdash \delta \qquad \dfrac{U', z, V', y \vdash \alpha}{U', z, V' \vdash \lambda y.\alpha}\,[/R]}{U', v, T', V' \vdash [\lambda y.\alpha][z := v(\delta)]}\,[/L]$$

by:

$$\frac{\dfrac{T' \vdash \delta \qquad U', z, V', y \vdash \alpha}{U', v, T', V', y \vdash \alpha[z := v(\delta)]}\,[/L]}{U', v, T', V' \vdash \lambda y.[\alpha[z := v(\delta)]]}\,[/R]$$

The terms $[\lambda y.\alpha][z := v(\delta)]$ and $\lambda y.[\alpha[z := v(\delta)]]$ are identical, since z and y are different variables.

C: Third, whenever a $\backslash L$ or $/L$ inference yields the right-hand side premise of another $\backslash L$ or $/L$ inference, and both inferences have *different* active categories,[5] we can reverse the order of the inferences and shift the latter inference to the left-hand side premise (cf. (25)) or to the right-hand side premise (cf. (26)) of the former one. We only treat the $\langle /L, \backslash L \rangle$-case. (The cases $\langle /L, /L \rangle$, $\langle \backslash L, /L \rangle$ and $\langle \backslash L, \backslash L \rangle$ are analogous.)

Suppose that the conclusion $X, W, d\backslash c, Z \vdash e$ has been derived by applying $\backslash L$ to left-hand side premise $W \vdash d$ and right-hand side premise $X, c, Z \vdash e$, and that the right-hand side premise $X, c, Z \vdash e$ is the result of

[5]The notions 'same' and 'different' active category are defined as follows: in a structure

$$\frac{T_2 \vdash b_2 \qquad \dfrac{T_1 \vdash b_1 \quad U, a, V \vdash c}{T_3 \vdash c}\,[\$L]}{T_4 \vdash c}\,[\#L]$$

(where $\$$ and $\# \in \{\backslash, /\}$), the inferences $\$L$ and $\#L$ have the *same* active category iff

(*i*) $\$ = \backslash$, $\# = \backslash$, $T_3 = U, T_1, b_1\backslash a, V$ and $T_4 = U, T_1, T_2, b_2\backslash(b_1\backslash a), V$; or

(*ii*) $\$ = \backslash$, $\# = /$, $T_3 = U, T_1, b_1\backslash a, V$ and $T_4 = U, T_1, (b_1\backslash a)/b_2, T_2, V$; or

(*iii*) $\$ = /$, $\# = \backslash$, $T_3 = U, a/b_1, T_1, V$ and $T_4 = U, T_2, b_2\backslash(a/b_1), T_1, V$; or

(*iv*) $\$ = /$, $\# = /$, $T_3 = U, a/b_1, T_1, V$ and $T_4 = U, (a/b_1)/b_2, T_2, T_1, V$;

otherwise, the inferences $\$L$ and $\#L$ have *different* active categories.

applying $/L$ with an active category a/b different from c – that is: $X, c, Z = P, a/b, Q, R$, and the category a/b occurs in X or Z:

$$\dfrac{W \vdash d \quad \dfrac{Q \vdash b \quad P, a, R \vdash e}{X, c, Z \vdash e}[/L]}{X, W, d\backslash c, Z \vdash e}[\backslash L]$$

Then there are three possibilities: c occurs in Q (see (25)), or c does not occur in Q, and then c occurs in P (see (26)) or in R (analogous to (26)). In all cases we can reverse the order of the rules while keeping the same semantics. If c occurs in Q, then $Q = Q_1, c, Q_2$, and we can replace:

(25)
$$\dfrac{W \vdash d \quad \dfrac{Q_1, c, Q_2 \vdash b \quad P, a, R \vdash e}{P, a/b, Q_1, c, Q_2, R \vdash e}[/L]}{P, a/b, Q_1, W, d\backslash c, Q_2, R \vdash e}[\backslash L]$$

by:

$$\dfrac{\dfrac{W \vdash d \quad Q_1, c, Q_2 \vdash b}{Q_1, W, d\backslash c, Q_2 \vdash b}[\backslash L] \quad P, a, R \vdash e}{P, a/b, Q_1, W, d\backslash c, Q_2, R \vdash e}[/L]$$

Semantically, this entails the replacement of:

$$\dfrac{W' \vdash \delta \quad \dfrac{Q'_1, z, Q'_2 \vdash \beta \qquad P', x, R' \vdash \epsilon}{P', y, Q'_1, z, Q'_2, R' \vdash \epsilon[x := y(\beta)]}[/L]}{P', y, Q'_1, W', v, Q'_2, R' \vdash [\epsilon[x := y(\beta)]][z := v(\delta)]}[\backslash L]$$

by:

$$\dfrac{\dfrac{W' \vdash \delta \qquad Q'_1, z, Q'_2 \vdash \beta}{Q'_1, W', v, Q'_2 \vdash \beta[z := v(\delta)]}[\backslash L] \quad P', x, R' \vdash \epsilon}{P', y, Q'_1, W', v, Q'_2, R' \vdash \epsilon[x := y(\beta[z := v(\delta)])]}[/L]$$

The terms $[\epsilon[x := y(\beta)]][z := v(\delta)]$ and $\epsilon[x := y(\beta[z := v(\delta)])]$ are identical, because z does not occur in ϵ and z and y are different variables.

If c occurs in P, then $P = P_1, c, P_2$ and we can replace:

(26)
$$\dfrac{W \vdash d \quad \dfrac{Q \vdash b \quad P_1, c, P_2, a, R \vdash e}{P_1, c, P_2, a/b, Q, R \vdash e}[/L]}{P_1, W, d\backslash c, P_2, a/b, Q, R \vdash e}[\backslash L]$$

by:

$$\dfrac{Q\vdash b \qquad \dfrac{W\vdash d \quad P_1,c,P_2,a,R\vdash e}{P_1,W,d\backslash c,P_2,a,R\vdash e}\,[\backslash L]}{P_1,W,d\backslash c,P_2,a/b,Q,R\vdash e}\,[/L]$$

In the semantics, this amounts to the replacement of:

$$\dfrac{W'\vdash \delta \qquad \dfrac{Q'\vdash \beta \qquad P'_1,z,P'_2,x,R'\vdash \epsilon}{P'_1,z,P'_2,y,Q',R'\vdash \epsilon[x:=y(\beta)]}\,[/L]}{U',P'_1,v,P'_2,y,Q',R'\vdash [\epsilon[x:=y(\beta)]][z:=v(\delta)]}\,[\backslash L]$$

by:

$$\dfrac{Q'\vdash \beta \qquad \dfrac{W'\vdash \delta \qquad P'_1,z,P'_2,x,R'\vdash \epsilon}{P'_1,W',v,P'_2,x,R'\vdash \epsilon[z:=v(\delta)]}\,[\backslash L]}{P'_1,W',v,P'_2,y,Q',R'\vdash [\epsilon[z:=v(\delta)]][x:=y(\beta)]}\,[/L]$$

The terms $[\epsilon[x:=y(\beta)]][z:=v(\delta)]$ and $[\epsilon[z:=v(\delta)]][x:=y(\beta)]$ are identical, since x and z are different variables, x does not occur in $v(\delta)$, and z does not occur in $y(\beta)$.[6]

Observation **A** entails that for every proof of a sequent, there is a semantically equivalent proof π in which all axiom instances are atomic: $at \vdash at$. For note that each non-atomic axiom instance can be decomposed in the way sketched above, and that every decomposition strictly decreases the sum of the degrees of the axiom instances. Hence every proof can be transformed into such an equivalent proof π after finitely many decompositions.

[6] Note (i) that transformation (26) cannot arise when the right-hand side premise of the first inference is an axiom, because the left-hand side of $P_1,c,P_2,a,R\vdash e$ contains at least the two categories c and a; and (ii) that if transformation (26) is executed in a configuration such as the one indicated below, where on the left-hand side $@L$ and $\$L$ have the same active category and $\$L$ and $\#L$ have different active categories, then on the right-hand-side $@L$ and $\#L$ (as well as $\#L$ and $\$L$) have different active categories:

$$\dfrac{T_5\vdash c \qquad \dfrac{T_3\vdash b \qquad \dfrac{T_1\vdash a \quad T_2\vdash d}{T_4\vdash d}\,[@L]}{T_6\vdash d}\,[\$L]}{T_7\vdash d}\,[\#L] \quad \rightsquigarrow_{(26)} \quad \dfrac{T_3\vdash b \qquad \dfrac{T_5\vdash c \qquad \dfrac{T_1\vdash a \quad T_2\vdash d}{T_4\vdash d}\,[@L]}{T'_6\vdash d}\,[\#L]}{T_7\vdash d}\,[\$L]$$

Moreover, given such a π, we can use observations **B** and **C** for obtaining a further normalized proof π'' in which (i) no *major* (right-hand side) premise of a $\backslash L$ or $/L$ inference is the conclusion of a $\backslash R$ or $/R$ inference (this corresponds to **B**);[7] and (ii) the same left-hand side category remains active whenever one goes down from axioms via major premises of $\backslash L$ and $/L$ inferences (this corresponds to **C**).

In order to see this, we first define the notion 'maximal major path' (mmp) for *Cut*-free **L** proofs π: an *mmp* is a sequence $\langle \sigma_0, \ldots, \sigma_p \rangle$ $(p \in \mathbb{N})$ of sequent occurrences in π, such that (i) σ_0 is an axiom instance; (ii) for all i, $0 \leq i < p$, σ_{i+1} is the conclusion of an inference with sole ($\backslash R$ or $/R$) or major ($\backslash L$ or $/L$) premise σ_i; and (iii) σ_p is not the sole or major premise of an inference.[8]

It is not hard to see that every sequent occurrence in a proof π is on exactly one mmp, and that every proof π contains as many mmps as it contains axiom instances, i.e., at least one.

Now, consider a proof π with $n+1$ mmps, and focus on the main mmp $\mu = \langle \sigma_0, \ldots, \sigma_p \rangle$ which culminates in the final conclusion σ_p of π. Let q be the number of $\backslash L$ and $/L$ inferences in μ, and r the number of $\backslash R$ and $/R$ inferences in μ. So, $\sigma_p = \sigma_{q+r}$, some sequent $T_{q+r} \vdash c_{q+r}$, and the sequent σ_0 is an axiom $c_0 \vdash c_0$. Moreover, for $0 \leq i \leq q+r$: (a) each σ_{i+1} obtained by $\backslash R$ or $/R$ is concluded from a subproof π_i of π with conclusion σ_i, and (b) each σ_{i+1} obtained by $\backslash L$ or $/L$ is concluded from two subproofs of π: π'_j (where $1 \leq j \leq q$) and π_i (with conclusion σ_i):

$$(a) \quad \dfrac{\left.\begin{array}{c} \vdots \\ \sigma_i \end{array}\right] \pi_i}{\sigma_{i+1}} \qquad\qquad (b) \quad \dfrac{\pi'_j \qquad \left.\begin{array}{c} \vdots \\ \sigma_i \end{array}\right] \pi_i}{\sigma_{i+1}}$$

The left-hand side subproofs π'_1 through π'_q have a total number of n mmps, which is one less than the number of mmps in the whole proof π.

Next, let $k(\mu) \leq r \cdot q$ be the number of pairs of sequents σ_i and σ_j in μ such that (i) $1 \leq i < j \leq q+r$; (ii) σ_i is the conclusion of a $\backslash R$ or $/R$ inference; and (iii) σ_j is the conclusion of a $\backslash L$ or $/L$ inference.

Note that an application of transformation (24) to a $\backslash R$ or $/R$ inference which yields the major premise of a $\backslash L$ or $/L$ inference in an mmp ν always

[7] Consequently, every major premise of a $\backslash L$ or $/L$ inference must be an (atomic) axiom instance $at \vdash at$ or the conclusion of another $\backslash L$ or $/L$ inference. Since $\backslash L$ and $/L$ identify the goal categories of their major premise and conclusion, every $\backslash L$ and $/L$ inference in π'' necessarily derives a conclusion sequent with an atomic goal category: $T \vdash at$.

[8] Therefore, σ_p must be either the *minor* (left-hand side) premise of a $\backslash L$ or $/L$ inference, or the final conclusion of π.

leads to an mmp ν' which (i) has the same length, (ii) has the same σ_{q+r}, (iii) builds on the same left-hand side subproofs $\pi'_1, \ldots \pi'_q$, and (iv) assigns the same interpretation to σ_{q+r} as ν; but that $k(\nu') = k(\nu) - 1$:

$$\dfrac{\pi'_j \qquad \dfrac{\pi_{i-1}}{\sigma_i}}{\sigma_{i+1}} \qquad \leadsto_{(24)} \qquad \dfrac{\dfrac{\pi'_j \qquad \pi_{i-1}}{\sigma'_i}}{\sigma_{i+1}}$$

So, after $k(\mu)$ applications of transformation (24) we obtain a semantically equivalent proof π' of $T_{q+r} \vdash c_{q+r}$ such that in the main mmp μ' of π' no $\backslash R$ or $/R$ inference precedes a $\backslash L$ or $/L$ inference, i.e., $k(\mu') = 0$:

$$\left.\begin{array}{c} \dfrac{\pi'_1 \qquad c_0 \vdash c_0 \ [\text{AXIOM}]}{T'_1 \vdash c'_1} \\ \vdots \\ \dfrac{\pi'_q \qquad \ddots}{T'_q \vdash c'_q} \end{array}\right] \ \backslash L \text{ and } /L$$

$$\left.\begin{array}{c} \vdots \\ \dfrac{}{T'_{(q+r)-1} \vdash c'_{(q+r)-1}} \\ \dfrac{}{T_{q+r} \vdash c_{q+r}} \end{array}\right] \ \backslash R \text{ and } /R$$

Concentrate on the sequence $\nu = \langle c_0 \vdash c_0, \ldots, T'_q \vdash c'_q \rangle$ which is the main mmp of the subproof of π' with conclusion $T'_q \vdash c'_q$, and consider the first inference $\$L$ in ν which has a different active category than the inference $\#L$ which yields the major premise of $\$L$ (where $\$$ and $\# \in \{\backslash, /\}$). We can apply one of the proof transformations (25) and (26). If (26) is applicable, then inference $\$L$ reappears one place higher up in the main mmp, now yielding the major premise of $\#L$, and again having a different active category than the inference which now yields its major premise (cf. footnote 6 (ii)):

$$\dfrac{\pi'_{j+1} \qquad \dfrac{\pi'_j \qquad \pi_{i-1}}{\sigma_i}[\#L]}{\sigma_{i+1}}[\$L] \qquad \leadsto_{(26)} \qquad \dfrac{\pi'_j \qquad \dfrac{\pi'_{j+1} \qquad \pi_{i-1}}{\sigma'_i}[\$L]}{\sigma_{i+1}}[\#L]$$

Observe that an application of transformation (26) to an mmp of length q always leads to an mmp with (i) the same length, (ii) the same σ_q, (iii) the same left-hand side subproofs $\pi'_1, \ldots \pi'_q$ (though appearing in a different order), and (iv) the same interpretation for σ_q. Since the new situation is structurally the same as the old one, transformation (26) can be applied again and again, until transformation (25) is applicable – for $\$L$ yielding conclusion σ_i $(2 \leq i \leq q)$, this happens after at most i steps: when $\$L$ yields σ_2, transformation (25) must be applied (cf. footnote 6 (i)).

Transformation (25) *removes* inference $\$L$ from the mmp and *adds* it, together with its left-hand side subproof π'_{j+1}, to the left-hand side subproof π'_j of $\#L$, where the number of mmps in the newly created left-hand side subproof π''_j is equal to the sum of the number of mmps in π'_j and π'_{j+1}:[9]

$$\dfrac{\pi'_{j+1} \quad \dfrac{\pi'_j \qquad \pi_{i-1}}{\sigma_i}[\#L]}{\sigma_{i+1}}[\$L] \quad \rightsquigarrow_{(25)} \quad \pi''_j \left[\dfrac{\dfrac{\pi'_{j+1} \qquad \pi'_j}{\sigma'_i}[\$L] \qquad \pi_{i-1}}{\sigma_{i+1}}[\#L] \right.$$

The resulting mmp ν' is one sequent shorter than ν, but its last sequent σ'_{q-1} equals σ_q of ν, and receives the same interpretation. The same procedure can be reapplied to the first inference $\$L'$ in ν' which has a different active category than the inference $\#L'$ which yields its major premise. If this is done s times, where $0 \leq s \leq q-1$ and s is the number of $\backslash L$ and $/L$ inferences having a different active category than the first inference in ν, we obtain a semantically equivalent proof π'' of $T_{q+r} \vdash c_{q+r}$ such that in μ'', the main mmp of π'', no application of $\backslash R$ or $/R$ precedes an application of $\backslash L$ or $/L$, and all applications of $\backslash L$ and $/L$ have the same active category:

$$\dfrac{\dfrac{\vdots}{T'_{(q+r)-1} \vdash c'_{(q+r)-1}}}{T_{q+r} \vdash c_{q+r}} \quad \text{with} \quad \vdots = \dfrac{\pi''_{q-s} \qquad \dfrac{\dfrac{\pi''_1 \qquad c_0 \vdash c_0\ [\text{AXIOM}]}{T'_1 \vdash c'_1}}{\vdots}}{T'_q \vdash c'_q}$$

$\backslash L$ and $/L$: same active category

$\backslash R$ and $/R$

The finite procedure which led from π to π'' can be repeated for the subproofs $\pi''_1, \ldots, \pi''_{q-s}$ of π''. Moreover, the task of normalizing the proofs π''_1 through π''_{q-s} involves less work than our initial task of normalizing π, because we started with a proof with $n+1$ mmps, whereas the total number of mmps in π''_1 through π''_{q-s} equals n.

The above considerations can be summarized in the form of the calculus **L***, which is given in (27) below. This calculus introduces the symbol '*'

[9]So that the new series of left-hand side subproofs $\pi'_1,\ldots,\pi'_{j-1},\pi''_j,\pi'_{j+2},\ldots,\pi'_q$ of ν' still has a total number of n mmps.

which controls the activity of categories in derivations. Unlike the symbol '$\sharp$', which was merely added to the **L** derivations (14) through (21) in order to enhance their readability, the symbol '*' is a substantial ingredient of **L*** derivations, since there is a special rule which deals with its behaviour:

(27) SYNTAX: SEMANTICS:

$$at^* \vdash at \;[\text{AXIOM}] \qquad x \vdash x$$

$$\frac{T \vdash b^* \quad U, a^*, V \vdash at}{U, a/b^*, T, V \vdash at}\,[/L] \qquad \frac{T' \vdash \beta \qquad U', u, V' \vdash \gamma}{U', w, T', V' \vdash \gamma[u := w(\beta)]}$$

$$\frac{T \vdash b^* \quad U, a^*, V \vdash at}{U, T, b\backslash a^*, V \vdash at}\,[\backslash L] \qquad \frac{T' \vdash \beta \qquad U', u, V' \vdash \gamma}{U', T', w, V' \vdash \gamma[u := w(\beta)]}$$

$$\frac{T, b \vdash a^*}{T \vdash a/b^*}\,[/R] \qquad \frac{T', v \vdash \alpha}{T' \vdash \lambda v.\alpha}$$

$$\frac{b, T \vdash a^*}{T \vdash b\backslash a^*}\,[\backslash R] \qquad \frac{v, T' \vdash \alpha}{T' \vdash \lambda v.\alpha}$$

$$\frac{U, a^*, V \vdash at}{U, a, V \vdash at^*} \qquad \frac{U', u, V' \vdash \gamma}{U', u, V' \vdash \gamma}\,[\,*\,]$$

The calculus in (27) observes the same conventions as regards categories, types, terms and variables as (10) above, with the exceptions that at represents an arbitrary atomic category, that γ refers to a term of type $\text{TYPE}(at)$, and that x denotes a variable of type $\text{TYPE}(at)$.

Important properties of the calculus **L*** are the following. Let $\gamma' = \gamma$ denote equivalence of terms γ' and γ (we will use $\gamma' \equiv \gamma$ for syntactic identity of terms modulo renaming bound variables):

Theorem 1:
1. $T \vdash_{\mathbf{L*}} c^*$ if and only if $T \vdash_{\mathbf{L}} c$.
2. If $T{:}T' \vdash_{\mathbf{L*}} c^*{:}\gamma$, then $T{:}T' \vdash_{\mathbf{L}} c{:}\gamma$.
3. If $T{:}T' \vdash_{\mathbf{L}} c{:}\gamma$, there is a γ' such that $\gamma' = \gamma$ and $T{:}T' \vdash_{\mathbf{L*}} c^*{:}\gamma'$.

Proof: We have seen that if $T{:}T' \vdash_{\mathbf{L}} c{:}\gamma$, then there is a *Cut*-free **L** proof π of $T : T' \vdash c : \gamma'$ such that $\gamma' = \gamma$ and π has the following properties: (a) all axiom instances in π are atomic; (b) no major premise of a $\backslash L$ or $/L$

inference in π is the conclusion of a $\backslash R$ or $/R$ inference; and (c) the same category remains active whenever one goes down from axioms via major premises of $\backslash L$ and $/L$ inferences in π.

But this is sufficient, for there is an $\mathbf{L^*}$ proof π' of $T{:}T' \vdash c^*{:}\gamma'$ iff there is a *Cut*-free $\mathbf{L}$ proof π of $T{:}T' \vdash c{:}\gamma'$ with the properties (a) through (c):

Observe (a') that $\mathbf{L^*}$ axioms $at^* \vdash at$ involve only atomic categories; (b') that the major premise of a $\backslash L$ or $/L$ inference in $\mathbf{L^*}$ can only be an axiom or the conclusion of another $\backslash L$ or $/L$ inference (the asterisk must be on the left-hand side); and (c') that if a $\backslash L$ or $/L$ inference yields the major premise of another $\backslash L$ or $/L$ inference, then they have the same (asterisked) active left-hand side category. On account of (a') through (c'), every *Cut*-free $\mathbf{L}$ proof π of a sequent $T{:}T' \vdash c{:}\gamma'$ having the properties (a) through (c) can be turned into an $\mathbf{L^*}$ proof π' of $T : T' \vdash c^* : \gamma'$ by (i) adding an asterisk to the left-hand side category of every axiom instance; (ii) adding an asterisk to the active left-hand side category in every conclusion sequent of a $\backslash L$, $/L$, $\backslash R$ and $/R$ inference; and (iii) replacing every sequent of the form $U{:}U', a^*{:}u, V : V' \vdash at{:}\gamma$ which is not the major premise of a $\backslash L$ or $/L$ inference by the following inference:

(28) $$\frac{U{:}U', a^*{:}u, V{:}V' \vdash at : \gamma}{U{:}U', a{:}u, V{:}V' \vdash at^* : \gamma}\,[\,*\,]$$

And, conversely, every $\mathbf{L^*}$ proof π' of $T{:}T' \vdash c^*{:}\gamma'$ can be turned into a *Cut*-free $\mathbf{L}$ proof π of $T{:}T' \vdash c{:}\gamma'$ with properties (a) through (c) by (i) replacing every inference of the form (28) by the sequent $U : U', a : u, V : V' \vdash at : \gamma$; and (ii) deleting all remaining asterisks. □

4 Spurious Ambiguity

In the present section we will show that $\mathbf{L^*}$ is a solution to the spurious ambiguity problem: if π_1 is an $\mathbf{L^*}$ proof of $T : T' \vdash c : \gamma$ and π_2 is an $\mathbf{L^*}$ proof of $T : T' \vdash c : \gamma'$ such that $\gamma = \gamma'$, then $\pi_1 = \pi_2$.[10] As $\gamma = \gamma'$ entails that $\lambda T'.\gamma = \lambda T'.\gamma'$, this claim is a corollary of the following:

[10]Above we merely required that in the context of an $\mathbf{L}$ (or $\mathbf{L^*}$) proof all variables x and w assigned to an axiom instance or introduced in the conclusion of a $\backslash L$ or $/L$ inference be different. Accordingly, we will identify all derivations which differ only with respect to the particular variables they assign. Thus for sequents σ and τ, we let $\sigma = \tau$ iff $\sigma = c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ and $\tau = c_1 : v_1', \ldots, c_n : v_n' \vdash c : \gamma'$; and for proofs π and ρ, we let $\pi = \rho$ if and only if

- $\pi = \sigma$, $\rho = \tau$, where σ and τ are (axiom) sequents such that $\sigma = \tau$;

Theorem 2:
If π_1 is an **L*** proof of $T\!:\!T' \vdash c\!:\!\gamma'$ and π_2 is an **L*** proof of $T\!:\!T'' \vdash c\!:\!\gamma''$ such that $\lambda T'.\gamma' = \lambda T''.\gamma''$, then $\pi_1 = \pi_2$.

Proof: Suppose π_1 and π_2 are two different proofs of the same conclusion $T \vdash c$. Since axioms and non-axiom theorems in **L*** never have the same degree (cf. definition (13) (iii) above), no sequent can be both an axiom and a non-axiom theorem. This entails that there must be at least one non-axiom sequent $T^\circ \vdash c^\circ$ which is derived from different premises in π_1 and π_2. Inspection of **L*** shows that the goal c° of this sequent cannot be of the form $b\backslash a^*$ or a/b^*, for the rules $\backslash R$ and $/R$ which yield conclusions with such goals have the property that their conclusion sequents, $T \vdash b\backslash a^*$ and $T \vdash a/b^*$, completely determine the form of their premise sequents: $b, T \vdash a^*$ and $T, b \vdash a^*$, respectively. So, c° must be (i) an asterisked atomic category at^*; or (ii) a non-asterisked atomic category at.

(i) If $c^\circ = at^*$, then $T^\circ \vdash c^\circ$ must have been obtained with the * rule in both π_1 and π_2. Hence $T^\circ = U, a, V, b, W$ and the situation is as follows in π_1 and π_2:

(29) $$\frac{U\!:\!U', a^*\!:\!u', V\!:\!V', b\!:\!v', W\!:\!W' \vdash at\!:\!\gamma'}{U\!:\!U', a\!:\!u', V\!:\!V', b\!:\!v', W\!:\!W' \vdash at^*\!:\!\gamma'}\,[\,*\,] \text{ and}$$

$$\frac{U\!:\!U'', a\!:\!u'', V\!:\!V'', b^*\!:\!v'', W\!:\!W'' \vdash at\!:\!\gamma''}{U\!:\!U'', a\!:\!u'', V\!:\!V'', b\!:\!v'', W\!:\!W'' \vdash at^*\!:\!\gamma''}\,[\,*\,]$$

It turns out that $\lambda U'\lambda u'\lambda V'\lambda v'\lambda W'.\gamma'$ and $\lambda U''\lambda u''\lambda V''\lambda v''\lambda W''.\gamma''$ are non-equivalent terms (see Fact 1 below).

- $\pi = \dfrac{\pi'}{\sigma}$, $\rho = \dfrac{\rho'}{\tau}$, $\pi' = \rho'$ and $\sigma = \tau$; or
- $\pi = \dfrac{\pi' \quad \pi''}{\sigma}$, $\rho = \dfrac{\rho' \quad \rho''}{\tau}$, $\pi' = \rho'$, $\pi'' = \rho''$ and $\sigma = \tau$.

Alternatively, we could enforce one particular assignment of different variables to each derivation π, for instance by requiring (i) that the sequence of axiom leafs of π equal $at_1{}^* : v_1 \vdash at_1 : v_1, \ldots, at_n{}^* : v_n \vdash at_n : v_n$, where for all i, $1 \leq i \leq n$: v_1 is the i-th variable of type $\text{TYPE}(at_i)$; and (ii) that for all instances of $\backslash L$ and $/L$ in π the variable w be the i-th variable of type $(\text{TYPE}(b),\text{TYPE}(a))$ iff u is the i-th variable of type $\text{TYPE}(a)$. Note that this would not restrict the semantic potential of the calculus, for it is easily checked that all π_1 and π_2 such that $\pi_1 = \pi_2$ in the sense just defined are semantically equivalent: if $\pi_1 = \pi_2$, then π_1 is a proof of $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ and π_2 is a proof of $c_1 : v'_1, \ldots, c_n : v'_n \vdash c : \gamma'$ such that $\lambda v_1 ... \lambda v_n.\gamma \equiv \lambda v'_1 ... \lambda v'_n.\gamma'$ (and hence: $\lambda v_1 ... \lambda v_n.\gamma = \lambda v'_1 ... \lambda v'_n.\gamma'$). The more liberal option has been chosen since it links up better with the alternative approaches to be discussed in the next section.

(ii) If $c^\circ = at$, then the left-hand side T° of $T^\circ \vdash c^\circ$ must contain one asterisked category $d*$. Because $T^\circ \vdash c^\circ$ is not an axiom, $d*$ must be a non-atomic asterisked category $a/b*$ (or $b\backslash a*$), and, accordingly, $T^\circ \vdash x^\circ$ must have been obtained using $/L$ (or $\backslash L$) in both π_1 and π_2. We only consider $/L$, since the $\backslash L$ case is analogous. Then $d*$ equals $a/b*$; $T^\circ = U, a/b*, V, W, X$, where W is a non-empty sequence of categories; and the situation is as follows in π_1 and π_2:

(30)
$$\frac{V\!:\!V' \vdash b*\!:\!\beta' \qquad U\!:\!U', a*\!:\!u', W\!:\!W', X\!:\!X' \vdash at\!:\!\gamma'}{U\!:\!U', a/b*\!:\!w', V\!:\!V', W\!:\!W', X\!:\!X' \vdash at\!:\!\gamma'[u'\!:=\!w'(\beta')]}\,[/L] \text{ and}$$

$$\frac{V\!:\!V'', W\!:\!W'' \vdash b*\!:\!\beta'' \qquad U\!:\!U'', a*\!:\!u'', X\!:\!X'' \vdash at\!:\!\gamma''}{U\!:\!U'', a/b*\!:\!w'', V\!:\!V'', W\!:\!W'', X\!:\!X'' \vdash at\!:\!\gamma''[u''\!:=\!w''(\beta'')]}\,[/L]$$

The term $\lambda U'\lambda w'\lambda V'\lambda W'\lambda X'.\gamma'[u'\!:=\!w'(\beta')]$ can be shown to be non-equivalent to $\lambda U''\lambda w''\lambda V''\lambda W''\lambda X''.\gamma''[u''\!:=\!w''(\beta'')]$ (see Fact 2 below).

Consequently, the intermediate conclusion $T^\circ \vdash c^\circ$ gets non-equivalent interpretations in π_1 and π_2.

Moreover, it can be shown that any non-equivalence of the terms which different proofs assign to some intermediate sequent is inherited by the terms assigned to the conclusions drawn from that sequent (see Fact 3 below). So, if the intermediate conclusion $T^\circ \vdash c^\circ$ has non-equivalent interpretations in π_1 and π_2, then $T \vdash c$, the final conclusion of both proofs, will receive non-equivalent interpretations in π_1 and π_2 as well. □

In footnote 3 it was observed that terms γ assigned to sequent goals in *Cut*-free **L** proofs are always in β-normal form, i.e., they do not have subterms of the form $[\lambda v.\gamma'](\gamma'')$. In the proof of Theorem 1 we have seen that the terms assigned to sequent goals in **L*** proofs constitute a subset of the terms assigned to sequent goals in *Cut*-free **L** proofs and, hence, are in β-normal form as well. However, they also meet the stronger requirement of being in what we will call '$\beta\bar{\eta}$-normal form' (as can be seen by an easy induction on the length of **L*** derivations):

> A term γ is in *$\bar{\eta}$-normal form* ($\bar{\eta}$-nf) iff every occurrence of a subterm $\gamma' \not\equiv \lambda v.\gamma''$ of functional type (a, b) is applied to some term γ''' in γ, i.e., it occurs as $\gamma'(\gamma''')$ in γ.[11]

[11] Let *η-expansion* ('$\bar{\eta}$-conversion') be defined as the replacement of a subterm occurrence

A term γ is in *$\beta\bar{\eta}$-nf* iff γ is in β-nf and γ is in $\bar{\eta}$-nf.

As was noted in footnote 4, equivalence in typed λ-calculus amounts to identity of $\beta\eta$-nfs, i.e., two terms γ and δ are equivalent ($\gamma = \delta$) iff their $\beta\eta$-nfs $\gamma^{\beta\eta}$ and $\delta^{\beta\eta}$ are syntactically identical modulo renaming bound variables ($\gamma^{\beta\eta} \equiv \delta^{\beta\eta}$). Together with the FACT mentioned in footnote 11, this yields that $\gamma = \delta$ iff $\gamma^{\beta\bar{\eta}} \equiv \delta^{\beta\bar{\eta}}$. Terms assigned to sequent goals in **L*** proofs are in $\beta\bar{\eta}$-nf, so they are equivalent just in case they are identical, a fact which will be exploited in the corollaries of the following lemma. Let the function $\#$ count the number of arguments of $c \in$ CAT, i.e., $\#(at) = 0$ and $\#(a/b) = \#(b\backslash a) = \#(a) + 1$.

Lemma:
If $U\!:\!U', c^*\!:\!w, V\!:\!V' \vdash at : \gamma$, then there are terms $\gamma_1, \ldots, \gamma_{\#(c)}$ such that $\gamma \equiv w(\gamma_{\#(c)})...(\gamma_1)$ and w does not occur in $\gamma_1, \ldots, \gamma_{\#(c)}$.

Proof: by induction on the degree of c.
• If $c \in$ ATOM, then $\#(c) = 0$ and $U : U', c^* : w, V : V' \vdash at : \gamma$ is an axiom sequent $at^*\!:\!w \vdash at\!:\!w$, so $\gamma \equiv w$.
• If $c = a/b$ (or $b\backslash a$), then $U\!:\!U', c^*\!:\!w, V\!:\!V' \vdash at\!:\!\gamma$ is the conclusion of a $/L$ (or, analogously, a $\backslash L$) inference:

$$\frac{V_1\!:\!V_1' \vdash b^*\!:\!\beta \qquad U\!:\!U', a^*\!:\!u, V_2\!:\!V_2' \vdash at\!:\!\gamma}{U\!:\!U', a/b^*\!:\!w, V_1\!:\!V_1', V_2\!:\!V_2' \vdash at\!:\!\gamma[u\!:=\!w(\beta)]}\,[/L]$$

Applying the induction hypothesis to γ, we know that there are terms $\gamma_1, \ldots, \gamma_{\#(a)}$ such that $\gamma \equiv u(\gamma_{\#(a)})...(\gamma_1)$ and the variable u does not occur in $\gamma_1, \ldots, \gamma_{\#(a)}$. Note that $\#(a/b) = \#(a)+1$, and that the term $\gamma[u\!:=\!w(\beta)]$ is identical to $w(\beta)(\gamma_{\#(a)})...(\gamma_1)$, where w, being different from the variables in β and γ, does not occur in $\gamma_1, \ldots, \gamma_{\#(a)}, \beta$. $\Box$

$\gamma' \not\equiv \lambda v.\gamma''$ of functional type (a, b) which is not applied to any term γ''' by $\lambda v.\gamma'(v)$, where v does not occur freely in γ'. One can show that η-expansion is strongly normalizing and Church-Rosser: for every term γ there is a unique term $\gamma^{\bar{\eta}}$ which is in $\bar{\eta}$-normal form, and which can be obtained from γ by performing a finite number of η-expansions (where renaming of bound variables is allowed). This is established by induction on the complexity of (the type of) γ, using the facts that (i) variables of non-functional (atomic) type are in $\bar{\eta}$-nf; (ii) η-expansion of variables of type (a, b) only introduces variables of less complex type; and (iii) η-expansion of complex terms $\lambda v.\gamma'$ and $\gamma'(\gamma'')$ only involves η-expansion of less complex terms. As η-expansion respects β-normality, we have the following FACT: for every γ and δ: $\gamma^{\beta\eta} \equiv \delta^{\beta\eta}$ iff $\gamma^{\beta\bar{\eta}} \equiv \delta^{\beta\bar{\eta}}$.

Let T' be a sequence of variables $v_1, \ldots, v_n$, and T a sequence of categories $c_1, \ldots, c_n$. We will say that T' is *of type* T iff for all i, $1 \leq i \leq n$: v_i is of type $\text{TYPE}(c_i)$. If $T' = v'_1, \ldots, v'_n$ is a sequence of different variables and $T'' = v''_1, \ldots, v''_n$ is a sequence of variables of the same type as T', then $\gamma[T' := T'']$ denotes the result of performing the simultaneous substitution $[v'_1 := v''_2, \ldots, v'_n := v''_n]$ in γ.

Note that for sequences T', T'' and T''' of different variables of the same type such that T' and T'' occur in γ' and γ'', respectively, and T''' occurs in γ' nor γ'', we have: $\lambda T'.\gamma' \equiv \lambda T''.\gamma''$ iff $\gamma'[T' := T'''] \equiv \gamma''[T'' := T''']$, since

- $\lambda T'''.[\gamma'[T' := T''']] \equiv \lambda T'''.[\gamma''[T'' := T''']]$ iff $\gamma'[T' := T'''] \equiv \gamma''[T'' := T''']$;
- $\lambda T'.\gamma' \equiv \lambda T'''.[\gamma'[T' := T''']]$ by renaming bound variables; and
- $\lambda T''.\gamma'' \equiv \lambda T'''.[\gamma''[T'' := T''']]$ by renaming bound variables.

Fact 1. Consider (29). By the above Lemma: $\gamma' \equiv u'(\gamma'_{\#(a)})...(\gamma'_1)$ and $\gamma'' \equiv v''(\gamma''_{\#(b)})...(\gamma''_1)$. Let $U''', u''', V''', v''', W'''$ be a sequence of different variables of type U, a, V, b, W which do not occur in γ' and γ''. Abbreviate U', u', V', v', W' as S'; U'', u'', V'', v'', W'' as S''; and $U''', u''', V''', v''', W'''$ as S'''. Then $\lambda S'.\gamma' \not\equiv \lambda S''.\gamma''$, since $\gamma'[S' := S'''] \equiv [u'(\gamma'_{\#(a)})...(\gamma'_1)][S' := S''']$ $\equiv u'''(\gamma'_{\#(a)}[S' := S'''])...(\gamma'_1[S' := S''']) \not\equiv v'''(\gamma''_{\#(b)}[S'' := S'''])...(\gamma''_1[S'' := S'''])$ $\equiv [v''(\gamma''_{\#(b)})...(\gamma''_1)][S'' := S'''] \equiv \gamma''[S'' := S''']$. Therefore, $\lambda S'.\gamma' \neq \lambda S''.\gamma''$.

Fact 2. Consider (30). Again, by the Lemma: $\gamma' \equiv u'(\gamma'_{\#(a)})...(\gamma'_1)$ and $\gamma'' \equiv u''(\gamma''_{\#(a)})...(\gamma''_1)$, where u' and u'' do not occur in $\gamma'_1, \ldots, \gamma'_{\#(a)}$ and $\gamma''_1, \ldots, \gamma''_{\#(a)}$, respectively. Hence $\gamma'[u' := w'(\beta')] \equiv w'(\beta')(\gamma'_{\#(a)})...(\gamma'_1)$ and $\gamma''[u'' := w''(\beta'')] \equiv w''(\beta'')(\gamma''_{\#(a)})...(\gamma''_1)$. Since the sequence of categories W is non-empty, we have that $W = W_1, c, W_2$; $W' = W'_1, v', W'_2$; and $W'' = W''_1, v'', W''_2$. Note that for $1 \leq i \leq \#(a)$: v' occurs freely in some γ'_i but not in β', whereas v'' occurs freely in β'' but not in any γ''_i. Let $U''', w''', V''', W''', X'''$ be a sequence of different variables of type $U, a/b, V, W, X$ not occurring in $\gamma'[u' := w'(\beta')]$ and $\gamma''[u'' := w''(\beta')]$, and let $W''' = W'''_1, v''', W'''_2$. Denote U', w', V', W', X' by S'; U'', w'', V'', W'', X'' by S''; and $U''', w''', V''', W''', X'''$ by S'''.

We now have that $\lambda S'.[\gamma'[u' := w'(\beta')]] \not\equiv \lambda S''.[\gamma''[u'' := w''(\beta'')]]$, since $[\gamma'[u' := w'(\beta')]][S' := S'''] \not\equiv [\gamma''[u'' := w''(\beta'')]][S'' := S''']$:

- $[\gamma'[u' := w'(\beta')]][S' := S'''] \equiv [w'(\beta')(\gamma'_{\#(a)})...(\gamma'_1)][S' := S'''] \equiv$ $w'''(\beta'[S' := S'''])(\gamma'_{\#(a)}[S' := S'''])...(\gamma'_1[S' := S''']);$

- $[\gamma''[u'' := w''(\beta'')]][S'' := S'''] \equiv [w''(\beta'')(\gamma''_{\#(a)})...(\gamma''_1)][S'' := S'''] \equiv w'''(\beta''[S'' := S'''])(\gamma''_{\#(a)}[S'' := S'''])...(\gamma''_1[S'' := S'''])$; and
- the variable v''' occupies the position of v' in $[\gamma'[u' := w'(\beta')]][S' := S''']$ and of v'' in $[\gamma''[u'' := w''(\beta'')]][S'' := S''']$, so v''' occurs freely in some $\gamma'_i[S' := S''']$ but not in $\beta'[S' := S''']$, and it occurs freely in $\beta''[S'' := S''']$ but not in any $\gamma''_i[S'' := S''']$.

Therefore: $\lambda S'.[\gamma'[u' := w'(\beta')]] \not\equiv \lambda S''.[\gamma''[u'' := w''(\beta'')]]$.

Fact 3. Consider the following two inferences:

$$\frac{T : T' \vdash b* : \beta' \qquad U : U', a* : u', V : V' \vdash at : \gamma'}{U : U', a/b* : w', T : T', V : V' \vdash at : \gamma'[u' := w'(\beta')]}\,[/L] \text{ and}$$

$$\frac{T : T'' \vdash b* : \beta'' \qquad U : U'', a* : u'', V : V'' \vdash at : \gamma''}{U : U'', a/b* : w'', T : T'', V : V'' \vdash at : \gamma''[u'' := w''(\beta'')]}\,[/L]$$

By the Lemma: $\gamma' \equiv u'(\gamma'_{\#(a)})...(\gamma'_1)$, $\gamma'[u' := w'(\beta')] \equiv w'(\beta')(\gamma'_{\#(a)})...(\gamma'_1)$, $\gamma'' \equiv u''(\gamma''_{\#(a)})...(\gamma''_1)$ and $\gamma''[u'' := w''(\beta'')] \equiv w''(\beta'')(\gamma''_{\#(a)})...(\gamma''_1)$. Let U''', w''', T''', V''' be a sequence of different variables of type $U, a/b, T, V$ which do not occur in $\gamma'[u' := w'(\beta')]$ and $\gamma''[u'' := w''(\beta'')]$. Abbreviate U', w', T', V' as S'; U'', w'', T'', V'' as S''; and U''', w''', T''', V''' as S'''.

Now, suppose that $\lambda S'.[\gamma'[u' := w'(\beta')]] \equiv \lambda S''.[\gamma''[u'' := w''(\beta'')]]$. This entails that $[\gamma'[u' := w'(\beta')]][S' := S'''] \equiv [\gamma''[u'' := w''(\beta'')]][S'' := S''']$, that is: $[w'(\beta')(\gamma'_{\#(a)})...(\gamma'_1)][S' := S'''] \equiv [w''(\beta'')(\gamma''_{\#(a)})...(\gamma''_1)][S'' := S''']$.

So, $\beta'[S' := S'''] \equiv \beta''[S'' := S''']$ and for all i, $1 \leq i \leq \#(a)$: $\gamma'_i[S' := S'''] \equiv \gamma''_i[S'' := S''']$. Since the variables U', w', V' and U'', w'', V'' do not occur in β' and β'', respectively, we have that $\beta'[S' := S'''] \equiv \beta'[T' := T''']$ and $\beta''[S'' := S'''] \equiv \beta''[T'' := T''']$. So, $\lambda T'.\beta' \equiv \lambda T''.\beta''$. Moreover, since the variables w', T' and w'', T'' do not occur in any γ_i and γ''_i, respectively, we have that for all i: $\gamma'_i[S' := S'''] \equiv \gamma'_i[U', V' := U''', V''']$ and $\gamma''_i[S'' := S'''] \equiv \gamma''_i[U'', V'' := U''', V''']$. Finally, since u' and u'' are different from the variables in $\gamma'_1, \ldots, \gamma'_{\#(a)}$ and $\gamma''_1, \ldots, \gamma''_{\#(a)}$, respectively, we can conclude that $\lambda U'\lambda u'\lambda V'.u'(\gamma'_{\#(a)})...(\gamma'_1) \equiv \lambda U''\lambda u''\lambda V''.u''(\gamma''_{\#(a)})...(\gamma''_1)$.

Consequently, if $\lambda T'.\beta' \not\equiv \lambda T''.\beta''$ or $\lambda U'\lambda u'\lambda V'.\gamma' \not\equiv \lambda U''\lambda u''\lambda V''.\gamma''$, then $\lambda U'\lambda w'\lambda T'\lambda V'.[\gamma'[u' := w'(\beta')]] \not\equiv \lambda U''\lambda w''\lambda T''\lambda V''.[\gamma''[u'' := w''(\beta'')]]$.

Furthermore, note (i) that the $\backslash L$ case is analogous; (ii) that the semantics of the * rule is identity; and (iii) that the cases $\backslash R$ and $/R$ are trival as well. So, non-equivalence of terms assigned to some sequent is always inherited by the terms assigned to the conclusions drawn from that sequent.

Returning to the examples treated earlier, it turns out that the unambiguous sequent $c/c, c/c, c \vdash c^*$ is indeed derived only once in **L***, whereas it has two *Cut*-free derivations, (14) and (15), in **L**:

(31)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{c^* \vdash c}{c \vdash c^*}\,[*] \qquad c^* \vdash c}{c/c^*, c \vdash c}\,[/L]}{c/c, c \vdash c^*}\,[*] \qquad c^* \vdash c}{c/c^*, c/c, c \vdash c}\,[/L]}{c/c, c/c, c \vdash c^*}\,[*]$$

Modulo * inferences, derivation (31) has the structure of (14). There is no **L*** proof with the structure of (15), for the main mmp of (15) contains two different active left-hand side categories.

The two-way ambiguous sequent $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$ which has the six *Cut*-free **L** derivations (16) through (21) is derived twice in **L***, cf. (32) and (33) below. These derivations are structurally similar and semantically equivalent to (16) and (19), respectively. The remaining *Cut*-free **L** derivations of the sequent do not have **L*** counterparts: (17) and (20) because of a non-atomic axiom instance $n\backslash s \vdash n\backslash s$; (18) due to a $\backslash R$ inference which yields the major premise of a $/L$ inference; and (21) on account of an mmp with two different active left-hand side categories.

(32)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n^* \vdash n}{n \vdash n^*}\,[*] \qquad \dfrac{\dfrac{n^* \vdash n}{n \vdash n^*}\,[*] \qquad s^* \vdash s}{n, n\backslash s^* \vdash s}\,[\backslash L]}{n, (n\backslash s)/n^*, n \vdash s}\,[/L]}{n, (n\backslash s)/n, n \vdash s^*}\,[*]}{n, (n\backslash s)/n \vdash s/n^*}\,[/R] \qquad s^* \vdash s}{n, (n\backslash s)/n, (s/n)\backslash s^* \vdash s}\,[\backslash L]}{n, (n\backslash s)/n, (s/n)\backslash s \vdash s^*}\,[*]}{(n\backslash s)/n, (s/n)\backslash s^* \vdash n\backslash s^*}\,[\backslash R] \qquad s^* \vdash s}{s/(n\backslash s)^*, (n\backslash s)/n, (s/n)\backslash s \vdash s}\,[/L]}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s^*}\,[*]$$

(33)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{n^* \vdash n}{n \vdash n^*}\,[*] \quad \dfrac{\dfrac{n^* \vdash n}{n \vdash n^*}\,[*] \quad s^* \vdash s}{n, n\backslash s^* \vdash s}\,[\backslash L]}{n, (n\backslash s)/n^*, n \vdash s}\,[/L]}{n, (n\backslash s)/n, n \vdash s^*}\,[*]}{(n\backslash s)/n, n \vdash n\backslash s^*}\,[\backslash R] \quad s^* \vdash s}{s/(n\backslash s)^*, (n\backslash s)/n, n \vdash s}\,[/L]}{s/(n\backslash s), (n\backslash s)/n, n \vdash s^*}\,[*]}{s/(n\backslash s), (n\backslash s)/n \vdash s/n^*}\,[/R] \quad s^* \vdash s}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s^* \vdash s}\,[\backslash L]}{s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s^*}\,[*]$$

5 Partial Deduction and Proof Nets

In the present section we will discuss two alternative approaches to the problem of spurious ambiguity which were proposed in Moortgat (1990) and Roorda (1991). First, in Section 5.1, we give a formalization of the proposals in Moortgat (1990), and prove that the resulting proof representations 'summarize' **L*** derivations of sequents with atomic goal categories and are, hence, devoid of spurious ambiguity. Next, section 5.2 is devoted to a comparison of **L*** with the proof nets proposed in Roorda (1991). It is shown that every **L*** proof corresponds to a non-singleton *set* of proof nets.

5.1 Partial Deduction

Moortgat (1990) suggests that *partial deduction* offers a way of tackling the problem of spurious ambiguity for **L**. The basic idea of partial deduction is that theorem proving is preceded by a compilation of the *lexicon*, i.e., the set of categories which occur in the left-hand side of the sequents to be derived and interpreted. This set of categories serves as the basis for the generation of axioms and rules which are to replace the original axioms and rules of **L**.[12] The idea is that since they have more internal structure than the rules of **L**, the new rules restrict the derivational freedom of the original system, thus eliminating spurious ambiguity.

[12]Moortgat says that 'partial deduction of the lexicon adds *non-trivial* AXIOMS to the *trivial* axioms and logical rules' of **L**, such that the resulting system 'can do *without* the original logical rules' (1990, p. 388; small caps added, italics *sic*), but in the sequel we will see that the addition involves both axioms and rules.

Unfortunately, Moortgat (1990) does not contain a formal definition of partial deduction, let alone a proof that the method is sound and complete – i.e., that it eliminates all spurious ambiguity while preserving all **L**-derivable sequents and interpretations.

The paper only treats two examples, viz., the categories $(a_1\backslash a_2)/a_3$ and $(s/(n\backslash s))\backslash s$, where a_1, a_2, a_3, n and s are atomic.

The first order[13] category $(a_1\backslash a_2)/a_3$ gives rise to a single derived inference rule:[14]

(34) $$\frac{T_2 : T_2' \vdash a_3 : \gamma_2 \qquad\qquad T_1 : T_1' \vdash a_1 : \gamma_1}{T_1 : T_1', (a_1\backslash a_2)/a_3 : v_0, T_2 : T_2' \vdash a_2 : v_0(\gamma_2)(\gamma_1)}\,[(a_1\backslash a_2)/a_3]$$

The higher order category $(s/(n\backslash s))\backslash s$, on the other hand, results in a *set* consisting of two derived inference rules. One corresponds to the category $(s/(n\backslash s))\backslash s$ itself, and the other to the (only) argument $n\backslash s$ of the (only) argument $s/(n\backslash s)$ of $(s/(n\backslash s))\backslash s$:

(35) $$\frac{T_1 : T_1', n\backslash s : v_1 \vdash s : \gamma_1}{T_1 : T_1', (s/(n\backslash s))\backslash s : v_0 \vdash s : v_0(\lambda v_1.\gamma_1)}\,[(s/(n\backslash s))\backslash s]$$

$$\frac{T_1 : T_1' \vdash n : \gamma_1}{T_1 : T_1', n\backslash s : v_0 \vdash s : v_0(\gamma_1)}\,[n\backslash s]$$

We will show that a formal definition can be given which (i) is consistent with the examples treated by Moortgat; and (ii) indeed solves the problem of spurious ambiguity for sequents with atomic goal categories.

First, we define two functions from categories to sets of categories. For $c \in \textsc{cat}$, $A(c)$ is the set consisting of the arguments of c, while $A^A(c)$ is the set consisting of the arguments of the arguments of c:

$$A(at) = \emptyset;\ A(a/b) = A(b\backslash a) = \{b\} \cup A(a);$$
$$A^A(at) = \emptyset;\ A^A(a/b) = A^A(b\backslash a) = A(b) \cup A^A(a).$$

[13]The *order* of a category c, $o(c)$, is defined as follows: $o(c) = 0$ if $c \in \textsc{atom}$; $o(a/b) = o(b\backslash a) = \max(o(a), o(b) + 1)$. For example: $o(n) = 0$, $o(n\backslash s) = 1$, $o(s/(n\backslash s)) = 2$ and $o((s/(n\backslash s))\backslash s) = 3$, whereas $o((s\backslash(n/s))/s) = 1$.

[14]Different from Moortgat (1990), but in keeping with our modification (10) of his semantics (5), the left-hand side categories $(a_1\backslash a_2)/a_3$, $(s/(n\backslash s))\backslash s$ and $n\backslash s$ are assigned a variable v_0 instead of a (possibly complex) term.

For a set of categories C, let C^+ be the set which includes C and contains, for $n \in \mathbb{N}$, any argument of (an argument of an argument of)^n an argument of a category $c \in C$. More formally:

> Let $C \subseteq$ CAT. C^+ is the smallest set such that
> (i) $C \subseteq C^+$, and (ii) for all $c \in C^+$: $A^A(c) \subseteq C^+$.

If C is the set of categories which appear in the left-hand side of sequents, then C^+ is the set of categories from which new axioms and rules are derived.

That C^+ is the required set can be seen from the examples (34) and (35). A derived inference rule associated with a category c deals with occurrences of c in the left-hand side of sequents: a category c consisting of n arguments $c_1, \ldots c_n$ and a (final) value at yields a derived inference rule consisting of a conclusion with left-hand side category c and goal category at, plus n premises with goal categories $c_1, \ldots c_n$ – that is: provided that $c_1, \ldots c_n$ are atomic categories, as in (34). If, as in (35), an argument category c_i of c itself consists of arguments $c'_1, \ldots c'_m$ and final value at', then the relevant premise has at' as its goal category, while the categories $c'_1, \ldots c'_m$ appear in its left-hand side. Now, the presence of arguments of arguments of c in the left-hand side of premises of the derived inference associated with c entails that these categories should be associated with a derived inference rule of their own.

For the example categories $(a_1\backslash a_2)/a_3$ and $(s/(n\backslash s))\backslash s$, the above definitions yield the following:

> $A((a_1\backslash a_2)/a_3) = \{a_1, a_3\}$; $A^A((a_1\backslash a_2)/a_3) = \emptyset$;
> so $\{(a_1\backslash a_2)/a_3\}^+ = \{(a_1\backslash a_2)/a_3\}$.
> $A((s/(n\backslash s))\backslash s) = \{s/(n\backslash s)\}$; $A^A((s/(n\backslash s))\backslash s) = \{n\backslash s\}$;
> $A(n\backslash s) = \{n\}$; $A^A(n\backslash s) = \emptyset$;
> so $\{(s/(n\backslash s))\backslash s\}^+ = \{(s/(n\backslash s))\backslash s, n\backslash s\}$.

Next, let f be the following function from interpreted sequents to interpreted sequents:[15]

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[15] Strictly speaking, f is a partial function, because terms assigned to goal categories a/b or $b\backslash a$ are not necessarily of the form $\lambda v.\gamma$.

Below we will also employ the following straightforward extension of f to interpreted sequents with asterisked goal categories:

$$f(T{:}T' \vdash at^* : \gamma) = T{:}T' \vdash at^* : \gamma;$$

$f(T\!:\!T' \vdash at\!:\!\gamma) = T\!:\!T' \vdash at\!:\!\gamma$
$f(T\!:\!T' \vdash a/b\!:\!\lambda v.\gamma) = f(T\!:\!T', b\!:\!v \vdash a\!:\!\gamma)$; and
$f(T\!:\!T' \vdash b\backslash a\!:\!\lambda v.\gamma) = f(b\!:\!v, T\!:\!T' \vdash a\!:\!\gamma)$.

Syntactically, the function f peels off the argument categories from the goal category of the sequent to which it is applied. Semantically, it strips the term assigned to the goal category of its lambda abstractions. Together with the variables of the abstractions, these argument categories are added to the left-hand side of the sequent. An induction on $\#(c)$[16] shows that if a proof π of $T\!:\!T' \vdash c\!:\!\gamma$ is of the form indicated below, where $I_1, \ldots I_{\#(c)} \in \{\backslash R, /R\}$, then the conclusion of the subproof π' of π is the sequent $f(T\!:\!T' \vdash c\!:\!\gamma)$:

$$\cfrac{\cfrac{\pi'}{\vdots}\,[I_1]}{T\!:\!T' \vdash c\!:\!\gamma}\,[I_{\#(c)}]$$

The function g assigns an axiom to every atomic category at and a derived inference rule to every compound category $b\backslash a$ and a/b:

$g(at) = at\!:\!v_0 \vdash at\!:\!v_0$;

$$g(a/b) = \frac{f(T_{m+1}\!:\!T'_{m+1} \vdash b\!:\!\lambda v_1 ... \lambda v_{\#(b)}.\gamma_{m+1}) \qquad \sigma_m \qquad \ldots \qquad \sigma_1}{\mathbf{X}, a/b\!:\!v_0, T_{m+1}\!:\!T'_{m+1}, \mathbf{Y} \vdash at\!:\!v_0(\lambda v_1 ... \lambda v_{\#(b)}.\gamma_{m+1})(\delta_m)...(\delta_1)}$$

and

$$g(b\backslash a) = \frac{f(T_{m+1}\!:\!T'_{m+1} \vdash b\!:\!\lambda v_1 ... \lambda v_{\#(b)}.\gamma_{m+1}) \qquad \sigma_m \qquad \ldots \qquad \sigma_1}{\mathbf{X}, T_{m+1}\!:\!T'_{m+1}, b\backslash a\!:\!v_0, \mathbf{Y} \vdash at\!:\!v_0(\lambda v_1 ... \lambda v_{\#(b)}.\gamma_{m+1})(\delta_m)...(\delta_1)}$$

iff $g(a)$ consists of the conclusion $\mathbf{X}, a : v_0, \mathbf{Y} \vdash at : v_0(\delta_m)...(\delta_1)$ and the premises $\sigma_m, \ldots, \sigma_1$. (Note that $m = \#(a)$.)

The expressions $T_1 : T'_1, \ldots, T_{\#(c)} : T'_{\#(c)}$ occurring in a rule $g(c)$ will range over non-empty finite sequences of pairs consisting of a category and a

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$f(T\!:\!T' \vdash a/b^* : \lambda v.\gamma) = f(T\!:\!T', b\!:\!v \vdash a^*\!:\!\gamma)$; and
$f(T\!:\!T' \vdash b\backslash a^* : \lambda v.\gamma) = f(b\!:\!v, T\!:\!T' \vdash a^*\!:\!\gamma)$.

Note that since the terms assigned to goal categories in **L*** derivations are in $\bar{\eta}$-nf (see section 4 above), the restriction of f to **L***-derivable interpreted sequents is a total function.

[16]Recall that the function $\#$ counts the number of arguments of $c \in$ CAT: $\#(at) = 0$ and $\#(a/b) = \#(b\backslash a) = \#(a) + 1$.

variable.[17] As before, we will require that in the context of a proof all variables v_0 assigned to an axiom instance or introduced in the conclusion of a derived inference rule be different.

Let us see what these definitions amount to for the example categories discussed above. We start with the category $(a_1\backslash a_2)/a_3$. Since $\#(a_1) = \#(a_3) = 0$:

$$f(T_i\!:\!T_i' \vdash a_1\!:\!\lambda v_1...\lambda v_{\#(a_1)}.\gamma_i) = T_i\!:\!T_i' \vdash a_1\!:\!\gamma_i; \text{ and}$$
$$f(T_i\!:\!T_i' \vdash a_3\!:\!\lambda v_1...\lambda v_{\#(a_3)}.\gamma_i) = T_i\!:\!T_i' \vdash a_3\!:\!\gamma_i.$$

Since $g(a_2) = a_2\!:\!v_0 \vdash a_2\!:\!v_0$, we have:

$$g(a_1\backslash a_2) = \frac{T_1\!:\!T_1' \vdash a_1\!:\!\gamma_1}{T_1, a_1\backslash a_2\!:\!v_0 \vdash a_2\!:v_0(\gamma_1)}\text{, and}$$

$$g((a_1\backslash a_2)/a_3) = \frac{T_2\!:\!T_2' \vdash a_3\!:\!\gamma_2 \qquad\qquad T_1\!:\!T_1' \vdash a_1\!:\!\gamma_1}{T_1\!:\!T_1', (a_1\backslash a_2)/a_3, T_2\!:\!T_2' \vdash a_2\!:\!v_0(\gamma_2)(\gamma_1)}.$$

Note that $g((a_1\backslash a_2)/a_3)$ is the same rule as (34). Now consider the second example, $(s/(n\backslash s))\backslash s$. Since $\#(s/(n\backslash s)) = 1$:

$$f(T_i\!:\!T_i' \vdash s/(n\backslash s)\!:\!\lambda v_1...\lambda v_{\#(s/(n\backslash s))}.\gamma_i) = T_i\!:\!T_i', n\backslash s\!:\!v_1 \vdash s\!:\!\gamma_i.$$

Hence, and because $g(s) = s\!:\!v_0 \vdash s\!:\!v_0$:

$$g((s/(n\backslash s))\backslash s) = \frac{T_1\!:\!T_1', n\backslash s\!:\!v_1 \vdash s\!:\!\gamma_1}{T_1\!:\!T_1', (s/(n\backslash s))\backslash s\!:\!v_0 \vdash s\!:\!v_0(\lambda v_1.\gamma_1)}.$$

Since $\#(n) = 0$:

$$f(T_i\!:\!T_i' \vdash n\!:\!\lambda v_1...\lambda v_{\#(n)}.\gamma_i) = T_i\!:\!T_i' \vdash n\!:\!\gamma_i.$$

[17]Since the expressions $T_1 : T_1', \ldots, T_{\#(c)} : T_{\#(c)}'$ and $\gamma_1, \ldots, \gamma_{\#(c)}$ are distinguished by their subscripts, each $T_i\!:\!T_i'$ and γ_i can be instantiated independently. Strictly speaking, the variables $v_1, \ldots v_{\#(b_i)}$ and $v_1, \ldots v_{\#(b_j)}$ occurring in different premises σ_i and σ_j of a rule $g(c)$, as well as their counterparts in the conclusion of $g(c)$, should be distinguished for the same reason, for example in the following way:

$v_1^i, \ldots v_{\#(b_i)}^i$ and $v_1^j, \ldots v_{\#(b_j)}^j$.

Readability considerations, however, restrained us from doing so.

Moreover, $g(s) = s\!:\!v_0 \vdash s\!:\!v_0$, so that:

$$g(n\backslash s) = \quad \frac{T_1\!:\!T_1' \vdash n\!:\!\gamma_1}{T_1\!:\!T_1', n\backslash s\!:\!v_0 \vdash s\!:\!v_0(\gamma_1)}.$$

The rules $g((s/(n\backslash s))\backslash s)$ and $g(n\backslash s)$ are identical to the ones in (35). Thus, our formal definition is consistent with the examples of Moortgat (1990).

Call an interpreted sequent $c_1\!:\!v_1, \ldots c_n\!:\!v_n \vdash c\!:\!\gamma$ a **PD** ('Partial Deduction') theorem if and only if it is derivable using the axioms and rules in the set $\{g(c) \mid c \in \{c_1, \ldots, c_n\}^+\}$.

Notice that all **PD** theorems have atomic goal categories. We will show that there exists a surjection h from **L*** proofs of sequents $T:T' \vdash at^*:\gamma$ with asterisked atomic goal categories to **PD** proofs of $T\!:\!T' \vdash at\!:\!\gamma$.

On account of Theorem 1 (see section 3 above), this entails that partial deduction preserves all (interpretations of) **L**-derivable sequents with atomic goal categories.

Moreover, on account of Theorem 2 (see section 4 above), the existence of h entails that partial deduction is devoid of spurious ambiguity in that it derives every interpretation only once. The function h is defined as follows. Let $I_1, \ldots I_{\#(b)} \in \{\backslash R, /R\}$:

$$(a)\quad h\left(\frac{at^*\!:\!w \vdash at\!:\!w}{at\!:\!w \vdash at^*\!:\!w}\,[\,*\,]\right) \;=\; at\!:\!w \vdash at\!:\!w;$$

$$(b)\quad h\left(\frac{\dfrac{\dfrac{\dfrac{\pi'}{\vdots}\,[I_1]}{T\!:\!T' \vdash b^*\!:\!\beta}\,[I_{\#(b)}] \qquad \pi''}{U\!:\!U', a/b^*\!:\!w, T\!:\!T', V\!:\!V' \vdash at\!:\!\gamma[v\!:=\!w(\beta)]}\,[/L]}{U\!:\!U', a/b\!:\!w, T\!:\!T', V\!:\!V' \vdash at^*\!:\!\gamma[v\!:=\!w(\beta)]}\,[\,*\,]\right) =$$

$$\frac{h(\pi') \qquad \pi_n \qquad \ldots \qquad \pi_1}{U\!:\!U', a/b\!:\!w, T\!:\!T', V\!:\!V' \vdash at\!:\!\gamma[v\!:=\!w(\beta)]}\,[g(a/b)] \quad \text{iff}$$

$$h\left(\frac{\pi''}{U\!:\!U', a\!:\!v, V\!:\!V' \vdash at^*\!:\!\gamma}\,[\,*\,]\right) = \frac{\pi_n \qquad \ldots \qquad \pi_1}{U\!:\!U', a\!:\!v, V\!:\!V' \vdash at\!:\!\gamma}\,[g(a)];$$

$$(c)\quad h\left(\dfrac{\dfrac{\dfrac{\dfrac{\pi'}{\vdots}[I_1]}{T:T' \vdash b^*:\beta}[I_{\#(b)}] \qquad \pi''}{U:U',T:T',b\backslash a^*:w,V:V' \vdash at:\gamma[v:=w(\beta)]}[\backslash L]}{U:U',T:T',b\backslash a:w,V:V' \vdash at^*:\gamma[v:=w(\beta)]}[\,*\,]\right) =$$

$$\dfrac{h(\pi') \qquad \pi_n \qquad \ldots \qquad \pi_1}{U:U',T:T',b\backslash a:w,V:V' \vdash at:\gamma[v:=w(\beta)]}[g(b\backslash a)] \quad \text{iff}$$

$$h\left(\dfrac{\pi''}{U:U',a:v,V:V' \vdash at^*:\gamma}[\,*\,]\right) = \dfrac{\pi_n \qquad \ldots \qquad \pi_1}{U:U',a:v,V:V' \vdash at:\gamma}[g(a)].$$

On the level of proofs, the function h closely follows the pattern of the rule-building function g. Due to clause (a) of the definition of h, the most simple **L*** proofs π of sequents with asterisked atomic goal categories are mapped to **PD** proofs $h(\pi)$ which instantiate the value of g for atomic categories. Moreover, clause (b) maps more complex **L*** proofs π of sequents with asterisked atomic goal categories to **PD** proofs $h(\pi)$ in which the last rule applied is $g(a/b)$, where a/b is a category occurring in the left-hand side of the conclusion sequent of π. The $\#(a/b)$ subproofs of $h(\pi)$ which yield the premises of this application of the rule $g(a/b)$ are determined by two subproofs – π' and π'' – of π: π' is an **L*** proof of a sequent with asterisked atomic goal category, while π'' extended with $*$ ('$\pi'' + *$') is an **L*** proof of a sequent with asterisked atomic goal category such that $h(\pi'' + *)$ is a **PD** proof in which the last rule applied is $g(a)$. Now, given the way in which $g(a/b)$ is defined (viz., in terms of $g(a)$), the conclusion of $h(\pi')$ plus the premises of $g(a)$ in $h(\pi'' + *)$ can serve as the premises of an application of $g(a/b)$ which infers the non-asterisked counterpart of the conclusion of π. (An isomorphic story can be told about clause (c).)

We will now prove the following facts: (1) for every **L*** proof π of a sequent $Z:Z' \vdash at^*:\delta$ with asterisked atomic goal category, $h(\pi)$ is a **PD** proof of $Z:Z' \vdash at:\delta$; and (2) for every **PD** proof π, there is an **L*** proof π^* such that $h(\pi^*) = \pi$.

(1) That for every **L*** proof π of a sequent $Z:Z' \vdash at^*:\delta$ with asterisked atomic goal, $h(\pi)$ is a **PD** proof of $Z:Z' \vdash at:\delta$ is shown by induction on d, the degree of the conclusion $Z:Z' \vdash at^*:\delta$ of π. Note that in view of its goal category at^*, the sequent $Z:Z' \vdash at^*:\delta$, i.e., $U:U',c:w,W:W' \vdash at^*:\delta$,

must have been derived by the * rule from $U:U', c^*:w, W:W' \vdash at:\delta$.

• $d = 0$. Then the sequent $U:U', c^*:w, W:W' \vdash at:\delta$ must be an L* axiom $at^*:w \vdash at:w$. This case is covered by clause (a) of the definition of h which states that $h(\pi) = at:w \vdash at:w$, i.e., an application of $g(at)$.

• $d > 0$. There are two possibilities: $c = a/b$ or $c = b\backslash a$. We only consider the former case. (The $b\backslash a$ case is analogous.) If $c = a/b$, then $W:W' = T:T', V:V'$ and $\delta \equiv \gamma[v:=w(\beta)]$, because $U:U', c^*:w, W:W' \vdash at:\delta$, i.e., $U:U', a/b^*:w, T:T', V:V' \vdash at:\gamma[v:=w(\beta)]$, must have been inferred by $/L$ from the premises (a) $T:T' \vdash b^*:\beta$ and (b) $U:U', a^*:v, V:V' \vdash at:\gamma$.

(a) As for the subproof of $T:T' \vdash b^*:\beta$: such a sequent must have been inferred using * (if $b = at$), $\backslash R$ (if $b = b''\backslash b'$) or $/R$ (if $b = b'/b''$). Besides, premises of $\backslash R$ and $/R$ inferences have asterisked goal categories as well. So this subproof must have the following form, where $I_1, \ldots I_{\#(b)} \in \{\backslash R, /R\}$:

$$\pi' \left[\dfrac{\dfrac{\dfrac{\pi_0}{\sigma_0}\,[\,*\,]}{\vdots}\,[I_1]}{T:T' \vdash b^*:\beta}\,[I_{\#(b)}] \right.$$

Note that since every I_i $(1 \leq i \leq \#(b))$ adds an abstraction λu_i to the term assigned to the goal category of its premise, the term β must be of the form $\lambda u_1 ... \lambda u_{\#(b)}.\beta'$. So, using the observation concerning f made above, we know that σ_0 is the sequent $f(T:T' \vdash b^*:\lambda u_1 \ldots \lambda u_{\#(b)}.\beta')$. This sequent has an asterisked atomic goal and its degree is smaller than the degree of the conclusion of π. Hence we can apply the induction hypothesis to π' and infer that $h(\pi')$ is a **PD** proof of $f(T:T' \vdash b:\lambda u_1 \ldots \lambda u_{\#(b)}.\beta')$ – i.e., $f(T:T' \vdash b^*:\lambda u_1 \ldots \lambda u_{\#(b)}.\beta')$ minus asterisk.

(b) As for the subproof π'' of $U:U', a^*:v, V:V' \vdash at:\gamma$: note that we can apply the Lemma of section 4 above: $\gamma \equiv v(\gamma_{\#(a)})...(\gamma_1)$, where v does not occur in $\gamma_1, \ldots, \gamma_{\#(a)}$. Moreover, the derivation π''' consisting of π'' plus an additional * inference is an **L*** proof of a sequent with an asterisked atomic goal category:

$$\pi''' = \dfrac{\pi''}{U:U', a:v, V:V' \vdash at^*:\gamma}\,[\,*\,]$$

The degree of $U:U', a:v, V:V' \vdash at^*:\gamma$ is smaller than the degree of the conclusion of π. Hence $h(\pi''')$ is a **PD** proof of $U:U', a:v, V:V' \vdash at:\gamma$ by the induction hypothesis. Since $\gamma \equiv v(\gamma_{\#(a)})...(\gamma_1)$, we know that $g(a)$ must be the last inference of $h(\pi''')$, a proof which, therefore, has the following form:

$$h(\pi''') = \frac{\pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U:U', a:v, V:V' \vdash at:v(\gamma_{\#(a)})...(\gamma_1)}\ [g(a)]$$

Now, according to clause (b) of the definition of h,

$$h(\pi) = \frac{h(\pi') \qquad \pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U:U', a/b:w, T:T', V:V' \vdash at:\gamma[v:=w(\beta)]}\ [g(a/b)],$$

which is indeed a **PD** proof, for recall that

$$g(a/b) = \frac{f(T_{m+1}:T'_{m+1} \vdash b:\lambda v_1...\lambda v_{\#(b)}.\gamma_{m+1}) \qquad \sigma_{\#(a)} \qquad \cdots \qquad \sigma_1}{\mathbf{X}, a/b:v_0, T_{m+1}:T'_{m+1}, \mathbf{Y} \vdash at:v_0(\lambda v_1...\lambda v_{\#(b)}.\gamma_{m+1})(\delta_{\#(a)})...(\delta_1)},$$

where $\sigma_{\#(a)}, \ldots, \sigma_1$ and $\mathbf{X}, a:v_0, \mathbf{Y} \vdash at:v_0(\delta_{\#(a)})...(\delta_1)$ are the premises and conclusion of $g(a)$, and note that

- apparently, the conclusions of the subproofs $\pi_{\#(a)}, \ldots, \pi_1$ match the respective premises $\sigma_{\#(a)}, \ldots, \sigma_1$ of the rule $g(a)$ in such a way that $at:v(\gamma_{\#(a)})...(\gamma_1)$, $U:U'$, $a:v$ and $V:V'$ match $at:v_0(\delta_{\#(a)})...(\delta_1)$, $\mathbf{X}$, $a:v_0$ and $\mathbf{Y}$, respectively; and

- obviously, the conclusion $f(T:T' \vdash b:\lambda u_1...\lambda u_{\#(b)}.\beta')$ of $h(\pi')$ matches the first premise $f(T_{m+1}:T'_{m+1} \vdash b:\lambda v_1...\lambda v_{\#(b)}.\gamma_{m+1})$ of the rule $g(a/b)$ in such a way that $T:T'$ and $\lambda u_1...\lambda u_{\#(b)}.\beta'$ match $T_{m+1}:T'_{m+1}$ and $\lambda v_1...\lambda v_{\#(b)}.\gamma_{m+1}$, respectively.

But this means that the conclusion of $g(a/b)$ is matched by the sequent $U:U', a/b:w, T:T', V:V' \vdash at:w(\lambda u_1...\lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)$. Note, finally, that $w(\lambda u_1 \ldots \lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1) \equiv \gamma[v:=w(\beta)]$, because $\gamma \equiv v(\gamma_{\#(a)})...(\gamma_1)$, the variable v does not occur in $\gamma_1, \ldots, \gamma_{\#(a)}$, and $\beta \equiv \lambda u_1 \ldots \lambda u_{\#(b)}.\beta'$. $\square$

(2) We show that for every **PD** proof π there is an $\mathbf{L^*}$ proof π^* such that $h(\pi^*) = \pi$ by induction on d, the degree of the conclusion $Z:Z' \vdash at:\delta$ of the proof π.

• $d = 0$. Then Z only contains atomic categories, so for $c \in$ ATOM: π must match $g(c) = at:v_0 \vdash at:v_0$. Consequently, $\pi = at:v \vdash at:v$, and:

$$at\!:\!v \vdash at\!:\!v \;=\; h\left(\frac{at^*\!:\!v \vdash at\!:\!v}{at\!:\!v \vdash at^*\!:\!v}\,[\,*\,]\right)$$

• $d > 0$. Then the conclusion of π must have been obtained by a derived inference rule $g(c)$ such that the degree of c is greater than 0. So:

$$\pi = \frac{\pi_{\#(a)+1} \quad \pi_{\#(a)} \quad \cdots \quad \pi_1}{U\!:\!U', b\backslash a\!:\!w, W\!:\!W' \vdash at\!:\!\delta}\,[g(b\backslash a)], \text{ or}$$

$$\pi = \frac{\pi_{\#(a)+1} \quad \pi_{\#(a)} \quad \cdots \quad \pi_1}{U\!:\!U', a/b\!:\!w, W\!:\!W' \vdash at\!:\!\delta}\,[g(a/b)],$$

We will only consider the case of $g(a/b)$. (The case of $g(b\backslash a)$ is analogous.) Then $W\!:\!W' = T\!:\!T', V\!:\!V'$ and $\gamma \equiv w(\lambda u_1 ... \lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)$, where (i) the conclusion of $\pi_{\#(a)+1}$ is $f(T : T' \vdash b : \lambda u_1 ... \lambda u_{\#(b)}.\beta')$; and (ii) the conclusions of the subproofs $\pi_{\#(a)}, \ldots, \pi_1$ match the respective premises $\sigma_{\#(a)}, \ldots, \sigma_1$ of the rule $g(a)$ in such a way that application of that rule to these sequents yields $U\!:\!U', a\!:\!v, V\!:\!V' \vdash at\!:\!v(\gamma_{\#(a)})...(\gamma_1)$.

Since the degree of the sequents $f(T : T' \vdash b : \lambda u_1 ... \lambda u_{\#(b)}.\beta')$ and $U : U', a : v, V : V' \vdash at : v(\gamma_{\#(a)})...(\gamma_1)$ is smaller than the degree of the conclusion of π, we can apply the induction hypothesis to $\pi_{\#(a)+1}$ and $\rho =$

$$\frac{\pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U\!:\!U', a\!:\!v, V\!:\!V' \vdash at\!:\!v(\gamma_{\#(a)})...(\gamma_1)}\,[g(a)],$$

and conclude that there are **L*** proofs ${\pi_{\#(a)+1}}^*$ and ρ^* such that $h({\pi_{\#(a)+1}}^*)$ $= \pi_{\#(a)+1}$ and $h(\rho^*) = \rho$.

Because $\pi_{\#(a)+1}$ and ρ are proofs of $f(T : T \vdash b : \lambda u_1 ... \lambda u_{\#(b)}.\beta')$ and $U\!:\!U', a\!:\!v, V\!:\!V' \vdash at\!:\!v(\gamma_{\#(a)})...(\gamma_1)$, respectively, we know that ${\pi_{\#(a)+1}}^*$ and ρ^* are proofs of the asterisked sequents $f(T : T \vdash b^* : \lambda u_1 ... \lambda u_{\#(b)}.\beta')$ and $U\!:\!U', a\!:\!v, V\!:\!V' \vdash at^*\!:\!v(\gamma_{\#(a)})...(\gamma_1)$, respectively.

As for ${\pi_{\#(a)+1}}^*$: it can be shown by induction on $\#(c)$ that any **L*** proof π^* of $f(T : T \vdash c^* : \lambda u_1 ... \lambda u_{\#(c)}.\beta')$ can be extended to an **L*** proof π^{*+} of $T\!:\!T \vdash c^*\!:\!\lambda u_1 ... \lambda u_{\#(c)}.\beta'$ by adding $\#(c)$ $\backslash R$ and $/R$ inferences to π^*. Hence there exists such an **L*** proof ${\pi_{\#(a)+1}}^{*+}$ of $T\!:\!T \vdash b^*\!:\!\lambda u_1 ... \lambda u_{\#(b)}.\beta'$.

As for ρ^*: since $U\!:\!U', a\!:\!v, V\!:\!V' \vdash at^*\!:\!v(\gamma_{\#(a)})...(\gamma_1)$, the conclusion of ρ^*, has an asterisked atomic goal which is assigned a term with leading variable v, we know (by the Lemma of section 4) that

$$\rho^* = \quad \frac{\rho^{*\prime}}{U\!:\!U', a\!:\!v, V\!:\!V' \vdash at^*\!:\!v(\gamma_{\#(a)})...(\gamma_1)}\,[\,*\,],$$

where $U\!:\!U', a^*\!:\!v, V\!:\!V' \vdash at\!:\!v(\gamma_{\#(a)})...(\gamma_1)$ is the conclusion of $\rho^{*\prime}$. Hence π^*, the proof below, is an **L*** derivation, where $I_1, \ldots, I_{\#(b)} \in \{\backslash R, /R\}$:

$$\dfrac{\left.\begin{array}{c}\dfrac{\dfrac{{\pi_{\#(a)+1}}^*}{\vdots}\,[I_1]}{T\!:\!T' \vdash b^*\!:\!\lambda u_1...\lambda u_{\#(b)}.\beta'}\,[I_{\#(b)}]\end{array}\right]{\pi_{\#(a)+1}}^{*+} \qquad \rho^{*\prime}}{\dfrac{U\!:\!U', a/b^*\!:\!w, T\!:\!T', V\!:\!V' \vdash at\!:\!w(\lambda u_1...\lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)}{U\!:\!U', a/b\!:\!w, T\!:\!T', V\!:\!V' \vdash at^*\!:\!w(\lambda u_1...\lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)}\,[\,*\,]}\,[/L]$$

Finally, because $h(\rho^*) = \rho =$

$$\frac{\pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U\!:\!U', a\!:\!v, V\!:\!V' \vdash at\!:\!v(\gamma_{\#(a)})...(\gamma_1)}\,[g(a)],$$

clause (b) of the definition of h tells us that $h(\pi^*) =$

$$\frac{h({\pi_{\#(a)+1}}^*) \qquad \pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U\!:\!U', a/b\!:\!w, T\!:\!T', V\!:\!V' \vdash at\!:\!w(\lambda u_1...\lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)}\,[g(a/b)]$$

And because $h({\pi_{\#(a)+1}}^*) = \pi_{\#(a)+1}$, we have that $h(\pi^*) =$

$$\frac{\pi_{\#(a)+1} \qquad \pi_{\#(a)} \qquad \cdots \qquad \pi_1}{U\!:\!U', a/b\!:\!w, T\!:\!T', V\!:\!V' \vdash at\!:\!w(\lambda u_1...\lambda u_{\#(b)}.\beta')(\gamma_{\#(a)})...(\gamma_1)}\,[g(a/b)],$$

which is nothing else but π. $\Box$

By way of illustration, we return to the sample sequents $c/c, c/c, c \vdash c$ and $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$ treated in the previous sections. The categories c/c and c generate one rule and one axiom:

$$\frac{T_1\!:\!T_1' \vdash c\!:\!\gamma_1}{c/c\!:\!v_0, T_1\!:\!T_1' \vdash c\!:\!v_0(\gamma_1)}\,[g(c/c)]$$

$$c\!:\!v_0 \vdash c\!:\!v_0 \quad [g(c)]$$

These yield one **PD** derivation of $c/c, c/c, c \vdash c$, viz., (36). Observe that (36) = h(31):

(36)
$$\cfrac{\cfrac{c{:}x \vdash c{:}x}{c/c{:}y, c{:}x \vdash c{:}y(x)}\,[g(c/c)]}{c/c{:}z, c/c{:}y, c{:}x \vdash c{:}z(y(x))}\,[g(c/c)]$$

The categories $s/(n\backslash s), (s/n)\backslash s$ and $(n\backslash s)/n$ generate the following rules and axiom:

$$\frac{n : v_1, T_1 : T_1' \vdash s : \gamma_1}{s/(n\backslash s) : v_0, T_1 : T_1' \vdash c : v_0(\lambda v_1.\gamma_1)}\,[g(s/(n\backslash s))]$$

$$\frac{T_1 : T_1', n : v_1 \vdash s : \gamma_1}{T_1 : T_1', (s/n)\backslash s : v_0 \vdash c : v_0(\lambda v_1.\gamma_1)}\,[g((s/n)\backslash s)]$$

$$n : v_0 \vdash n : v_0 \quad [g(n)]$$

$$\frac{T_2 : T_2' \vdash n : \gamma_2 \qquad\qquad T_1 : T_1' \vdash n : \gamma_1}{T_1 : T_1', (n\backslash s)/n : v_0, T_2 : T_2' \vdash c : v_0(\gamma_2)(\gamma_1)}\,[g((n\backslash s)/n)]$$

These yield two proofs of $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$, viz., (37) and (38), where (37) = h(32) and (38) = h(33):

(37)
$$\cfrac{\cfrac{\cfrac{n : w' \vdash n : w' \qquad\qquad n : u' \vdash n : u'}{n : u', (n\backslash s)/n : v, n : w' \vdash s : v(w')(u')}\,[g((n\backslash s)/n)]}{n : u', (n\backslash s)/n : v, (s/n)\backslash s : w \vdash s : w(\lambda w'.v(w')(u'))}\,[g((s/n)\backslash s)]}{s/(n\backslash s) : u, (n\backslash s)/n : v, (s/n)\backslash s : w \vdash s : u(\lambda u'.w(\lambda w'.v(w')(u')))}\,[g(s/(n\backslash s))]$$

(38)
$$\cfrac{\cfrac{\cfrac{n : w' \vdash n : w' \qquad\qquad n : u' \vdash n : u'}{n : u', (n\backslash s)/n : v, n : w' \vdash s : v(w')(u')}\,[g((n\backslash s)/n)]}{s/(n\backslash s) : u, (n\backslash s)/n : v, n : w' \vdash s : u(\lambda u'.v(w')(u'))}\,[g(s/(n\backslash s))]}{s/(n\backslash s) : u, (n\backslash s)/n : v, (s/n)\backslash s : w \vdash s : w(\lambda w'.u(\lambda u'.v(w')(u')))}\,[g((s/n)\backslash s)]$$

5.2 Proof Nets

Roorda (1991) introduces so-called *proof nets* – which were invented for linear logic by Girard (1987) – for the Lambek calculus. Proof nets are proposed as a solution to the problem of spurious ambiguity, in that they replace 'inessential' order of rule applications in sequent proofs by 'parallellism'.

In the present section we will assess the merits of this proposal. We start with the definition of proof nets given in Roorda (1991), which is listed in (a) through (g) below. Next, an alternative, equivalent definition is presented in (a') through (h'). While Roorda's definition recursively introduces connectives, the alternative one proceeds by adding links. This different set-up has two advantages. First, it allows a direct, inductive definition of the semantic interpretation of proof nets – see the definition given in (a'') through (h''). And second, it brings out the correspondence between applying rules in **L*** and adding links to proof nets: there will turn out to be a one-many relationship between **L*** proofs and proof nets.

Basically, a proof net is a structure of linked signed categories. For the product-free Lambek calculus, the following links are distinguished:

(39) $$\frac{}{\overset{-}{at}\quad\overset{+}{at}}\,ax \qquad \frac{}{\overset{+}{at}\quad\overset{-}{at}}\,ax \qquad \frac{\overset{+}{a}\quad\overset{-}{b}}{\overset{+}{b\backslash a}}\,\backslash 1 \qquad \frac{\overset{-}{b}\quad\overset{+}{a}}{\overset{+}{a/b}}\,/1 \qquad \frac{\overset{+}{b}\quad\overset{-}{a}}{\overset{-}{b\backslash a}}\,\backslash 2 \qquad \frac{\overset{-}{a}\quad\overset{+}{b}}{\overset{-}{a/b}}\,/2$$

Below, the expressions X, Y and Z will denote sequences of signed categories, and $tc(C)$ will be used for the sequence of so-called *terminal* (signed) *categories* of a proof net C.

PN, the set of proof nets, is the smallest set satisfying the clauses (a) through (g) below, which also specify the sequence $tc(C)$ for each $C \in$ PN:[18]

[18] This definition is based on the definition in Roorda (1991, p. 30), but there are two differences: (i) the condition 'X or Y is non-empty' has been added to clause (b) and (c), to the effect that every proof net has at least one terminal category signed '$-$'. This mirrors the 'non-empty antecedent property' of **L**, i.e., the fact that the left-hand side of **L**-derivable sequents is non-empty, a property which was effectuated in (4) above by requiring that T denote a non-empty sequence of categories (Roorda (1991, p. 37) captures this aspect by a 'proof net condition' on the assignment of lambda terms); and (ii) anticipating the semantic interpretation of proof nets, the various cases collapsed by Roorda have been disentangled. (Note, for that matter, that Roorda collapses too much: his clauses **2**A and **2**B – which comprise the clauses (d) and (g), and (e) and (f), respectively – have the same condition: 'if β and γ are proof nets with terminal formulas tA resp. uBv' (t, u, v and A, B range over sequences of signed categories and signed categories,

(a) $\overline{\overset{-}{at}\ \ \overset{+}{at}} \in$ PN and $\overline{\overset{+}{at}\ \ \overset{-}{at}} \in$ PN;
$tc(\overline{\overset{-}{at}\ \ \overset{+}{at}}) = \overset{-}{at}, \overset{+}{at}$ and $tc(\overline{\overset{+}{at}\ \ \overset{-}{at}}) = \overset{+}{at}, \overset{-}{at}$;

(b) if $A \in$ PN and $tc(A) = X, \overset{+}{a}, \overset{-}{b}, Y$ and X or Y is non-empty,
then $C \in$ PN, where C results from applying $\backslash 1$ to $\overset{+}{a}$ and $\overset{-}{b}$;
$tc(C) = X, \overset{+}{b\backslash a}, Y$;

(c) if $A \in$ PN and $tc(A) = X, \overset{-}{b}, \overset{+}{a}, Y$ and X or Y is non-empty,
then $C \in$ PN, where C results from applying $/1$ to $\overset{-}{b}$ and $\overset{+}{a}$;
$tc(C) = X, \overset{+}{a/b}, Y$;

(d) if $A \in$ PN and $tc(A) = X, \overset{-}{a}, Y$ and $B \in$ PN and $tc(B) = Z, \overset{+}{b}$,
then $C \in$ PN, where C results from applying $\backslash 2$ to $\overset{+}{b}$ and $\overset{-}{a}$;
$tc(C) = X, Z, \overset{-}{b\backslash a}, Y$;

(e) if $A \in$ PN and $tc(A) = X, \overset{+}{b}, Y$ and $B \in$ PN and $tc(B) = \overset{-}{a}, Z$,
then $C \in$ PN, where C results from applying $\backslash 2$ to $\overset{+}{b}$ and $\overset{-}{a}$;
$tc(C) = X, \overset{-}{b\backslash a}, Z, Y$;

(f) if $A \in$ PN and $tc(A) = X, \overset{-}{a}, Y$ and $B \in$ PN and $tc(B) = \overset{+}{b}, Z$,
then $C \in$ PN, where C results from applying $/2$ to $\overset{-}{a}$ and $\overset{+}{b}$;
$tc(C) = X, \overset{-}{a/b}, Z, Y$;

(g) if $A \in$ PN and $tc(A) = X, \overset{+}{b}, Y$ and $B \in$ PN and $tc(B) = Z, \overset{-}{a}$,
then $C \in$ PN, where C results from applying $/2$ to $\overset{-}{a}$ and $\overset{+}{b}$;
$tc(C) = X, Z, \overset{-}{a/b}, Y$.

The proof nets defined in clause (a) correspond to the axioms of **L** (where observation **A** of section 3 above – entailing the possibility of having only axiom sequents with atomic categories – has been incorporated), while the clauses (b), (c), (d) + (e), and (f) + (g) are the proof net counterparts of the **L** rules $\backslash R$, $/R$, $\backslash L$ and $/L$, respectively.

On account of the clauses (d), (e), (f) and (g), two separate proof nets

respectively). Since one has to be 'strict on order and adjacency of premises in proof nets' (1991, p. 30), however, this will not work for clause **2**B, which needs a separate condition, viz., 'if β and γ are proof nets with terminal formulas At resp. uBv'.)

can be combined into one: application of $\backslash 2$ and $/2$ has the effect of *inserting* proof net B into proof net A.

Note that for every proof net C, the sequence $tc(C)$ consists of exactly one category signed '+' and at least one category signed '−'. We will call the unique positively signed category in $tc(C)$ the *goal* of C, which will be denoted by $go(C)$. A proof net C *corresponds* to a sequent $T \vdash c$ iff

$$T = c_1, \ldots, c_i, c_{i+1}, \ldots, c_n \text{ and } tc(C) = \overset{-}{c}_{i+1}, \ldots, \overset{-}{c}_n, \overset{+}{c}, \overset{-}{c}_1, \ldots, \overset{-}{c}_i.$$

By way of illustration of the definition, consider proof net (40), which corresponds to the **L**-derivable sequent $a/b, b/c \vdash a/c$; $tc(40)$ is the sequence listed in (40′).

(40) $$\frac{\overset{-}{b}\;\overset{+}{c}}{\overset{-}{b/c}} \qquad \frac{\overset{-}{c}\;\overset{+}{a}}{\overset{+}{a/c}} \qquad \frac{\overset{-}{a}\;\overset{+}{b}}{\overset{-}{a/b}}$$

(40′) $\overset{-}{b/c}, \overset{+}{a/c}, \overset{-}{a/b}$

Proof net (40) can be built up in two ways from the basic proof nets in (41), which have the sequences of terminal categories listed in (42):

(41) $(i)\ \overline{\overset{-}{b} \qquad\qquad \overset{+}{b}}\,ax \qquad (ii)\ \overline{\overset{+}{c} \quad \overset{-}{c}}\,ax \qquad (iii)\ \overline{\overset{+}{a} \quad \overset{-}{a}}\,ax$

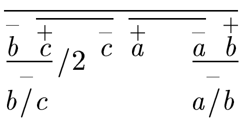
(42) $tc(i) = \overset{-}{b}, \overset{+}{b}$; $tc(ii) = \overset{+}{c}, \overset{-}{c}$; $tc(iii) = \overset{+}{a}, \overset{-}{a}$.

A first possibility is that we add a $/2$ link to $\overset{-}{a}$ in (iii) and $\overset{+}{b}$ in (i), due to clause (g) (where (i) and (iii) instantiate A and B, respectively). Note that (iii) is inserted into (i). The resulting proof net (43) has the terminal categories indicated in (43′):

(43) $$\overset{-}{b} \qquad\quad \overset{+}{a} \quad \frac{\overset{-}{a}\;\overset{+}{b}}{\overset{-}{a/b}}/2$$

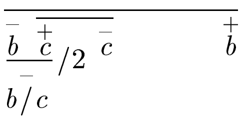
(43′) $\overset{-}{b}, \overset{+}{a}, \overset{-}{a/b}$

Next, we can add a $/2$ link to $\overset{-}{b}$ in (43) and $\overset{+}{c}$ in (ii), due to clause (f) (where (43) and (ii) instantiate A and B, respectively): (ii) is inserted into (43). The resulting proof net (44) has the terminal categories indicated in (44′):

(44) 

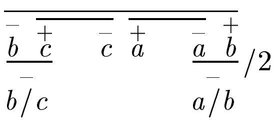
(44′) $\overset{-}{b/c}, \overset{-}{c}, \overset{+}{a}, \overset{-}{a/b}$

Secondly, we may add a /2 link to $\overset{-}{b}$ in (i) and $\overset{+}{c}$ in (ii), due to clause (f) (where (i) and (ii) instantiate A and B, respectively): (ii) is inserted into (i). The resulting proof net (45) has the terminal categories indicated in (45′):

(45) 

(45′) $\overset{-}{b/c}, \overset{-}{c}, \overset{+}{b}$

Next, we can add a /2 link to $\overset{-}{a}$ in (iii) and $\overset{+}{b}$ in (45), due to clause (g) (where (45) and (iii) instantiate A and B, respectively): (iii) is inserted into (45). The resulting proof net (46) is identical to (44):

(46) 

(46′) $\overset{-}{b/c}, \overset{-}{c}, \overset{+}{a}, \overset{-}{a/b}$

Finally, the addition of a /1 link (by clause (c)) to the categories $\overset{-}{c}$ and $\overset{+}{a}$ in proof net (44) (= (46)) yields proof net (40).

The above definition of the set PN is equivalent to the following one (in the sense that PN = PN_2), in which *two* sets, PN_1 and PN_2, are defined as the smallest sets such that:

(a') $\overset{-}{at}\ \overset{+}{at} \in \text{PN}_1$ and $\overset{+}{at}\ \overset{-}{at} \in \text{PN}_1$;

$tc(\overset{-}{at}\ \overset{+}{at}) = \overset{-}{at},\overset{+}{at}$ and $tc(\overset{+}{at}\ \overset{-}{at}) = \overset{+}{at},\overset{-}{at}$;

(b') if $A \in \text{PN}_2$ and $tc(A) = X, \overset{+}{a}, \overset{-}{b}, Y$ and X or Y is non-empty,

then $C \in \text{PN}_2$, where C results from applying \1 to $\overset{+}{a}$ and $\overset{-}{b}$;

$tc(C) = X, \overset{+}{b\backslash a}, Y$;

(c') if $A \in \text{PN}_2$ and $tc(A) = X, \overset{-}{b}, \overset{+}{a}, Y$ and X or Y is non-empty, then $C \in \mathrm{PN}_2$, where C results from applying $/1$ to $\overset{-}{b}$ and $\overset{+}{a}$;

$tc(C) = X, \overset{+}{a/b}, Y$;

(d') if $A \in \mathrm{PN}_1$ and $tc(A) = X, \overset{-}{a}, Y$ and $B \in \mathrm{PN}_2$ and $tc(B) = Z, \overset{+}{b}$,

then $C \in \mathrm{PN}_1$, where C results from applying $\backslash 2$ to $\overset{+}{b}$ and $\overset{-}{a}$;

$tc(C) = X, Z, \overset{-}{b\backslash a}, Y$;

(e') if $A \in \mathrm{PN}_2$ and $tc(A) = X, \overset{+}{b}, Y$ and $B \in \mathrm{PN}_1$ and $tc(B) = \overset{-}{a}, Z$,

then $C \in \mathrm{PN}_1$, where C results from applying $\backslash 2$ to $\overset{+}{b}$ and $\overset{-}{a}$;

$tc(C) = X, \overset{-}{b\backslash a}, Z, Y$;

(f') if $A \in \mathrm{PN}_1$ and $tc(A) = X, \overset{-}{a}, Y$ and $B \in \mathrm{PN}_2$ and $tc(B) = \overset{+}{b}, Z$,

then $C \in \mathrm{PN}_1$, where C results from applying $/2$ to $\overset{-}{a}$ and $\overset{+}{b}$;

$tc(C) = X, \overset{-}{a/b}, Z, Y$;

(g') if $A \in \mathrm{PN}_2$ and $tc(A) = X, \overset{+}{b}, Y$ and $B \in \mathrm{PN}_1$ and $tc(B) = Z, \overset{-}{a}$,

then $C \in \mathrm{PN}_1$, where C results from applying $/2$ to $\overset{-}{a}$ and $\overset{+}{b}$;

$tc(C) = X, Z, \overset{-}{a/b}, Y$;

(h') if $A \in \mathrm{PN}_1$ and $tc(A) = X, \overset{+}{at}, Y$, then $C \in \mathrm{PN}_2$, where C results from applying ax to $\overset{+}{at}$ and the non-connected occurrence of $\overset{-}{at}$ in A;

$tc(C) = X, \overset{+}{at}, Y$.

Note (i) that for all $A \in \mathrm{PN}_1$: $go(A)$ is an atomic category; (ii) that every $A \in \mathrm{PN}_1$ contains exactly two occurrences of an atomic category at which are not connected by ax, viz., $go(A)$ and one negatively signed occurrence of at; and (iii) that if $C \in \mathrm{PN}_2$, then all occurrences of atomic categories in C are connected by ax. Hence clause (h') yields a $C \in \mathrm{PN}_2$ for every $A \in \mathrm{PN}_1$.

Let $C \in \mathrm{PN}_2$ be such that $go(C) = at$. The ax link which connects $go(C)$ with some negatively signed occurrence of at will be called the *goal link* of C, and C^- will denote the structure from which C is obtained by adding a goal link. Obviously, if $C \in \mathrm{PN}_2$ is obtained from $A \in \mathrm{PN}_1$ by clause (h'), then $A = C^-$.

Now, the set PN_1 is equal to $\{\, C^- \mid C \in \mathrm{PN} \,\&\, go(C) \in \mathrm{ATOM} \,\}$, while PN_2 is the same set as PN. This is shown by induction on the degree of $tc(C)$:

• Observe that (b) through (g) and (b') through (g') yield structures C with $d(tc(C)) > 0$. Consequently, if $d(tc(C)) = 0$, then: (i) $C \in \textsc{pn}$ must be due to (a), therefore $C = \overline{\overset{-}{at}\;\overset{+}{at}}$ or $C = \overline{\overset{+}{at}\;\overset{-}{at}}$; and (ii) $C \in \textsc{pn}_2$ must be due to (h'), i.e., C is obtained from $C^- \in \textsc{pn}_1$, where $d(tc(C^-)) = d(tc(C)) = 0$. So, $C^- \in \textsc{pn}_1$ must be due to (a'): $C^- = \overset{-}{at}\;\overset{+}{at}$ (and $C = \overline{\overset{-}{at}\;\overset{+}{at}}$) or $C^- = \overset{+}{at}\;\overset{-}{at}$ (and $C = \overline{\overset{+}{at}\;\overset{-}{at}}$).

• If $d(tc(C)) > 0$, then either (i) $go(C) = b\backslash a$ (or, analogously, a/b); or (ii) $go(C) \in \textsc{atom}$.

∘ (i) If $go(C) = b\backslash a$ and $C \in \textsc{pn}$, then $tc(C) = X, b\overset{+}{\backslash}a, Y$, and – by an 'easily verifiable' fact[19] – for some $A \in \textsc{pn}$ with $tc(A) = X, \overset{+}{a}, \overset{-}{b}, Y$: C can be obtained by applying (b) to A. Conversely, $C \in \textsc{pn}_2$ *must* be obtained by applying (b') to $A \in \textsc{pn}_2$. Since $A \in \textsc{pn}$ iff $A \in \textsc{pn}_2$ by induction hypothesis, we have that $C \in \textsc{pn}$ iff $C \in \textsc{pn}_2$.

∘ (ii) If $go(C) \in \textsc{atom}$, then $C \in \textsc{pn}$ is due to (d), (e), (f) or (g). Suppose (d) is involved. Then $tc(C) = X, Z, b\overset{-}{\backslash}a, Y$, and C is the result of applying $\backslash 2$ to $\overset{+}{b}$ in $B \in \textsc{pn}$ and $\overset{-}{a}$ in $A \in \textsc{pn}$, where $tc(B) = Z, \overset{+}{b}$ and $tc(A) = X, \overset{-}{a}, Y$. By induction hypothesis: $B \in \textsc{pn}_2$ and $A \in \textsc{pn}_2$. Since $go(B) = b$, we have that $at = go(A)$. Therefore, $A \in \textsc{pn}_2$ must have been obtained by (h') from $A^- \in \textsc{pn}_1$, i.e., A with missing goal link. By (d'), we can apply $\backslash 2$ to $\overset{-}{a}$ in $A^- \in \textsc{pn}_1$ and $\overset{+}{b}$ in $B \in \textsc{pn}_2$, yielding $C^- \in \textsc{pn}_1$, i.e., C with missing goal link, which, by (h'), can be added so as to obtain C as a member of $\textsc{pn}_2$. (The cases (e), (f) and (g) are analogous.) Conversely, $C \in \textsc{pn}_2$ must

[19] Viz., the proof net version of observation **B** made in section 3 above: If $C \in \textsc{pn}$ and $tc(C) = X, b\overset{+}{\backslash}a, Y$ (or $X, a\overset{+}{/}b, Y$), then there is an $A \in \textsc{pn}$ with $tc(A) = X, \overset{+}{a}, \overset{-}{b}, Y$ (or $X, \overset{-}{b}, \overset{+}{a}, Y$) such that C results from applying (b) (or (c)) to A. (Roorda (1991), p. 33)

have been obtained by (h') from $C^- \in$ PN$_1$. Note that $tc(C^-) = tc(C)$ and $d(tc(C)) > 0$, so $C^- \in$ PN$_1$ must be due to (d'), (e'), (f') or (g'). If (d') is responsible, then $tc(C^-) = X, Z, \overset{-}{b\backslash a}, Y$, and C^- results from applying $\backslash 2$ to $\overset{+}{b}$ in $B \in$ PN$_2$ and $\overset{-}{a}$ in $A' \in$ PN$_1$, where $tc(A') = X, \overset{-}{a}, Y$ and $tc(B) = Z, \overset{+}{b}$. By induction hypothesis: $A' = A^-$ for $A \in$ PN, while $B \in$ PN. Therefore, by (d), we can apply $\backslash 2$ to $\overset{+}{b}$ in $B \in$ PN and $\overset{-}{a}$ in $A \in$ PN, yielding C^- with additional goal link – i.e., C – as a member of PN. (The cases (e'), (f') and (g') are analogous.) $\square$

It was said above that the definition of the set of proof nets in terms of (a') through (h') has two pleasant properties: (i) it allows a direct, inductive definition of the semantic interpretation of proof nets, and (ii) it brings out the correspondence between the application of rules in **L*** and the addition of links to proof nets.

As for (i): let u and v be different variables; let the sets of variables occurring in A and B be disjoint; let w be a variable which does not occur in A and B; and let $A[v{:=}\gamma]$ denote the structure which results from $A \in$ PN$_1$ by replacing all occurrences of the variable v in A by the term γ. Then the semantic interpretation of proof nets can be defined as follows:

(a'') $\overset{-}{at}{:}u \ \ \overset{+}{at}{:}v \in$ PN$_1$ and $\overset{+}{at}{:}v \ \ \overset{-}{at}{:}u \in$ PN$_1$;
$tc(\overset{-}{at}{:}u \ \ \overset{+}{at}{:}v) = \overset{-}{at}{:}u, \overset{+}{at}{:}v$ and $tc(\overset{+}{at}{:}v \ \ \overset{-}{at}{:}u) = \overset{+}{at}{:}v, \overset{-}{at}{:}u$;

(b'') if $A \in$ PN$_2$ and $tc(A) = X, \overset{+}{a}{:}\alpha, \overset{-}{b}{:}v, Y$ and X or Y is non-empty,
then $C \in$ PN$_2$, where C results from applying $\backslash 1$ to $\overset{+}{a}{:}\alpha$ and $\overset{-}{b}{:}v$ in A;
$tc(C) = X, \overset{+}{b\backslash a}{:}\lambda v.\alpha, Y$;

(c'') if $A \in$ PN$_2$ and $tc(A) = X, \overset{-}{b}{:}v, \overset{+}{a}{:}\alpha, Y$ and X or Y is non-empty,
then $C \in$ PN$_2$, where C results from applying $/1$ to $\overset{-}{b}{:}v$ and $\overset{+}{a}{:}\alpha$ in A;
$tc(C) = X, \overset{+}{a/b}{:}\lambda v.\alpha, Y$;

(d'') if $A \in$ PN$_1$ and $tc(A) = X, \overset{-}{a}{:}v, Y$ and $B \in$ PN$_2$ and $tc(B) = Z, \overset{+}{b}{:}\beta$,
then $C \in$ PN$_1$, where C results from applying $\backslash 2$ to

$\overset{+}{b}{:}\beta$ in B and $\overset{-}{a}{:}w(\beta)$ in $A[v{:=}w(\beta)]$;

$tc(C) = X', Z, \overset{-}{b\backslash a}{:}w, Y'$, where $X', \overset{-}{a}{:}w(\beta), Y' = tc(A[v{:=}w(\beta)])$;

(e'') if $A \in \textsc{pn}_2$ and $tc(A) = X, \overset{+}{b}{:}\beta, Y$ and $B \in \textsc{pn}_1$ and $tc(B) = \overset{-}{a}{:}v, Z$,

then $C \in \textsc{pn}_1$, where C results from applying $\backslash 2$ to

$\overset{+}{b}{:}\beta$ in A and $\overset{-}{a}{:}w(\beta)$ in $B[v{:=}w(\beta)]$;

$tc(C) = X, \overset{-}{b\backslash a}{:}w, Z', Y$, where $\overset{-}{a}{:}w(\beta), Z' = tc(B[v{:=}w(\beta)])$;

(f'') if $A \in \textsc{pn}_1$ and $tc(A) = X, \overset{-}{a}{:}v, Y$ and $B \in \textsc{pn}_2$ and $tc(B) = \overset{+}{b}{:}\beta, Z$,

then $C \in \textsc{pn}_1$, where C results from applying $/2$ to

$\overset{-}{a}{:}w(\beta)$ in $A[v{:=}w(\beta)]$ and $\overset{+}{b}{:}\beta$ in B;

$tc(C) = X', \overset{-}{a/b}{:}w, Z, Y'$, where $X', \overset{-}{a}{:}w(\beta), Y' = tc(A[v{:=}w(\beta)])$;

(g'') if $A \in \textsc{pn}_2$ and $tc(A) = X, \overset{+}{b}{:}\beta, Y$ and $B \in \textsc{pn}_1$ and $tc(B) = Z, \overset{-}{a}{:}v$,

then $C \in \textsc{pn}_1$, where C results from applying $/2$ to

$\overset{-}{a}{:}w(\beta)$ in $B[v{:=}w(\beta)]$ and $\overset{+}{b}{:}\beta$ in A;

$tc(C) = X, Z', \overset{-}{a/b}{:}w, Y$, where $Z', \overset{-}{a}{:}w(\beta) = tc(B[v{:=}w(\beta)])$;

(h'') if $A \in \textsc{pn}_1$, $tc(A) = X, \overset{+}{at}{:}v, Y$ and $\overset{-}{at}{:}\gamma$ in A is not connected by ax,

then $C \in \textsc{pn}_2$, where C results from applying ax to

$\overset{+}{at}{:}\gamma$ and $\overset{-}{at}{:}\gamma$ in $A[v{:=}\gamma]$;

$tc(C) = tc(A[v{:=}\gamma])$.

The proof nets C defined in (b''), (c''), (d''), (e''), (f''), (g''), and (h'') will be called $\backslash 1(A)$, $/1(A)$, $\backslash 2(A^{w/v}, B)$, $\backslash 2(A, B^{w/v})$, $/2(A^{w/v}, B)$, $/2(A, B^{w/v})$ and $ax(A)$, respectively.

Though it is not exactly simple, the above definition is straightforward in comparison with the indirect semantic interpretation procedure given in Roorda (1991, pp. 34–38):

First, the set of *proof frames* PF is defined as the smallest set such that:

(A) if X is a list of signed atomic categories, then $X \in \textsc{pf}$; $tc(X) = X$;

(B) if $A \in \textsc{pf}$ and $tc(A) = X, \overset{+}{a}, \overset{-}{b}, Y$, then $C \in \textsc{pf}$, where C results from applying $\backslash 1$ to $\overset{+}{a}$ and $\overset{-}{b}$ in A; $tc(C) = X, \overset{+}{b\backslash a}, Y$;

(C) if $A \in$ PF and $tc(A) = X, \overset{-}{b}, \overset{+}{a}, Y$, then $C \in$ PF, where C results from applying $/1$ to $\overset{-}{b}$ and $\overset{+}{a}$ in A; $tc(C) = X, \overset{+}{a/b}, Y$;

(D) if $A \in$ PF and $tc(A) = X, \overset{+}{b}, \overset{-}{a}, Y$, then $C \in$ PF, where C results from applying $\backslash 2$ to $\overset{+}{b}$ and $\overset{-}{a}$ in A; $tc(C) = X, \overset{-}{b\backslash a}, Y$;

(E) if $A \in$ PF and $tc(A) = X, \overset{-}{a}, \overset{+}{b}, Y$, then $C \in$ PF, where C results from applying $/2$ to $\overset{-}{a}$ and $\overset{+}{b}$ in A; $tc(C) = X, \overset{-}{a/b}, Y$.

Moreover, a *proof structure* is defined as a proof frame together with a linking of all its atomic categories by *ax* links, and a *planar* proof structure is defined as a proof structure of which the *ax* links do not cross.[20]

Second, for every category occurrence in a proof structure C a lambda term is constructed:

(I) every category occurrence in $tc(C)$ is assigned a different variable;

(II) the assignment of terms to the conclusions of $\backslash 2$, $/2$, $\backslash 1$ and $/1$ links in C is extended to the premises of these links in the following way:

$$\frac{\overset{+}{b}:? \quad \overset{-}{a}:\text{¿}}{\overset{-}{b\backslash a}:\gamma}\,\backslash 2 \qquad \frac{\overset{-}{a}:\text{¿} \quad \overset{+}{b}:?}{\overset{-}{a/b}:\gamma}\,/2$$

In $\backslash 2$ and $/2$ links, the category $\overset{+}{b}$ is assigned a fresh variable u, and the term assigned to $\overset{-}{a}$ is $\gamma(u)$.

$$\frac{\overset{+}{a}:\text{¿} \quad \overset{-}{b}:?}{\overset{+}{b\backslash a}:\delta}\,\backslash 1 \qquad \frac{\overset{-}{b}:? \quad \overset{+}{a}:\text{¿}}{\overset{+}{a/b}:\delta}\,/1$$

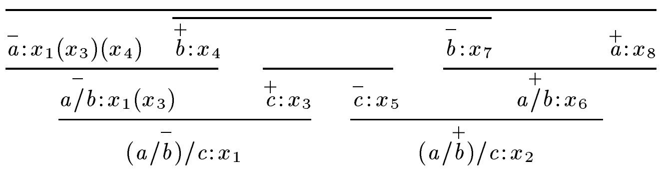
[20]Though a proof net also consists of linked signed categories of which the atomic ones are all connected by non-crossing *ax* links, the set of planar proof structures is considerably larger than the set PN defined earlier. This is due to the fact that on account of the clauses (B) through (E) above, the links $\backslash 1$, $/1$, $\backslash 2$ and $/2$ are allowed to connect any pair of adjacent (and appropriately signed) categories in a proof structure. The clauses (*b*) through (*g*), on the other hand, have the effect that the links $\backslash 1$ and $/1$ only connect adjacent *terminal* categories of a *single* proof structure, whereas the links $\backslash 2$ and $/2$ only connect adjacent *terminal* categories of *separate* proof structures.

In $\backslash 1$ and $/1$ links, the respective terms assigned to $\overset{-}{b}$ and $\overset{+}{a}$ are two different fresh variables u and v, and the conclusion term δ is *rewritten*[21] as $\lambda u.v$.

(III) every *ax* link in C which connects $\overset{+}{at}{:}\gamma$ and $\overset{-}{at}{:}\delta$ induces a substitution[22] of γ by δ. This substitution is *global*: it involves all occurrences of γ in C.

Third, Roorda supplies a list of five[23] 'proof net conditions' which a proof structure interpreted in the way indicated in (I) through (III) must observe in order to be a proof net, and shows that the set of proof nets corresponding to **L**-derivable sequents is identical to the set of planar proof structures which satisfy these conditions. Except for the first one (which requires that there be exactly one terminal category signed '+'), these conditions constrain the

[21]Positively signed categories are always assigned variables, so this rewriting of δ is essentially a *substitution*. Moreover, though Roorda (1991) does not mention this, the substitution will have to be *global*: just as the substitution specified in (III) below, it must affect *all* occurrences of δ in the proof structure. This can be seen as follows:



Stage (I) yields the initial assignment of x_1 and x_2; the clause for $/2$ in stage (II) takes care of the assignment of $x_1(x_3)$, x_3, $x_1(x_3)(x_4)$ and x_4. Now, the clause for $/1$ assigns x_5 and x_6, substitutes x_2 by $\lambda x_5.x_6$, assigns x_7 and x_8, and substitutes x_6 by $\lambda x_7.x_8$. If the latter substitution is local and does not affect the occurrence of x_6 in $\lambda x_5.x_6$, then after (III) (which – globally – replaces x_8, x_3 and x_4 by $x_1(x_3)(x_4)$, x_5, and x_7, respectively) the undesirable term $\lambda x_5.x_6$ is obtained for the goal of the proof net. Global substitution, on the other hand, will eventually yield the right result, viz., $\lambda x_5 \lambda x_7.x_1(x_5)(x_7)$.

[22]Roorda says that γ and δ are *unified* but that γ is always a variable, so 'the unification is nothing else than the substitution [of γ by δ]' (1991, p. 35). The notion of unification is probably invoked for the reason that Roorda wants the process to fail, 'typically' when γ occurs in δ. 'In that case we draw the conclusion that the proof structure is not a proof net, and we do not provide a lambda term.' (1991, p. 35). (Note that a substitution under such circumstances would be impeccable.) However, the fact that this eventuality does not arise in $\text{PN}_1 \cup \text{PN}_2$ kept us from taking over this subtlety.

[23]In fact, a sixth proof net condition enforces the 'non-empty antecedent property' alluded to in footnote 18 above.

assignment of lambda terms to proof structures.

Their precise contents need not concern us here, since the equivalence of the interpretation procedures specified in (a'') through (h'') and (I) through (III) for members C of the set $\text{PN}_1 \cup \text{PN}_2$ can be established directly:[24] it is easy to see that both (I) through (III) and (a'') through (h'') license an assignment of lambda terms to category occurrences in $C \in \text{PN}_1 \cup \text{PN}_2$ iff:

- every $\overset{-}{c}$ occurring in $tc(C)$ is assigned a different variable;
- the assignment to every \1, /1, \1 and /2 link in C is as follows:

$$\frac{\overset{+}{b}\!:\!\beta \quad \overset{-}{a}\!:\!\alpha(\beta)}{b\overset{-}{\backslash} a\!:\!\alpha}\,\backslash 2 \qquad\qquad \frac{\overset{-}{a}\!:\!\alpha(\beta) \quad \overset{+}{b}\!:\!\beta}{a\overset{-}{/} b\!:\!\alpha}\,/2$$

$$\frac{\overset{+}{a}\!:\!\gamma \quad \overset{-}{b}\!:\!v}{b\overset{+}{\backslash} a\!:\!\lambda v.\gamma}\,\backslash 1 \qquad\qquad \frac{\overset{-}{b}\!:\!v \quad \overset{+}{a}\!:\!\gamma}{a\overset{+}{/} b\!:\!\lambda v.\gamma}\,/1\text{; and}$$

- if an ax link connects $\overset{-}{at}\!:\!\gamma$ and $\overset{+}{at}\!:\!\delta$, then γ is identical to δ.

Do proof nets consitute a solution to the problem of spurious ambiguity? Observe that the proof net semantics defined above in (b'') through (h'') is closely related to **L***, viz., via the function k from **L*** proofs to the power set of $\text{PN}_1 \cup \text{PN}_2$ which is defined below.

As before, we let $\pi[T\!:\!T' \vdash c\!:\!\gamma]$ represent that π is an **L*** proof with interpreted conclusion sequent $T\!:\!T' \vdash c\!:\!\gamma$.

- $k(at^*\!:\!v \vdash at\!:\!v) = \{\overset{-}{at}\!:\!v \;\; \overset{+}{at}\!:\!u \mid u \neq v\} \cup \{\overset{+}{at}\!:\!u \;\; \overset{-}{at}\!:\!v \mid u \neq v\}$;
- for $\pi[b\!:\!v, T\!:\!T' \vdash a^*\!:\!\alpha]$: $k(\dfrac{\pi}{T\!:\!T' \vdash b\backslash a^*\!:\!\lambda v.\alpha}[\backslash R]) = \{\backslash 1(A) \mid A \in k(\pi)\}$;
- for $\pi[T\!:\!T', b\!:\!v \vdash a^*\!:\!\alpha]$: $k(\dfrac{\pi}{T\!:\!T' \vdash a/b^*\!:\!\lambda v.\alpha}[/R]) = \{/1(A) \mid A \in k(\pi)\}$;
- for $\pi_1[T\!:\!T' \vdash b^*\!:\!\beta]$ and $\pi_2[U\!:\!U', a^*\!:\!v, V\!:\!V' \vdash at\!:\!\gamma]$:

[24] Strictly speaking, the set PN_1 transcends the range of the clauses (I) through (III), because its members fail to be proof structures. However, they can be incorporated easily, for instance by allowing (I) through (III) to operate on members of the set PO (of *proof objects*): the smallest set which includes the set PF defined above and is closed under the operation of adding ax links.

$$k\Big(\frac{\pi_1 \qquad\qquad\qquad\qquad \pi_2}{U:U',T:T',b\backslash a^*:w,V:V' \vdash at:\gamma[v:=w(\beta)]}[\backslash L]\Big) =$$

$$\{\backslash 2(A^{w/v},B) \mid A \in k(\pi_2) \,\&\, B \in k(\pi_1) \,\&\, tc(B) = \overset{-}{T}:T', \overset{+}{b}:\beta\} \cup$$

$$\{\backslash 2(A,B^{w/v}) \mid A \in k(\pi_1) \,\&\, B \in k(\pi_2) \,\&\, tc(B) = \overset{-}{a}:v,\ \overset{-}{V}:V',\ \overset{+}{at}:u,\ \overset{-}{U}:U'\};$$

$$k\Big(\frac{\pi_1 \qquad\qquad\qquad\qquad \pi_2}{U:U',a/b^*:w,T:T',V:V' \vdash at:\gamma[v:=w(\beta)]}[/L]\Big) =$$

$$\{/2(A^{w/v},B) \mid A \in k(\pi_2) \,\&\, B \in k(\pi_1) \,\&\, tc(B) = \overset{+}{b}:\beta,\ \overset{-}{T}:T'\} \cup$$

$$\{/2(A,B^{w/v}) \mid A \in k(\pi_1) \,\&\, B \in k(\pi_2) \,\&\, tc(B) = \overset{-}{V}:V',\ \overset{+}{at}:u,\ \overset{-}{U}:U',\ \overset{-}{a}:v\};$$

- for $\pi[U:U',a^*:v,V:V' \vdash at:\gamma]$: $k\Big(\dfrac{\pi}{U:U',a:v,V:V' \vdash at^*:\gamma}[*]\Big) =$

  $\{ax(A) \mid A \in k(\pi)\}$.

Next, for (non-asterisked) interpreted sequents $T:T' \vdash c:\gamma$, we define the set $cp(T:T' \vdash c:\gamma)$ of *cyclic permutations* of $T:T' \vdash c:\gamma$, as follows:

$$cp(T:T' \vdash c:\gamma) = \{\overset{-}{T_2}:T_2', \overset{+}{c}:\gamma, \overset{-}{T_1}:T_1' \mid T:T' = T_1:T_1', T_2:T_2'\}.$$

Note that a proof net C corresponds to a sequent $T : T' \vdash c : \gamma$ if and only if $tc(C) \in cp(T : T' \vdash c : \gamma)$. By a straightforward induction on the length of π (which will be omitted here) it can be proven that for **L*** proofs $\pi[T:T' \vdash c^*:\gamma]$ and $\rho[U:U,c^*:v,V:V' \vdash at:\gamma]$:

$$k(\pi) = \{C \in \mathrm{PN}_2 \mid tc(C) \in cp(T:T' \vdash c:\gamma)\}; \text{ and}$$

$$k(\rho) = \{A \in \mathrm{PN}_1 \mid \text{for some } u\text{: } tc(A) \in cp(U:U',c:v,V:V' \vdash at:u) \text{ and } \overset{-}{at}:\gamma \text{ occurs not connected by } ax \text{ in } A\}.$$

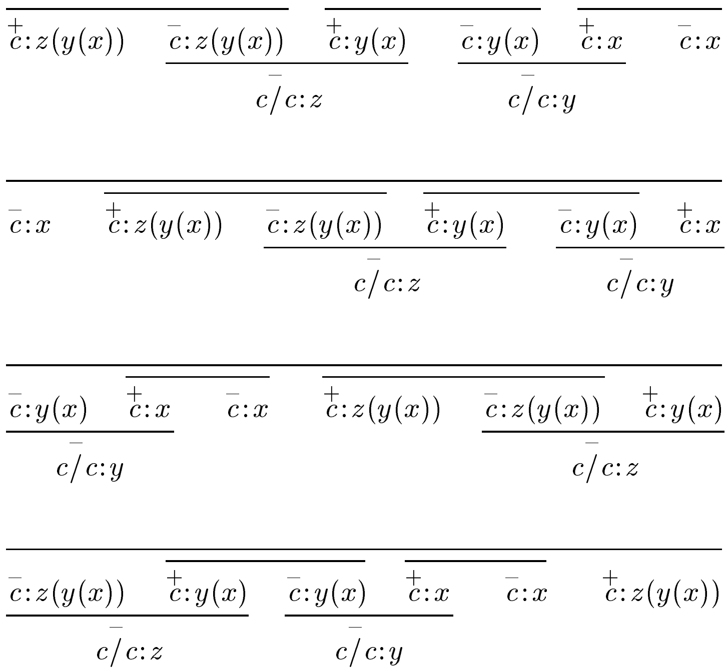
So, the function k assigns a subset of PN_2 to every **L*** proof of a sequent with an asterisked goal and a subset of PN_1 to every **L*** proof of a sequent with a non-asterisked goal.

Moreover, the set $\{k(\pi) \mid \pi \text{ is an } \mathbf{L^*} \text{ proof}\}$ partitions $\mathrm{PN}_1 \cup \mathrm{PN}_2$, for (i) it is obvious that every $C \in \mathrm{PN}_1 \cup \mathrm{PN}_2$ is a member of at least one $k(\pi)$; and (ii) Theorem 2 (see section 4 above) entails that every $C \in \mathrm{PN}_1 \cup \mathrm{PN}_2$ is a member of at most one $k(\pi)$.

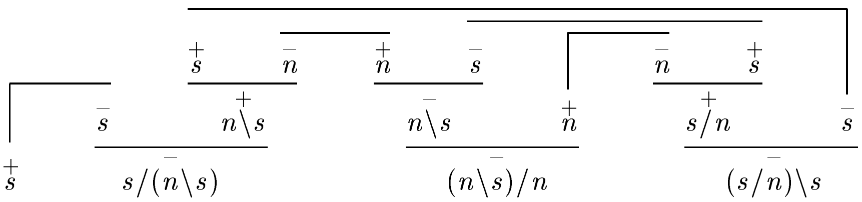
Now, since for every **L**-derivable sequent $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ there is one **L*** derivation of $c_1 : v_1, \ldots, c_n : v_n \vdash c^* : \gamma'$ such that $\gamma = \gamma'$ and since for a sequent $c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma$ the set $cp(c_1 : v_1, \ldots, c_n : v_n \vdash c : \gamma)$ contains $n+1$ members, the above means that every reading of an **L**-derivable sequent $c_1, \ldots, c_n \vdash c$ corresponds to $n+1$ proof nets.

Accordingly, there are four proof nets for the sequent $c/c, c/c, c \vdash c$, and eight proof nets for $s/(n\backslash s), (n\backslash s)/n, (s/n)\backslash s \vdash s$.[25] Note that the proof nets in (47) make up $k(31)$, and that the proof nets in (48) and (49) constitute $k(32)$ and $k(33)$, respectively.

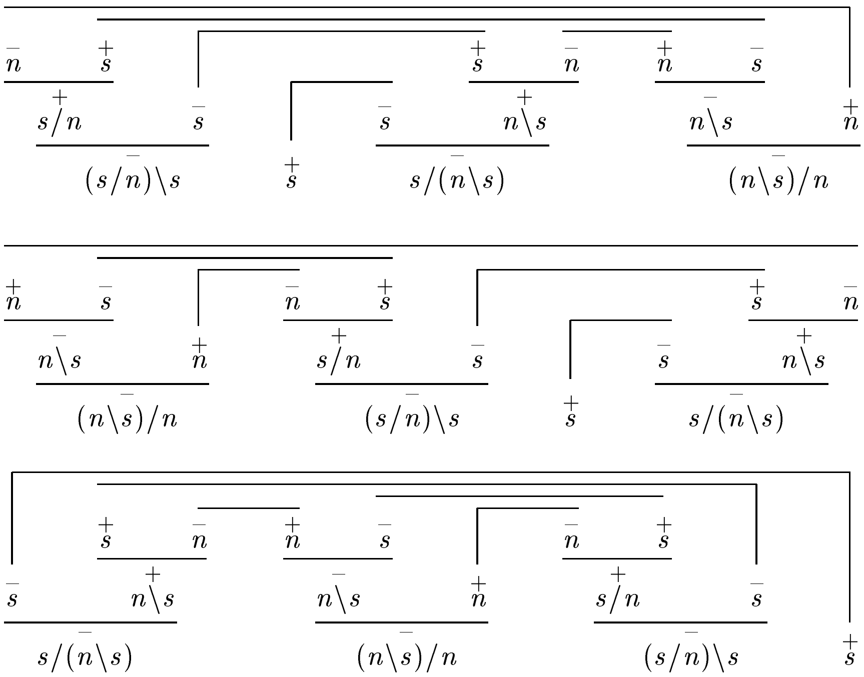
(47)



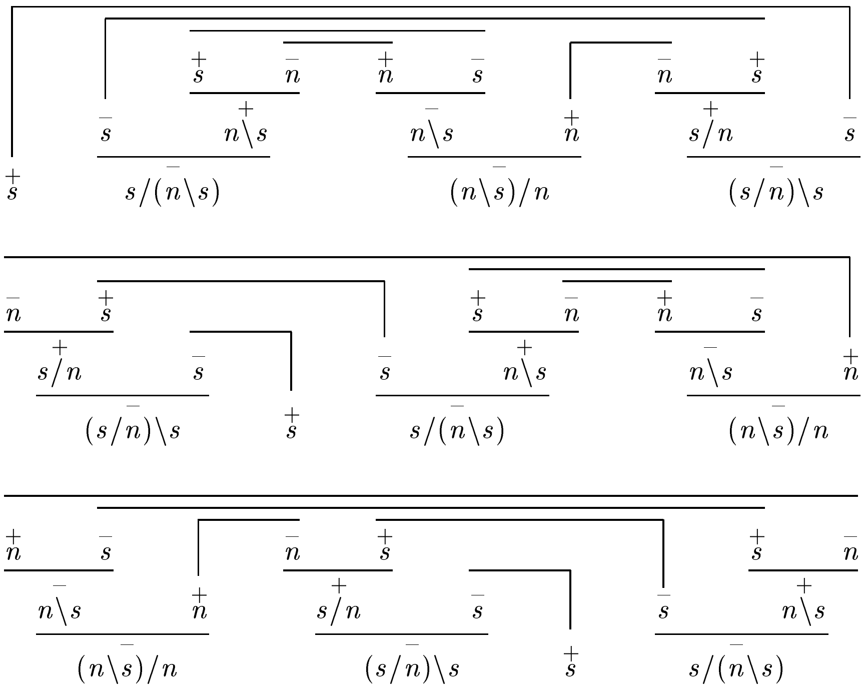
(48)

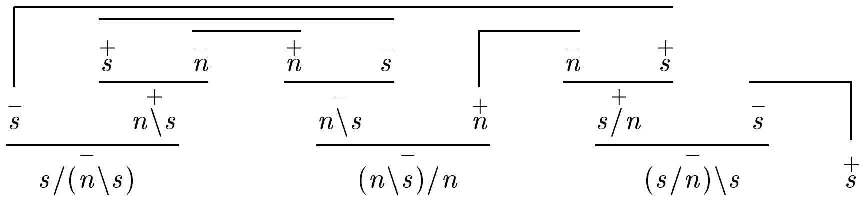


[25] As above (cf. footnote 10), we identify all proof nets which differ only with respect to the particular variables assigned.



(49)





If $s/(\overset{-}{n}\backslash s)$, $(n\backslash\overset{-}{s})/n$, and $(s/\overset{-}{n})\backslash s$ are assigned the variables u, v and w, then the term $u(\lambda u'.w(\lambda w'.v(u')(w')))$ is assigned to the positively signed terminal category s in the proof nets listed in (48), and $w(\lambda w'.u(\lambda u'.v(u')(w')))$ is assigned to the positively signed terminal category s in the proof nets listed in (49).

Roorda (1991, p. 27) claims that

> a proof net can be considered as a parallellized sequent proof and as such it lacks the spurious ambiguity of (even Cut-free) sequent proofs. A proof net is a concrete structure, not merely an abstract equivalence class of derivations [...].

However, we have just seen that the set of proof nets has to be partitioned into equivalence classes if it is to serve as a solution to the spurious ambiguity problem: for every reading of an **L**-derivable sequent $c_1, \dots, c_n \vdash c$, there are $n+1$ proof nets.

Of course, these $n+1$ proof nets are equivalent in that they make up a set of cyclic permutations. But on the one hand, one cannot isolate one specific representant – e.g., the proof net with its positive terminal category in rightmost position – from this set, since one has to be strict on order and adjacency of premises in proof nets, so that all these permutations are required for the definition (cf. clause (*b*) and (*c*)). And on the other hand, it is not possible to consider this set of permutations as a monolithic object, since sometimes a specific representant (e.g., the proof net with its positive terminal category in rightmost position – cf. B in clause (*d*)) has to be isolated from this set.

So, ironically, it is the proof net approach which has to resort to 'abstract' equivalence classes, whereas sequent normalization (as well as partial deduction) associates one concrete object with each interpretation of a derivable sequent: its **L*** (**PD**) proof.

6 Encoding Atomic Categories

Ponse (1988) proved that **LP** derivability in an atomic goal category can be mimicked by **LP** derivability using one atomic category only. **LP** refers to the Lambek calculus with Permutation, which is also known as Lambek-Van Benthem calculus.[26] In the present section we will show that (a generalization of) this result can be extended to **L**, the Lambek calculus proper: **L** derivability in *any* category can be mimicked by **L** derivability using one atomic category only. More formally:

Let the set $\textsc{at}$ consist of the distinct atomic categories $at_1, \ldots, at_k$, let at be an atomic category, and let $\textsc{cat}_{\textsc{at}}$ and $\textsc{cat}_{\{at\}}$ be the sets of categories based on $\textsc{at}$ and $\{at\}$, respectively. Then there is a substitution σ replacing every $at_i \in \textsc{at}$ by a $c_i \in \textsc{cat}_{\{at\}}$ such that for all $c_1, \ldots, c_n, c$ in $\textsc{cat}_{\textsc{at}}$:

(50) $c_1, \ldots, c_n \vdash_{\mathbf{L}} c$ if and only if $\sigma(c_1, \ldots, c_n) \vdash_{\mathbf{L}} \sigma(c)$.

If σ is a substitution and α is a category or a sequence of categories, then the expression $\sigma(\alpha)$ denotes the result of performing σ to α. The atomic category at will be abbreviated as t in the sequel.

The proof of theorem (50) is organized as follows. First, the calculus **L*** will be used for establishing the facts (51) and (53), which express useful properties of **L**-derivable sequents that will be exploited throughout. Next, a Lemma will be proven which concerns the non-derivability of certain sequents that involve categories built up from the categories $(t/t)/t$, $((t/t)/(t/t))/(t/t)$ and atomic categories different from t. This Lemma is then shown to entail Claim 1, which states that the categories $(t/t)/t$ and $((t/t)/(t/t))/(t/t)$ can be used to encode two atomic categories, viz., t and some other atomic category, also in the presence of yet other atomic categories. Finally, the substitution σ employed in Claim 1 is generalized in Claim 2: by means of a substitution $\sigma_{\langle t, at_1, \ldots, at_m \rangle}$, any finite number of atomic categories $t, at_1, \ldots, at_m$ can be encoded in terms of t.[27]

[26] See Van Benthem (1986; 1991).

[27] The above theorem follows from Claim 2, since the following substitutions will meet the requirement specified in (50):

- $\sigma_{\langle t, at_1, \ldots, at_k \rangle}$ if $t \notin \textsc{at}$ (note that any category based on $\textsc{at}$ is based on $\textsc{at} \cup \{t\}$);
- $\sigma_{\langle t, at_1, \ldots, at_{i-1}, at_{i+1}, \ldots, at_k \rangle}$ if $t \in \textsc{at}$ and $\textsc{at} = \{at_1, \ldots, at_{i-1}, t, at_{i+1}, \ldots, at_k\}$.

We start by putting each category c in $\textsc{cat}_{\textsc{at}}$ into an equivalence class $\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$. Let c and $c_1,\ldots,c_{p+q}$ be members of $\textsc{cat}_{\textsc{at}}$ (where $p+q\geq 0$), and let $at\in\textsc{at}$. Then:

$c\in\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$ iff

(a) $c=at$ and $p+q=0$;
(b) $c=c_p\backslash c'$ and $c'\in\lfloor c_{p-1}\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$; or
(c) $c=c'/c_{p+q}$ and $c'\in\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q-1}\rfloor$.

The sets $\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$ partition $\textsc{cat}_{\textsc{at}}$.[28] Note that the following claims hold:

(51) If $c\in\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$, then
$T\vdash_{\mathbf{L}} c$ iff $c_1,\ldots,c_p,T,c_{p+q},\ldots,c_{p+1}\vdash_{\mathbf{L}} at$.

(52) $U,c^*,V\vdash_{\mathbf{L*}} at$ and $c\in\lfloor c_p\backslash...\backslash c_1\backslash at'/c_{p+1}/.../c_{p+q}\rfloor$ iff
$at'=at$; $U=T_1,\ldots,T_p$; $V=T_{p+q},\ldots,T_{p+1}$;
and for all i, $1\leq i\leq p+q$: $T_i\vdash_{\mathbf{L*}} {c_i}^*$.

Both (51) and (52) are proven by induction on $p+q$.

As for (51):
• If $p+q=0$, then the claim is trivial.
• If $p+q>0$, then (i) $c=c_p\backslash c'$ and $c'\in\lfloor c_{p-1}\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$; or (ii) $c=c'/c_{p+q}$ and $c'\in\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q-1}\rfloor$. We only treat (i), since (ii) is analogous. Note that the following claims are equivalent: (1) $T\vdash_{\mathbf{L}} c_p\backslash c'$; (2) $T\vdash_{\mathbf{L*}} c_p\backslash c'^*$; (3) $c_p,T\vdash_{\mathbf{L*}} c'^*$; (4) $c_p,T\vdash_{\mathbf{L}} c'$; (5) $c_1,\ldots,c_p,T,c_{p+q},\ldots,c_{p+1}\vdash_{\mathbf{L}} at$. Theorem 1.1 (see section 3 above) yields the equivalence of (1) and (2) as well as (3) and (4); the equivalence of (2) and (3) is due to **L***; and (4) and (5) are equivalent on account of the induction hypothesis. □

As for (52):

[28]Different categories c and c' are members of the same set $\lfloor c_p\backslash...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$ iff c and c' have the same final atomic value (viz., at) and the same series of left-hand side $(c_p,\ldots,c_1)$ and right-hand side $(c_{p+1},\ldots,c_{p+q})$ arguments, but combine with these arguments in a different order. The set $\lfloor t\backslash t/(t/t)/t\rfloor$, for example, consists of three categories: (a) $((t\backslash t)/(t/t))/t$, (b) $(t\backslash(t/(t/t)))/t$ and (c) $t\backslash((t/(t/t))/t)$.

• $p+q=0$. Then $U, c^*, V \vdash_{\mathbf{L}*} at$ must be an axiom $at^* \vdash_{\mathbf{L}*} at$; so $at' = at$ and U and V are empty.
• $p+q>0$. Then (i) $c = c_p\backslash c'$ and $c' \in \lfloor c_{p-1}\backslash ...\backslash c_1\backslash at'/c_{p+1}/.../c_{p+q}\rfloor$; or (ii) $c = c'/c_{p+q}$ and $c' \in \lfloor c_p\backslash ...\backslash c_1\backslash at'/c_{p+1}/.../c_{p+q-1}\rfloor$. We only treat (ii), since (i) is analogous. The sequent $U, c'/c_{p+q}{}^*, V \vdash at$ must be derived by $/L$ in **L***. Hence $U, c'/c_{p+q}{}^*, V \vdash_{\mathbf{L}*} at$ iff $U, c'^*, V' \vdash_{\mathbf{L}*} at$ and $T_{p+q} \vdash_{\mathbf{L}*} c_{p+q}{}^*$, where $V = T_{p+q}, V'$. By induction hypothesis: $U, c'^*, V' \vdash_{\mathbf{L}*} at$ iff $at' = at$; $U = T_1, \ldots T_p$; $V' = T_{p+q-1}, \ldots T_{p+1}$; and for all i such that $1 \leq i \leq p+q-1$: $T_i \vdash_{\mathbf{L}*} c_i{}^*$. $\square$

Given (52), suppose that $T \vdash_{\mathbf{L}} at$. This holds iff $T \vdash_{\mathbf{L}*} at^*$ by Theorem 1.1 (see Section 3 above). The sequent $T \vdash at^*$ must have been derived by the $*$ rule in **L***. Therefore, $T = U, c, V$ and $U, c^*, V \vdash_{\mathbf{L}*} at$. Now, for some $c_1, \ldots, c_{p+q}, at'$: $c \in \lfloor c_p\backslash ...\backslash c_1\backslash at'/c_{p+1}/.../c_{p+q}\rfloor$. By (52), we have that $at' = at$; $U = T_1, \ldots, T_p$; $V = T_{p+q}, \ldots, T_{p+1}$; and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}*} c_i{}^*$, which is equivalent to $T_i \vdash_{\mathbf{L}} c_i$ by Theorem 1.1. Summing up:

(53) $T \vdash_{\mathbf{L}} at$ iff there is a $c \in \lfloor c_p\backslash ...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$ such that $T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$.

Let A be a set of categories. In the sequel we will say that c is an A-category iff c is built up from categories in A.[29] We will abbreviate sequences $t/t, \ldots, t/t$ consisting of n occurrences of the category t/t as $(t/t)^n$.

Lemma: let A be the set $\{at_1, \ldots, at_k, (t/t)/t, ((t/t)/(t/t))/(t/t)\}$, where $at_1, \ldots, at_k$ and t are distinct atomic categories; and let T be a non-empty sequence of A-categories. Then:

(**a**) $T, t \not\vdash_{\mathbf{L}} t$;
(**b**) for all $n \in \mathbb{N}$: $T, (t/t)^n \not\vdash_{\mathbf{L}} t$; and
(**c**) $T, t/t, t \not\vdash_{\mathbf{L}} t$.

Proof of (**a**) and (**b**): by induction on m, the number of occurrences of $at_1, \ldots, at_k$, $(t/t)/t$ and $((t/t)/(t/t))/(t/t)$ in T.

• $m=1$. Then $T = at_i$ $(1 \leq i \leq k)$; $T = (t/t)/t$; or $T = ((t/t)/(t/t))/(t/t)$:

[29] That is, the set of A-categories is the smallest set A' such that (i) $A \subseteq A'$; and (ii) if $c \in A'$ and $c' \in A'$, then $c/c' \in A'$ and $c'\backslash c \in A'$.

(**a**) $at_i, t \not\vdash_{\mathbf{L}} t$; $(t/t)/t, t \not\vdash_{\mathbf{L}} t$; and $((t/t)/(t/t))/(t/t), t \not\vdash_{\mathbf{L}} t$.

(**b**) That $T, (t/t)^n \not\vdash_{\mathbf{L}} t$ can be shown by *at*-count, a notion introduced in Van Benthem (1986). Let $at \in \text{AT}$. Then at-count$[c]$ is defined as follows: for $c \in \text{AT}$, at-count$[c] = 1$ if $c = at$, while at-count$[c] = 0$ if $c \neq at$; at-count$[a/b]$ $= at$-count$[b\backslash a] = at$-count$[a] - at$-count$[b]$. Moreover, at-count$[c_1, \ldots, c_n]$ $= at$-count$[c_1] + \ldots + at$-count$[c_n]$. A useful property of **L**-derivable sequents $T \vdash c$ is that for all $at \in \text{AT}$: at-count$[T] = at$-count$[c]$. (The proof proceeds by an easy induction on the length of the derivation of $T \vdash c$.)

Note that t-count$[(t/t)/t] = -1$; that t-count$[((t/t)/(t/t))/(t/t)] = 0$; and that for all $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$: t-count$[at_i] = t$-count$[(t/t)^n] = 0$. Hence t-count$[(t/t)/t, (t/t)^n] = -1$ and t-count$[((t/t)/(t/t))/(t/t), (t/t)^n] =$ t-count$[at_i, (t/t)^n] = 0$. On the other hand, t-count$[t] = 1$. So, for all $n \in \mathbb{N}$: $at_i, (t/t)^n \not\vdash_{\mathbf{L}} t$; $(t/t)/t, (t/t)^n \not\vdash_{\mathbf{L}} t$; and $((t/t)/(t/t))/(t/t), (t/t)^n \not\vdash_{\mathbf{L}} t$.

• $m > 1$. Note[30] that if c is an A-category and $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$, then (i) $c_{p+1} = c_{p+2} = t$, so $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/t/c_{p+3}/.../c_{p+q}\rfloor$; or (ii) c_{p+1} $= t$ and $c_{p+2} = c_{p+3} = t/t$, so $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/(t/t)/(t/t)/c_{p+4}/.../c_{p+q}\rfloor$.

(**a**) Suppose $T, t \vdash_{\mathbf{L}} t$. By (53), there is a $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$ such that $T, t = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$. Since T is non-empty, this c cannot be the rightmost category t in T, t. Hence c is an A-category in T and either (i) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/t/c_{p+3}/.../c_{p+q}\rfloor$; or (ii) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/(t/t)/(t/t)/c_{p+4}/.../c_{p+q}\rfloor$. Focus on T_{p+2}. On the one hand: if (i), then $c_{p+2} = t$, so $T_{p+2} \vdash_{\mathbf{L}} t$; and if (ii), then $c_{p+2} = t/t$, so $T_{p+2} \vdash_{\mathbf{L}} t/t$. On the other hand: T_{p+2} is non-empty, since $T_{p+2} \vdash_{\mathbf{L}} c_{p+2}$; T_{p+2} is a sequence of A-categories, since t in T, t is part of T_{p+1} (which must be non-empty since $T_{p+1} \vdash_{\mathbf{L}} c_{p+1}$); and T_{p+2} contains less occurrences of $at_1, \ldots, at_k$, $((t/t)/(t/t))/(t/t)$ and $(t/t)/t$ than T, since c occurs in T but not in T_{p+2}. Therefore, the induction hypothesis for (**b**) $(n = 0)$ yields that $T_{p+2} \not\vdash_{\mathbf{L}} t$, while the induction hypothesis for (**a**) yields that $T_{p+2}, t \not\vdash_{\mathbf{L}} t$. Because $t/t \in \lfloor t/t\rfloor$, the latter entails – by (51) – that $T_{p+2} \not\vdash_{\mathbf{L}} t/t$. So, both (i) and (ii) lead to contradiction, which means that $T, t \not\vdash_{\mathbf{L}} t$.

(**b**) Suppose $T, (t/t)^n \vdash_{\mathbf{L}} t$. By (53), there is a $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$ such that $T, (t/t)^n = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$:

[30] This is easily seen by induction on the number of occurrences of $at_1, \ldots, at_k$, $(t/t)/t$ and $((t/t)/(t/t))/(t/t)$ in c.

$T_i \vdash_{\mathbf{L}} c_i$. Since T is non-empty, this c cannot be a category in $(t/t)^n$. Hence c is an A-category in T and either (i) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/t/c_{p+3}/.../c_{p+q}\rfloor$; or (ii) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/(t/t)/(t/t)/c_{p+4}/.../c_{p+q}\rfloor$. Focus on T_{p+1}. On the one hand: both (i) and (ii) entail that $c_{p+1} = t$, so $T_{p+1} \vdash_{\mathbf{L}} t$. On the other hand: T_{p+1} cannot be of the form $(t/t)^m$ for $m \leq n$, since $(t/t)^m$ and t have different t-counts; hence T_{p+1} consists of a non-empty subsequence T' of T followed by $(t/t)^n$, where T' contains less occurrences of $at_1, \ldots, at_k$, $((t/t)/(t/t))/(t/t)$ and $(t/t)/t$ than T, since c occurs in T but not in T_{p+1}. Therefore, the induction hypothesis of (**b**) yields that $T', (t/t)^n \not\vdash_{\mathbf{L}} t$ in both cases. Since $T', (t/t)^n = T_{p+1}$, we have a contradiction. Consequently, $T, (t/t)^n \not\vdash_{\mathbf{L}} t$.

Proof of (**c**):
Suppose $T, t/t, t \vdash_{\mathbf{L}} t$. By (53), there is a $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$ such that $T, t/t, t = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$. Since T is non-empty, this c cannot be t/t or t in $T, t/t, t$. Hence c is an A-category in T and either (i) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/t/c_{p+3}/.../c_{p+q}\rfloor$; or (ii) $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/t/(t/t)/(t/t)/c_{p+4}/.../c_{p+q}\rfloor$.

Suppose (i). Then on the one hand: $c_{p+2} = t$, so $T_{p+2} \vdash_{\mathbf{L}} t$. But on the other hand: $T_{p+1} \vdash_{\mathbf{L}} t$ entails that T_{p+1} is non-empty and includes at least t. Hence t/t must be part of (a) T_{p+1} or (b) T_{p+2}. Suppose (a). Then T_{p+2} is a sequence of A-categories which is, moreover, non-empty since $T_{p+2} \vdash_{\mathbf{L}} c_{p+2}$, so that $T_{p+2} \vdash_{\mathbf{L}} t$ contradicts Lemma (**b**) $(n = 0)$. Suppose (b). Then T_{p+2} consists of a sequence T' of A-categories followed by t/t and T' must be non-empty since $t/t \not\vdash_{\mathbf{L}} t$, so that $T_{p+2} \vdash_{\mathbf{L}} t$ contradicts Lemma (**b**) $(n = 1)$.

Suppose (ii). Then on the one hand: $c_{p+3} = t/t$, so $T_{p+3} \vdash_{\mathbf{L}} t/t$ and $T_{p+3}, t \vdash_{\mathbf{L}} t$ by (51). But on the other hand: $T_{p+1} \vdash_{\mathbf{L}} t$ entails that T_{p+1} is non-empty and includes at least t. Hence t/t must be part of T_{p+1} or T_{p+2}. Anyway, T_{p+3} is a sequence of A-categories which is, moreover, non-empty since $T_{p+3} \vdash_{\mathbf{L}} c_{p+3}$, so that $T_{p+3}, t \vdash_{\mathbf{L}} t$ contradicts Lemma (**a**).

All cases lead to contradiction, so $T, t/t, t \not\vdash_{\mathbf{L}} t$. □

Corollary:
(1) There is no sequence S of A-categories such that
$S, t, t = T''', T'', T'$, where $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$ and $T' \vdash_{\mathbf{L}} t$.

Suppose the contrary. Then T''', T'' and T' are non-empty, so the second t in S, t, t is part of T', and the first t is part of T'' or T'. Either way T''' is a non-empty sequence of A-categories. But $T''' \vdash_{\mathbf{L}} t/t$ entails $T''', t \vdash_{\mathbf{L}} t$ by (51), and the latter contradicts Lemma (**a**).

(2) There is no sequence S of A-categories such that
$S, t/t, t/t, t = T'', T'$, where $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$.

Suppose the contrary. Then T'' and T' are non-empty, so the category t in $S, t/t, t/t, t$ is part of T', so that $T'' = S', (t/t)^m$, where $m \in \{0, 1, 2\}$ and S' is (a subsequence of) S. But then $T'' \not\vdash_{\mathbf{L}} t$, since $(t/t)^m \not\vdash_{\mathbf{L}} t$ by t-count, and for non-empty S': $S', (t/t)^m \not\vdash_{\mathbf{L}} t$ by Lemma (**b**).

(3) There is no non-empty sequence S of A-categories such that
$S, t, t = T'', T'$, where $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$.

Suppose the contrary. Then T'' and T' are non-empty, so the second t in S, t, t is part of T', and (i) $T'' = S, t$; or (ii) $T'' = S'$ and non-empty S' is (a subsequence of) S. Now, (ii) contradicts Lemma (**b**) ($n = 0$), and (i) contradicts Lemma (**a**) for non-empty S. Hence S is empty (and $T'' = T' = t$).

(4) There is no non-empty sequence S of A-categories such that
$S, t/t, t/t, t = T''', T'', T'$, where $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$ and $T' \vdash_{\mathbf{L}} t$.

Suppose the contrary. Then T''', T'' and T' are non-empty, and T''' is not a subsequence of S, since $T''' \vdash_{\mathbf{L}} t/t$ entails that $T''', t \vdash_{\mathbf{L}} t$ by (51), contradicting Lemma (**a**). So T''' includes the first t/t in $S, t/t, t/t, t$ – but not the second one, for then T'' or T' would have to be empty. Hence $T''' = S, t/t$ (so that $T'' = t/t$ and $T' = t$) and S is empty, since $S, t/t \vdash_{\mathbf{L}} t/t$ entails $S, t/t, t \vdash_{\mathbf{L}} t$ by (53), and the latter is impossible for non-empty S on account of Lemma (**c**).

Let t and at_0 be two distict atomic categories. The following claim shows that the compound categories $(t/t)/t$ and $((t/t)/(t/t))/(t/t)$ can be used for encoding t and at_0, respectively.

Claim 1:
Let the set $\text{AT} = \{t, at_0, at_1, \ldots, at_k\}$ consist of distinct atomic categories; and let σ be the substitution $[t := (t/t)/t; at_0 := ((t/t)/(t/t))/(t/t)]$. Then for all T, c in CAT_{AT}: $T \vdash_{\mathbf{L}} c$ iff $\sigma(T) \vdash_{\mathbf{L}} \sigma(c)$.

Proof: by induction on $d(T \vdash c)$, the degree of $T \vdash c$.

• $d(T \vdash c) = 0$. Then the categories T, c are members of the set $\text{AT} = \{t, at_0, at_1, \ldots, at_k\}$, while the categories $\sigma(T), \sigma(c)$ are members of the set $\text{AT}' = \{(t/t)/t, ((t/t)/(t/t))/(t/t), at_1, \ldots, at_k\}$, and the claim holds in view of the fact that both for $T, c \in \text{AT}$ and for $T, c \in \text{AT}'$ we have that $T \vdash_{\mathbf{L}} c$

entails that $T = c$. This is straightforward for $T, c \in \text{AT}$ (by at_i-count for $at_i \in \text{AT}$). For $T, c \in \text{AT}'$:

$\circ$ If $T \vdash_{\mathbf{L}} at_j$ and $1 \leq j \leq k$, then for $c' \in \lfloor c_p\backslash ...\backslash c_1\backslash at_j/c_{p+1}/.../c_{p+q}\rfloor$: $T = T_1, \ldots, T_p, c', T_{p+q}, \ldots, T_{p+1}$ (and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$) by (53). The only member of AT' in $\lfloor c_p\backslash ...\backslash c_1\backslash at_j/c_{p+1}/.../c_{p+q}\rfloor$ is at_j, and $at_j \in \lfloor at_j \rfloor$. Therefore, $p+q = 0$ and $T = at_j$.

$\circ$ If $T \vdash_{\mathbf{L}} (t/t)/t$, then $T, t, t \vdash_{\mathbf{L}} t$ by (51), since $(t/t)/t \in \lfloor t/t/t \rfloor$. By (53), for $c' \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$: $T, t, t = T_1, \ldots, T_p, c', T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$. For $c' \in \text{AT}'$, this entails (i) $c' = (t/t)/t$ and $c' \in \lfloor t/t/t \rfloor$; or (ii) $c' = ((t/t)/(t/t))/(t/t)$ and $c' \in \lfloor t/t/(t/t)/(t/t)\rfloor$. If (ii), then $T, t, t = ((t/t)/(t/t))/(t/t), S, t, t$ and $S, t, t = T''', T'', T'$, where $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$ and $T' \vdash_{\mathbf{L}} t$ – which is impossible by Corollary (1). So, assume (i). Then $T, t, t = (t/t)/t, S, t, t$ and $S, t, t = T'', T'$, where $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$ – which, by Corollary (3), entails that S is empty and, consequently, that $T = (t/t)/t$.

$\circ$ If $T \vdash_{\mathbf{L}} ((t/t)/(t/t))/(t/t)$, then $T, t/t, t/t, t \vdash_{\mathbf{L}} t$ by (51), due to the fact that $c \in \lfloor t/t/(t/t)/(t/t)\rfloor$. By (53), for $c' \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$: $T, t/t, t/t, t = T_1, \ldots, T_p, c', T_{p+q}, \ldots, T_{p+1}$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$, so that again (i) $c' = (t/t)/t$; or (ii) $c' = ((t/t)/(t/t))/(t/t)$. If (i), then $T, t/t, t/t, t = (t/t)/t, S, t/t, t/t, t$ and $S, t/t, t/t, t = T'', T'$, where $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$ – which is impossible by Corollary (2). So, assume (ii). Then $T, t/t, t/t, t = ((t/t)/(t/t))/(t/t), S, t/t, t/t, t$ and $S, t/t, t/t, t = T''', T'', T'$, where $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$ and $T' \vdash_{\mathbf{L}} t$ – which, by Corollary (4), entails that S is empty and, consequently, that $T = ((t/t)/(t/t))/(t/t)$.

$\bullet$ $d(T \vdash c) > 0$. If $c \in \text{CAT}_{\text{AT}}$ and $c \in \lfloor c_p\backslash ...\backslash c_1\backslash at/c_{p+1}/.../c_{p+q}\rfloor$, then:
(A) $at \in \{at_1, \ldots, at_k\}$ and $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash at/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$;
(B) $at = t$ and $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$; or
(C) $at = at_0$ and $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$.
Since $p+q > 0$ or $p+q = 0$, six cases can be distinguished:

$\circ$ $c \in \lfloor c_p\backslash ...\backslash c_1\backslash at_j/c_{p+1}/.../c_{p+q}\rfloor$, $1 \leq j \leq k$, and $p+q > 0$:
$T \vdash_{\mathbf{L}} c$ iff$_1$ $c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1} \vdash_{\mathbf{L}} at_j$
iff$_2$ $\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} \sigma(at_j) =$
$\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} at_j =$

$\sigma(c_1), \ldots, \sigma(c_p), \sigma(T), \sigma(c_{p+q}) \ldots, \sigma(c_{p+1}) \vdash_{\mathbf{L}} at_j$

iff$_3$ $\sigma(T) \vdash_{\mathbf{L}} \sigma(c)$.

'iff$_1$' and 'iff$_3$' hold by (51) (since $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash at_j / c_{p+1} / ... / c_{p+q} \rfloor$, while $\sigma(c) \in \lfloor \sigma(c_p) \backslash ... \backslash \sigma(c_1) \backslash at_j / \sigma(c_{p+1}) / ... / \sigma(c_{p+q)} \rfloor$ due to (A); and 'iff$_2$' holds by induction hypothesis ($d(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1} \vdash at_j) < d(T \vdash c)$, because $p + q > 0$).

$\circ$ $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash t / c_{p+1} / ... / c_{p+q} \rfloor$ and $p + q > 0$:

$T \vdash_{\mathbf{L}} c$ iff$_1$ $c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1} \vdash_{\mathbf{L}} t$

iff$_2$ $\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} \sigma(t) =$

$\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} (t/t)/t$

iff$_3$ $\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}), t, t \vdash_{\mathbf{L}} t =$

$\sigma(c_1), \ldots, \sigma(c_p), \sigma(T), \sigma(c_{p+q}) \ldots, \sigma(c_{p+1}), t, t \vdash_{\mathbf{L}} t$

iff$_4$ $\sigma(T) \vdash_{\mathbf{L}} \sigma(c)$.

'iff$_1$', iff$_3$' and iff$_4$' hold by (51) ($c \in \lfloor c_p \backslash ... \backslash c_1 \backslash t / c_{p+1} / ... / c_{p+q} \rfloor$, $(t/t)/t \in \lfloor t/t/t \rfloor$, and $\sigma(c) \in \lfloor \sigma(c_p) \backslash ... \backslash \sigma(c_1) \backslash t/t/t/\sigma(c_{p+1}) / ... / \sigma(c_{p+q)} \rfloor$ due to (B)); and 'iff$_2$' holds by induction hypothesis (since $p + q > 0$).

$\circ$ $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash at_0 / c_{p+1} / ... / c_{p+q} \rfloor$ and $p + q > 0$:

$T \vdash_{\mathbf{L}} c$ iff$_1$ $c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1} \vdash_{\mathbf{L}} at_0$

iff$_2$ $\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} \sigma(at_0) =$

$\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}) \vdash_{\mathbf{L}} ((t/t)/(t/t))/(t/t)$

iff$_3$ $\sigma(c_1, \ldots, c_p, T, c_{p+q} \ldots, c_{p+1}), t/t, t/t, t \vdash_{\mathbf{L}} t =$

$\sigma(c_1), \ldots, \sigma(c_p), \sigma(T), \sigma(c_{p+q}) \ldots, \sigma(c_{p+1}), t/t, t/t, t \vdash_{\mathbf{L}} t$

iff$_4$ $\sigma(T) \vdash_{\mathbf{L}} \sigma(c)$.

'iff$_1$', iff$_3$' and iff$_4$' hold by (51) (since $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash at_0 / c_{p+1} / ... / c_{p+q} \rfloor$, $((t/t)/(t/t))/(t/t) \in \lfloor t/t/(t/t)/(t/t) \rfloor$ and – on account of (C) – $\sigma(c) \in \lfloor \sigma(c_p) \backslash ... \backslash \sigma(c_1) \backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1}) / ... / \sigma(c_{p+q)} \rfloor$); and 'iff$_2$' holds by induction hypothesis (since $p + q > 0$).

$\circ$ $c \in \lfloor at_j \rfloor$ and $1 \leq j \leq k$:

$T \vdash_{\mathbf{L}} at_j$ iff$_1$ for $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash at_j / c_{p+1} / ... / c_{p+q} \rfloor$:

$T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$

and for all i, $1 \leq i \leq p + q$: $T_i \vdash_{\mathbf{L}} c_i$

iff$_2$ for $c \in \lfloor c_p \backslash ... \backslash c_1 \backslash at_j / c_{p+1} / ... / c_{p+q} \rfloor$:

$T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$

and for all i, $1 \leq i \leq p + q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$

iff$_3$ for $\sigma(c) \in \lfloor \sigma(c_p) \backslash ... \backslash \sigma(c_1) \backslash at_j / \sigma(c_{p+1}) / ... / \sigma(c_{p+q}) \rfloor$:

$\sigma(T) = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1})$

and for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$
iff$_4$ $\sigma(T) \vdash_{\mathbf{L}} at_j$.

Note that $at_j = \sigma(at_j)$, and that 'iff$_1$' holds by (53); 'iff$_2$' holds by induction hypothesis ($d(T \vdash_{\mathbf{L}} at_j) > 0$ entails that $p+q > 0$, hence $d(T_i \vdash_{\mathbf{L}} c_i) < d(T \vdash_{\mathbf{L}} at_j)$ for all i); 'iff$_3$' holds by (A); and 'iff$_4$' holds by (53).

∘ $c \in \lfloor t \rfloor$:

$T \vdash_{\mathbf{L}} t$ iff$_1$ for $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$:
$T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$
and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$
iff$_2$ for $c \in \lfloor c_p\backslash ...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$:
$T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$
and for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$
iff$_3$ for $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
$\sigma(T) = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1})$
and for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$
iff$_4$ $\sigma(T), t, t \vdash_{\mathbf{L}} t$
iff$_5$ $\sigma(T) \vdash_{\mathbf{L}} (t/t)/t$.

Note that $(t/t)/t = \sigma(t)$, and that 'iff$_1$' holds by (53); 'iff$_2$' holds by induction hypothesis; 'iff$_3$' holds by (B); 'iff$_5$' holds by (51) (since $(t/t)/t \in \lfloor t/t/t \rfloor$); and the 'only if' part of 'iff$_4$' is an application of (53) (since $t \vdash_{\mathbf{L}} t$). As for the 'if' part of 'iff$_4$': if the final value of $\sigma(c)$ is t, then either

$$\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor \text{ or}$$
$$\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor.$$

Hence if $\sigma(T), t, t \vdash_{\mathbf{L}} t$, then, by (53), either
(i) for some $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
– $\sigma(T), t, t = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1}), T''', T'', T'$,
– for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$, and
– $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$, and $T' \vdash_{\mathbf{L}} t$; or
(ii) for some $\sigma(c) \in \lfloor \sigma(c_p)\backslash ...\backslash \sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
– $\sigma(T), t, t = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1}), T'', T'$,
– for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$, and
– $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$.

However, (i) is impossible by Corollary (1), and Corollary (3) entails that (ii) is only possible if $T'' = T' = t$.

$\circ\ c \in \lfloor at_0 \rfloor$:

$T \vdash_{\mathbf{L}} at_0$ iff$_1$ for $c \in \lfloor c_p\backslash...\backslash c_1\backslash t/c_{p+1}/.../c_{p+q}\rfloor$:
$\quad T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$
$\quad$ and for all i, $1 \leq i \leq p+q$: $T_i \vdash_{\mathbf{L}} c_i$

iff$_2$ for $c \in \lfloor c_p\backslash...\backslash c_1\backslash at_0/c_{p+1}/.../c_{p+q}\rfloor$:
$\quad T = T_1, \ldots, T_p, c, T_{p+q}, \ldots, T_{p+1}$
$\quad$ and for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$

iff$_3$ for $\sigma(c) \in \lfloor \sigma(c_p)\backslash...\backslash\sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
$\quad \sigma(T) = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1})$
$\quad$ and for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$

iff$_4$ $\sigma(T), t/t, t/t, t \vdash_{\mathbf{L}} t$

iff$_5$ $\sigma(T) \vdash_{\mathbf{L}} ((t/t)/(t/t))/(t/t)$.

Note that $((t/t)/(t/t))/(t/t) = \sigma(at_0)$, and that 'iff$_1$' holds by (53); 'iff$_2$' holds by induction hypothesis; 'iff$_3$' holds by (C); 'iff$_5$' holds by (51) (since $((t/t)/(t/t))/(t/t) \in \lfloor t/t/(t/t)/(t/t)\rfloor$); and the 'only if' part of 'iff$_4$' is an application of (53) (since $t/t \vdash_{\mathbf{L}} t/t$ and $t \vdash_{\mathbf{L}} t$). As for the 'if' part of 'iff$_4$': again, if the final value of $\sigma(c)$ is t, then either

$$\sigma(c) \in \lfloor \sigma(c_p)\backslash...\backslash\sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor \text{ or}$$
$$\sigma(c) \in \lfloor \sigma(c_p)\backslash...\backslash\sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor.$$

Hence if $\sigma(T), t/t, t/t, t \vdash_{\mathbf{L}} t$, then, by (53), either

(i) for some $\sigma(c) \in \lfloor \sigma(c_p)\backslash...\backslash\sigma(c_1)\backslash t/t/t/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
- $\sigma(T), t/t, t/t, t = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1}), T'', T'$,
- for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$, and
- $T'' \vdash_{\mathbf{L}} t$ and $T' \vdash_{\mathbf{L}} t$; or

(ii) for some $\sigma(c) \in \lfloor \sigma(c_p)\backslash...\backslash\sigma(c_1)\backslash t/t/(t/t)/(t/t)/\sigma(c_{p+1})/.../\sigma(c_{p+q})\rfloor$:
- $\sigma(T), t/t, t/t, t = \sigma(T_1), \ldots, \sigma(T_p), \sigma(c), \sigma(T_{p+q}), \ldots, \sigma(T_{p+1}), T''', T'', T'$,
- for all i, $1 \leq i \leq p+q$: $\sigma(T_i) \vdash_{\mathbf{L}} \sigma(c_i)$, and
- $T''' \vdash_{\mathbf{L}} t/t$, $T'' \vdash_{\mathbf{L}} t/t$, and $T' \vdash_{\mathbf{L}} t$.

This time, (i) is impossible by Corollary (2), and Corollary (4) entails that (ii) is only possible if $T''' = T'' = t/t$ and $T' = t$. $\Box$

Finally, the following claim shows that one can generalize the substitution used in Claim 1 to encode any finite number of atomic categories. Let, for $c \in$ CAT and $n \in \mathbb{N}$:

$$\beta(c) = ((c/c)/(c/c))/(c/c);$$
$$\alpha(c) = (c/c)/c;$$

$\alpha^0(c) = c$; and
$\alpha^{n+1}(c) = \alpha^n(\alpha(c))$.

Claim 2: Let $A = \langle t, at_1, \ldots, at_m \rangle$ be a sequence of distinct atomic categories such that $m \geq 1$, and let σ_A be the substitution

$$[t := \alpha^m(t); at_1 := \beta(\alpha^{m-1}(t)); \ldots; at_m := \beta(\alpha^{m-m}(t))].$$

Then for all T, c in $\text{CAT}_{\{t,at_1,\ldots,at_m\}}$: $T \vdash_{\mathbf{L}} c$ iff $\sigma_A(T) \vdash_{\mathbf{L}} \sigma_A(C)$.

Proof: by induction on m.

• $m = 1$. Then Claim 2 comes down to Claim 1 (with at_0 and k instantiated as at_1 and 0, respectively).

• $m > 1$. Observe (i) that $\sigma_A(c) = \sigma'_A(\sigma''_A(c))$ for the substitutions $\sigma'_A = [t := \alpha^{m-1}(t); at_1 := \beta(\alpha^{(m-1)-1}(t)); \ldots; at_{m-1} := \beta(\alpha^{(m-1)-(m-1)}(t))]$ and $\sigma''_A = [t := \alpha(t); at_m := \beta(t)]$; and (ii) that $\sigma'_A(c) = \sigma_{A'}(c)$ for the sequence $A' = \langle t, at_1, \ldots, at_{m-1} \rangle$. Consequently, we have the following equivalences:
$\sigma_A(T) \vdash_{\mathbf{L}} \sigma_A(c)$ iff$_1$ $\sigma'_A(\sigma''_A(T)) \vdash_{\mathbf{L}} \sigma'_A(\sigma''_A(c))$
iff$_2$ $\sigma''_A(T) \vdash_{\mathbf{L}} \sigma''_A(c)$
iff$_3$ $T \vdash_{\mathbf{L}} c$.
'iff$_1$' holds by observation (i); 'iff$_2$' holds by induction hypothesis and observation (ii) (note that $m - 1 < m$, and that $\sigma''_A(c) \in \text{CAT}_{\{t,at_1,\ldots,at_{m-1}\}}$ if $c \in \text{CAT}_{\{t,at_1,\ldots,at_m\}}$); and 'iff$_3$' is another application of Claim 1 (with at_0 and k instantiated as at_m and $m - 1$, respectively). □

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